Physics of Tsunamis

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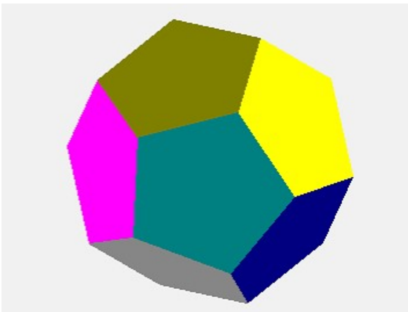

Ole Witt-Hansen 2011 (2016)

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1. Constructing a differential equation for the Tsunami wave

In all those years, where I have been teaching the physics of waves, I have emphasized, that the propagation of a wave is not a transportation of matter, but rather a periodic propagation of energy and momentum. And almost each time the class has confronted me with the fact that sea waves break and subsequently rush on the shore.
In these cases I normally choose the classic evasion technique, saying: Yes that is correct, but unfortunately it is far to complicated to comprehend at this level of education. True enough, if we are talking hydrodynamics, but if an audacious student asks: But do you understand it yourself? Then my answer has been evasive, in the sense that I merely explain that a rising water crest is created, because the dept of the water decreases.
But this is just an phenomenological ascertainment, it is not an explanation of physics.

When I have been seeking for an analytic treatment founded on theoretical physics I have only found some less conspicuous articles, so I decided to find out for myself.

At the University of Copenhagen we were in 1967 taught hydrodynamic from the book by Arnold Sommerfeld: Mechanics of Deformable Bodies, from 1964, but without a thorough knowledge of vector analysis, that is, the operator's of vector analysis *grad*, *div* and *rot* including the theorems of Stoke and Green, Sommerfeld's book is not really applicable.

In 2011 the "subject of the year" at the Danish gymnasium (senior high) was "catastrophes", and many students chose to write about the Tsunami in 2004, and I was their physic teacher.

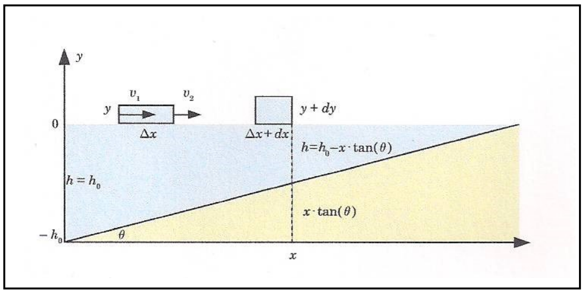
I certainly searched for extern assistance, but the only thing, that was really tangible, was a remark in Sommerfelds book, accompanying two formulas for the velocity of propagation in deep water and in medium deep water.

(1.1) Deep water: $v = \sqrt{\dfrac{g\lambda}{2\pi}} = \sqrt{\dfrac{g}{k}}$ Medium deep water: $v = \sqrt{gh}$

h is the depth of water, and g is the gravitational acceleration and λ is the wavelength.

In deep water the velocity of propagation is independent of the depth, but in medium water it is not, and that is the key to understand the physics of the tsunamis.
The derivation of the formulas (1.1) is however far from simple. The derivations are postponed until section 2.

One may obtain a phenomenological understanding of the creation of a tsunami, propagating towards lower water from the velocity formula. Namely the back end of a wave packet will steadily run faster than the front end, because the velocity decreases with the water depth.
In the long run the back of the wave packet will crawl up on the front of the wave, and create a higher wave crest. This happens continually and the result may become a wave crest which is rising and finally breaks before reaching the shore, flooding the surroundings of the coast line, that is, a tsunami.



To do a more formal analysis, it will be necessary with some simplifying assumptions.
First we assume that the wave has a rectangular shape, where the bottom floats on the surface of the water. Further we assume that the water depth decreases linearly with distance from a reference point. At $x = 0$, the seabed is at h_0 and at x it is $h = h_0 - x\tan\theta$

The velocities of the wave-packet (the box) are at the rear end and at the front end, v_1 and v_2 respectively, where $v_1 > v_2$.
Since the wave-packet (the box) holds the same amount of water when moving, but changing its shape, it follows that the area A of the two rectangles shown in the figure must be the same.

$A = y\Delta x = (\Delta x+dx)(y+dy)$, which gives: $\Delta x dy + y dx = 0$

Solved with respect to dy to give:

(1.1) $$dy = -\frac{y}{\Delta x}dx$$

For medium deep water, we have the formula:

(1.2) $$v = \sqrt{gh} = \sqrt{g(h_0 - x\tan\theta)}\ .$$

Because of the different velocities of the wave-packet at the rear and at the front, the wave-packet (the box) will be compressed an amount dx during the time dt.

$$dx = (v_2 - v_1)dt = \Delta v dt$$

When inserted in (1.1) gives:

(1.3) $$\begin{aligned} dy &= -\frac{y}{\Delta x}dx = -\frac{y}{\Delta x}\Delta v dt \quad \Leftrightarrow \\ dy &= -y\frac{\Delta v}{\Delta x}dt \approx y\frac{dv}{dx}dt \quad \Rightarrow \\ \frac{dy}{dt} &= -y\frac{dv}{dx} \end{aligned}$$

Furthermore $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = v\frac{dy}{dx}$$

This leads to the wanted differential equation.

(1.4) $$\frac{dy}{dx} = -\frac{y}{v}\frac{dv}{dx}$$

Substituting:

$$v = \sqrt{g(h_0 - x\tan\theta)} \quad \text{and} \quad \frac{dv}{dx} = -\frac{\sqrt{g}\tan\theta}{2\sqrt{h_0 - x\tan\theta}}$$

We finally arrive at a differential equation, which determine how the height of the wave crest depends on the distance to the shore.

(1.5) $$\frac{dy}{dx} = \tfrac{1}{2}\frac{y\cdot\tan\theta}{h_0 - x\tan\theta} \quad \Leftrightarrow \quad \frac{dy}{y} = \tfrac{1}{2}\frac{\tan\theta dx}{h_0 - x\tan\theta}$$

The last differential equation can then be integrated.

(1.6) $$\int_{y_0}^{y}\frac{dy}{y} = \tfrac{1}{2}\int_0^x \frac{\tan\theta dx}{h_0 - x\tan\theta} \quad \Leftrightarrow \quad [\ln y]_{y_0}^{y} = -\tfrac{1}{2}[\ln(h_0 - x\tan\theta)]_0^x$$

When the equation is solved for y it yields:

(1.7) $$y = y_0\sqrt{\frac{h_0}{h_0 - x\tan\theta}}$$

Even if we have used a highly simplified model, the solution reflects the behaviour of a huge wave approaching the shore.
The wave crest becomes infinite, when the depth approaches zero, but this is a mathematical fact, not a physical one.
One could easily plot y, the height of the wave crest, as a function of the distance to the shore, but it is more illustrative to plot a wave-packet rolling in towards the shore.

A harmonic wave can be written (as the real part of) $\phi(x,t) = e^{i(\omega t - kx)}$
Integrating over the wave number $k = 2\pi/\lambda$ one obtains a wave-packet.

(1.8) $$\psi = e^{i\omega t}\int e^{-ikx}dk = e^{i\omega t}\frac{e^{-ikx}}{-ix} = e^{i\omega t}\frac{\cos(kx) - i\sin(kx)}{-ix} \quad ,$$

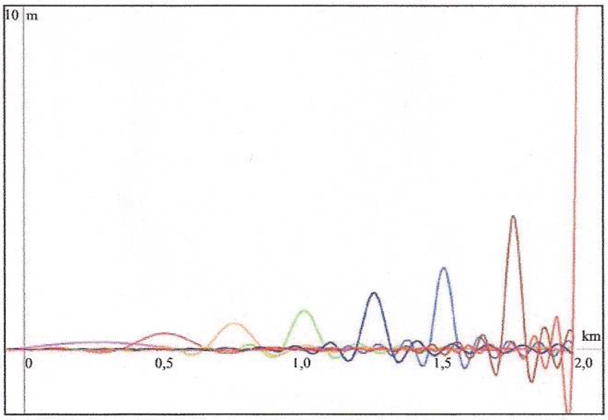
Which after rewriting using trigonometric formulas: $\mathrm{Re}\,\psi = \dfrac{\sin(kx - \omega t)}{x}$

At $t = 0$ the shape of the wave-packet is: $y_0\dfrac{\sin kx}{x}$, and the shape in x_0, may be found multiplying the height of the wave at x_0.

Since $\dfrac{\sin x}{x} \to 1 \quad for \quad x \to 0$ the maximum value of the wave packet is $y(x)$.

We the multiply (1.8) by the height of the wave crest y, we will get a model that illustrates what happens, when a Tsunami wave approaches the shore.

(1.9) $$\psi = y(x)\frac{\sin k(x-x_0)}{x-x_0}$$



The figure shows a wave approaching the shore. The rising wave is shown at various instants from 1.5 *km* from the shore until it reaches the shore, where it goes to infinity, in accordance with (1.9).
Although hydrodynamics is highly complex, the simple model developed above gives at least a qualitative answer on the dynamics of a tsunami.

2. The velocity of propagation in deep and medium deep water

The derivation of the two formulas for the velocity of propagation of a wave, require vector analysis, and we shall recall the notation:

$$\vec{\nabla}\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$$ Gradient of a scalar field φ.

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$ Divergence of a vector field ***v***.

$$\nabla^2\varphi = \vec{\nabla}\cdot\vec{\nabla}\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$ The Laplace operator

$$\vec{\nabla}\times\vec{v} = (\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})$$ The operator **rot.**

From the vector analysis we know that for a rotational free vector field i.e. $\vec{\nabla}\times\vec{v} = 0$, it is possible to define a potential from which the vector field is its gradient. Applying this to the velocity vector field ***v*** in a two dimensional field of flowing liquid without turbulence, we have

(2.1) $$\vec{\nabla}\times\vec{v} = 0 \quad \Rightarrow \quad \vec{v} = \vec{\nabla}\Phi \quad \Leftrightarrow \quad \vec{v} = (\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y})$$

If the velocity field is also divergence free: ($\vec{\nabla}\cdot\vec{v} = 0$), which means that the fluid is incompressible, then the potential will satisfy the Laplace equation, since:

$$\vec{\nabla}\cdot\vec{v} = 0 \quad \Leftrightarrow \quad \vec{\nabla}\cdot\vec{\nabla}\Phi = 0 \quad \Leftrightarrow \quad \nabla^2\Phi = 0$$

(2.2) $$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$$

For any analytic function: $f(z) = f(x+iy) = \Phi(x,y) + i\Psi(x,y)$, both the real and the imaginary part are solutions to the Laplace equation. This follows, because they satisfy the Cauchy-Riemann differential equations:

(2.3) $$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} \quad and \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

Using (2.3)

(2.4) $$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} = 0$$

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial y \partial x} = 0$$

If $\vec{v} = \vec{\nabla} \Phi$, then $\Phi(x,y) = \Phi_0$, will represent curves with the same velocity, and through the Cauchy-Riemann equation, it follows that:

$$\left(\frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y} \right) \text{ is orthogonal to } \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right),$$

So the curves $\Psi(x,y) = \Psi_0$ and $\Phi(x,y) = \Phi_0$, will be orthogonal curve systems, and for that reason $\Psi(x,y)$ will represent the streamlines.

The most general velocity potential, that represents the propagation of a wave, can be written:

(2.5) $$\Phi = e^{i(kx-\omega t)}(Ae^{-ky} + Be^{+ky})$$

Where we fix $y = 0$ at the surface.

The boundary condition at the bottom ($y = -h$), is that the vertical velocity is zero. ($v_y = 0$ or using the potential: $\frac{\partial \Phi}{\partial y}|_{y=-h} = 0$).

Inserted in (2.5) it leads to the equation:

$$-Ae^{kh} + Be^{-kh} = 0 .$$

Introducing the constant C by

$$\tfrac{1}{2}C = Ae^{kh} = Be^{-kh} \quad so\ that \quad A = \tfrac{1}{2}Ce^{-kh} \quad og \quad B = \tfrac{1}{2}Ce^{+kh},$$

the potential takes the form:

(2.6) $$\Phi = e^{i(kx-\omega t)} \tfrac{1}{2} C(e^{-k(h+y)} + e^{k(h+y)}) = e^{i(kx-\omega t)} C \cosh(k(y+h))$$

The derivation of the expression for the velocity field, takes its starting point in the Navier-Stokes equation.

(2.7) $$\rho \frac{d\vec{v}}{dt} + \vec{\nabla} p = \vec{F}$$

If the fluid is rotational free and divergence free i.e. $\vec{\nabla} \times \vec{v} = 0$ *and* $\vec{\nabla} \cdot \vec{v} = 0$, Then (using vector analysis) (2.7) can be reduced to the following form:

(2.8) $$-\rho \frac{\partial \vec{v}}{\partial t} + \tfrac{1}{2} \rho \vec{\nabla} v^2 + \vec{\nabla} p = \vec{F}$$

If F is a conservative force (the gravitational force) then F can be expressed as the gradient of a potential U.

(2.9) $$\vec{F} = -\vec{\nabla} U$$

Finally inventing the velocity potential $\vec{v} = \vec{\nabla} \Phi$, and moving the gradient outside a parenthesis we get Bernoulli's law.

(2.10) $$\vec{\nabla}(-\rho \frac{\partial \Phi}{\partial t} + \tfrac{1}{2} \rho (\vec{\nabla} \Phi)^2 + p + U) = 0$$

For moderate velocities, we may discard the term $(\vec{\nabla} \Phi)^2$ and if we preliminary are interested only in the surface profile of the wave, we may put the pressure $p = 0$.

Hereafter the equation becomes substantially simplified .

(2.11) $$-\rho \frac{\partial \Phi}{\partial t} + U = const$$

The constant may in principle depend on time, but for a non forced periodic movement, one can show that the only possibility is that it is zero. At the same time $U = \rho g y$, where ρ is the density of water and g is the gravitational acceleration. Inserting this in (2.11) gives:

(2.12) $$\frac{\partial \Phi}{\partial t} = gy$$

If we consider a progressive harmonic wave, the surface profile y is of the form:

(2.13) $$y = u(x,t) = a e^{i(kx - \omega t)},$$

Where a in general is a complex number possibly containing a phase.

Using (2.12) $\frac{\partial \Phi}{\partial t} = gy$ with (2.6) $\Phi = e^{i(kx-\omega t)} C \cosh(k(y+h))$ and (2.13) we find:

(2.14) $$-i\omega e^{i(kx-\omega t)} C \cosh(k(y+h)) = a g e^{i(kx-\omega t)} \Rightarrow -i\omega C \cosh(k(y+h)) = ag$$

To determine the velocity of propagation of the wave at the surface, we need yet another condition, which we choose that the velocity V_n of a point at the surface, normal to the surface must be the same as the corresponding velocity of the fluid element v_n , at the same point. Expressed by the velocity potential: $v_n = \frac{\partial \Phi}{\partial n}$.

We can obtain the velocity on the surface in the same point V_n as the velocity in the wave profiles up and down movements: $V_n = \frac{\partial u}{\partial t}$, where $u(x,t) = ae^{i(kx-\omega t)}$.

If the wavelength, is substantially larger than the amplitude, we may replace

$$v_n = \frac{\partial \Phi}{\partial n} \quad \text{by} \quad v_n = \frac{\partial \Phi}{\partial y}$$

Applying the last expression for the surface profile: $u(x,t)=ce^{i(kx-\omega t)}$ *,* and the velocity potential $\Phi = e^{i(kx-\omega t)} C \cosh(k(y+h))$, dropping the factor $e^{i(kx-\omega t)}$ it gives:

(2.15) $$\frac{\partial \Phi}{\partial y} = \frac{\partial u}{\partial t} \quad \Rightarrow \quad kC \sinh(ky + kh) = i\omega a$$

If we put y =0 (close to the surface) the two equations (2.14) $-i\omega C \cosh(k(y+h)) = ag$ and (2.15) $kC \sinh(ky + kh) = i\omega a$ become:

(2.16) $$-i\omega C \cosh(kh) = ag \quad \text{and} \quad kC \sinh(kh) = i\omega a:$$

Thus $$\frac{C}{a} = \frac{g}{-i\omega \cosh(kh)} = \frac{i\omega}{k \sinh(kh)} \quad \Rightarrow \quad v^2 = \frac{\omega^2}{k^2} = \frac{g}{k} \tanh(kh)$$

And indeed.

(2.17) $$v = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh(kh)}$$

We shall then look into two limiting cases:

Medium deep water: $kh << 1 \Rightarrow \tanh(kh) \approx kh$

Deep water: $kh >> 1 \Rightarrow \tanh(kh) \approx 1$

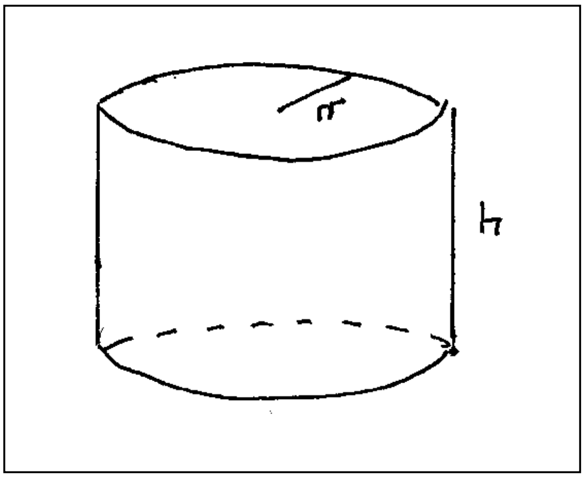
This gives the velocities:

(2.18) Deep water: $v = \sqrt{\frac{g}{k}}$ and medium deep water: $v = \sqrt{gh}$

As stated in the beginning.

3. Estimating the destructive force of a Tsunami

This section does not pretend to be a credible estimate of the destructive effect of a Tsunami, corresponding to a certain earthquake in a certain depth at a certain distance from shore, (which is probably impossible), it is only an attempt to give a qualitative understanding of the phenomena.



If an earthquake triggers a power P measured in *Watt* along the axis of a cylinder, with radius r and height h, then the radial intensity $I(r)$ measured in W/m^2, at the distance r from the axis, will be the power divided by the area of the cylinder.

(3.1) $$I(r) = \frac{P}{2\pi rh}$$

We shall the try to estimate the intensity of a tsunami, which is triggered at a distance of 2000 *km* from the earthquake.

But first, we shall estimate the power that is triggered by a lift of the seabed, as a consequence of an earthquake. We shall do some prerequisites more or less at random, and the formulas derived may be used with other initial conditions.

Consequently we suppose that the seabed is lifted $\Delta h = 1\ m$, at a circular area with $r = 10\ km$. We put the depth at that place to $h = 3\ km$. This will result in an increase in energy

(3.2) $$\Delta E = mg\Delta h = \rho_{water} V_{water} g\Delta h$$

The volume of water is $V_{water} = \pi r^2 h$. If the numbers above are inserted, the energy released is

(3.3) $$\Delta E = 9.3\ 10^{15}\ J$$

If the duration of the earthquake is $\Delta t = 5\ min = 300\ s$, it will correspond to a released power:

(3.4) $$P = \frac{\Delta E}{\Delta t} = 3.1\ 10^{13}\ W$$

We presume that the tsunami propagates from a depth of 100 *m* to the surface, and we subsequently calculate the intensity at a distance 2000 *km* from the epicentre.

(3.5) $$I(r) = \frac{P}{2\pi rh} \qquad I(2000km) = \frac{3{,}1\cdot 10^{13}}{2\pi \cdot 2\cdot 10^{6}\cdot 100} kW/m^2 = 24.6\ kW/m^2$$

In deep water the velocity is dependent on the wavelength, but we don't know the wavelength. When the tsunami approaches the shore however, we may use the formula for the velocity in medium deep water: $v = \sqrt{gh}$.

At a depth at 3.0 *km* it gives 170 *m/s*, and at a depth 10 *m* it is about 10 *m/s*.

The Power of a force, acting on a body with velocity v is: $P = \vec{F} \cdot \vec{v}$
It then follows that the Intensity = Power/area is equal to the pressure times the velocity: $I = p \cdot v$
And if we assume that the tsunami moves with velocity 10 *m/s* when it reaches the shore, this will trigger a pressure = force/m^2 amounting to:

(3.6) $$p = \frac{I}{v} = \frac{24.6\ kW/m^2}{10\ m/s} = 2.5\, kN/m^2$$

Which corresponds to a weight of 250 *kg* per m^2, being absolutely terrifying.

As mentioned above, the data are chosen much at random, but the calculation may be carried out with other data giving other results. However, when you look at a movie, observing the devastating force of a tsunami, the values obtained above do not seem unreasonable.
And now we understand better why!