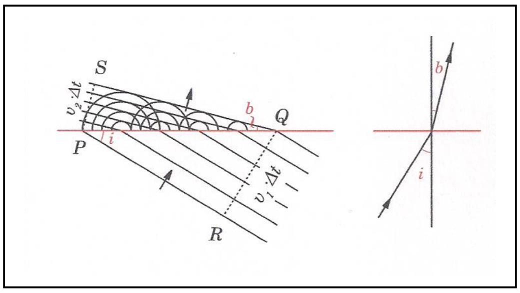
1. The law of refraction

The principle of least time belongs to the most fundamental concepts in physics.
Thus, this principle is the theoretical foundation of the Lagrange formalism, which leads to the equations of motion in analytical mechanics as well as to the geodesics in General Relativity.

Regarding waves and especially light, the principle states that light always travels from one point to another using a trajectory that takes the least time.

Trivially (but nontrivially in General Relativity) it means that light propagates along straight lines (or geodesics), when it propagates in the same material. When light propagates from one material to another, e.g. from air to glass or water, this is no longer the case. The trajectory that the light follows is given by the well known law of refraction.



To the right we have illustrated the phenomena.
The light comes from below, where the speed is v_1.
Then it traverses an interface to a material, where the speed is v_2. The figure is chosen so that: $v_1 > v_2$, reflecting the fact that the wavelength becomes smaller after the passage of the interface.
From the two right angle triangles *PQR* and *PQS* shown in the figure, we see that:

$$\sin i = \frac{v_1 t}{|PQ|} \qquad \textit{and} \qquad \sin b = \frac{v_2 t}{|PQ|}$$

Where *i* is the angle between the incident wave front and the interface and *b* is the angle between the refracted wave front and the interface. Or, as it is usually stated in geometric optics: The angles *i* and *b* are the angles between the the direction of the rays, and the normal to the interface.

By division of the two equations we obtain the law of refraction.

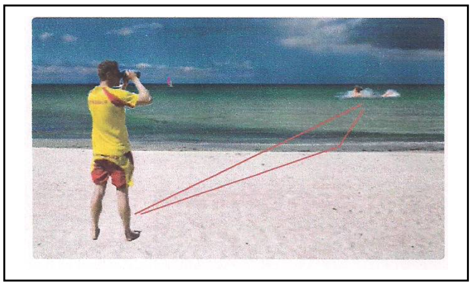
(1.1) $$\frac{\sin i}{\sin b} = \frac{v_1}{v_2} = n_{12}$$

where n_{12} is called the refractive index.

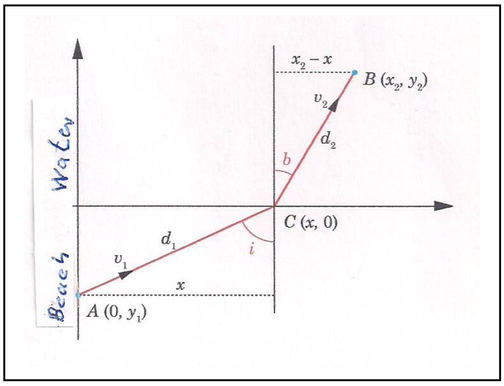
Our aim is to show that the law of refraction can be derived from the principle of least time.

2. Obeying the principle of least time.

In our argument, we shall however first consider another quite different, but as it turns out, a related problem.



A lifeguard standing on the beach spots a beautiful girl in the water crying for help. We assume that the position of the girl is not straight forward from the position of the lifeguard. The lifeguard moves faster on the beach than he can swim, so his problem is how long should he run on the beach before he enters the water to reach the girl in the least time. He should probably not choose a straight line. To decide on this issue, we insert a coordinate system as shown in the figure below.



The two distances that the lifeguard must run and swim are denoted d_1 and d_2. At the instant where the lifeguard begins to run, he has the position $A\ (0, y_1)$, and the girl has the position $B\ (x_2, y_2)$. The point, where the lifeguard enters the water has the coordinates $C\ (x, 0)$.
The lifeguard can run on the beach with the velocity v_1 , and he can swim in the water with the velocity v_2.
We then realize that this situation is a completely analogy to the light ray passing an interface, and for that reason we have kept the same designations for the two angles.

From the figure, we can see that $d_1 = |AC|$ and $d_2 = |BC|$.

According to the formula for the distances between two points in a plane, we have:

$$d_1 = \sqrt{x^2 + y_1^2} \quad \text{and} \quad d_2 = \sqrt{(x_2 - x)^2 + y_2^2}$$

And according to the trigonometric formulas for a right angle triangle, we have

$$\sin i = \frac{x}{d_1} = \frac{x}{\sqrt{x^2 + y_1^2}} \qquad \text{and} \qquad \sin b = \frac{x_2 - x}{d_2} = \frac{x_2 - x}{\sqrt{(x_2 - x)^2 + y_2^2}}$$

If you move the distance s in time t with the velocity v, then $s = vt$ and it then follows that $t = s/v$. The lifeguard moves on the beach the distance d_1 with velocity v_1 and he swims in the water the distance d_2 with velocity v_2. From this follows that the total time $t(x)$, it takes the lifeguard to reach the girl is:

(2.1) $$t(x) = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{x^2 + y_1^2}}{v_1} + \frac{\sqrt{(x_2 - x)^2 + y_2^2}}{v_1}$$

We shall then the use the ordinary analytic methods to find the minimum of $t(x)$, so we differentiate, and put the differential quotient to zero.

$$t'(x) = \frac{1}{v_1}\frac{2x}{2\sqrt{x^2+y_1^2}} + \frac{1}{v_2}\frac{-2(x_2-x)}{2\sqrt{(x_2-x)^2+y_2^2}} \quad \Rightarrow$$

$$t'(x) = \frac{1}{v_1}\frac{x}{\sqrt{x^2+y_1^2}} - \frac{1}{v_2}\frac{(x_2-x)}{\sqrt{(x_2-x)^2+y_2^2}}$$

If we try to solve the equation *t'*(*x*) = 0, we will (probably) get nowhere. But with the law of refraction in mind, we realize that:

$$\sin i = \frac{x}{d_1} = \frac{x}{\sqrt{x^2+y_1^2}} \qquad \text{and} \qquad \sin b = \frac{x_2-x}{d_2} = \frac{x_2-x}{\sqrt{(x_2-x)^2+y_2^2}}$$

And consequently *t'(x)* can be written as:

$$t'(x) = \frac{\sin i}{v_1} - \frac{\sin b}{v_2}$$

The equation *t'(x)=0* therefore has the solution:

$$\frac{\sin i}{v_1} - \frac{\sin b}{v_2} = 0 \qquad \Leftrightarrow \qquad \frac{\sin i}{\sin b} = \frac{v_1}{v_2}$$

So indeed the law of refraction is a consequence of the principle of least time.

I have examined this example for 15 years in the Danish high school, and especially since 2010, where CAS' almost have completely supplanted mathematical textbooks, paper, brainwork and pencils, I have used it as an example, demonstrating that no computer can compete with a theoretical survey.

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