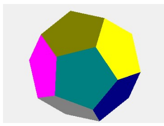
Newtonian Mechanics
Dynamics

Chapter 6 of the textbook
Elementary Physics 2

This is an article from my home page: www.olewitthansen.dk



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(1977) 2017

Acknowledgement

This paper is a digitalized translation from Danish of a chapter in a textbook on physics written in 1977, that was conceived and written for the second year of the Danish 3-year high school (called the gymnasium, for 16 -19 year-olds). From the same book is on my home page also available a translation of the chapters: Rotation and The Special Theory of relativity.

All three books: Elementary Physics 1 -3, are available on my home page, but in Danish.

Rereading it, I have found that it is still an elementary yet rigorous presentation of the theory of Newtonian dynamics, and it covers the central aspects of the theory. Therefore it can still be used as an undergraduate text on the subject of mechanics.

Also I have not found a similar straight and thorough presentation on the Internet, free from glittering pictures, with a contemporary call for using educational mathematical programs, instead of doing the analytical calculations by hand.

In the last 10 years, however, there has in Demark emerged a rising demand for this kind of textbook of physics, solidly founded on calculus, and where the basic understanding is the issue. Whether it is also the case in other parts of the world, I do not know, but I can gather it from the hitherto interest in my home page: www.olewitthansen.dk

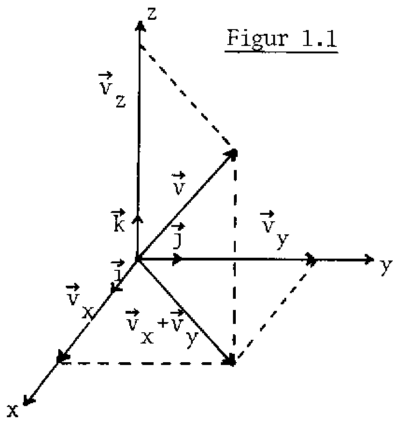
Even in 1970's it was somewhat above the standard level in the 9-12 year grade both regarding the mathematics, and today it would be entirely out of question to try to use it in the Danish Gymnasium.

Content

1. Moving in two and three dimensions

In a plane motion, some physical quantities, besides their numerical values and units, also have a direction. They are *vectors*, or more precisely they transform as vectors under coordinate transformations, so that the equations of motion are form invariant.

This is, for example, the case regarding position $\vec{r}$, velocity $\vec{v}$, acceleration $\vec{a}$ and force $\vec{F}$. Regarding motion in a plane, these quantities are usually described by their coordinates along two orthogonal directions with suffix x and y or (1) and (2).
In 3-dim space, the notations for the physical quantities are the same, the only difference is that they have three coordinates instead of two.

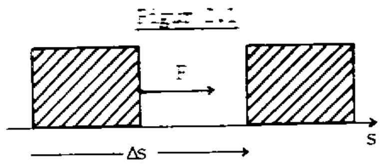


These coordinates are usually denoted (x, y, z), or indexed with (1), (2) and (3). This corresponds to a spatial coordinate system with three orthogonal axes, sometimes represented by three orthogonal base vectors $\vec{i}, \vec{j}, \vec{k}$ as depicted in the figure to the left.
Also the figure shows, how one determines the coordinates (v_x, y_x, v_z) to a vector $\vec{v}$.
All the rules for manipulating vectors in a plane, can literally be applied to vectors in 3-dim space.
For example the scalar product and the length of a vector are in 3-dim coordinates.

$$\vec{F}\cdot\vec{v} = F_x v_x + F_y v_y + F_z v_z \quad \text{and} \quad |\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$$

Generally we shall mostly consider motion in the plane, but vector equations have the advantage, that they remain the same in 3-dim space, but with their new interpretation.

2. Work



For a linear motion, we have that the work ΔW , which the force F performs at the displacement Δs is calculated as:

(2.2) $$\Delta W = F\Delta s$$

The SI unit for work is $Nm = J$. The formula holds, as long as the force F is constant along the displacement Δs .
For infinitesimal displacements ds, the force can always be considered as constant, and we may rewrite (2.2), with the help of differentials.

(2.3) $$dW = Fds \quad \Leftrightarrow \quad \frac{dW}{ds} = F(s) \quad \Leftrightarrow \quad W'(s) = F(s)$$

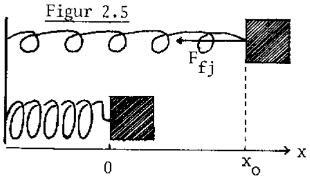
Where $F(s)$ is the force at the position s.

What (2.3) expresses is that the work W is an integral to the force F as a function of position s.
If the work is calculated from the position s_0, so that $W(s_0) = 0$, and $W_1(s)$ is an arbitrary integral to $F(s)$, then the work done by F on the distance can be calculated as: $W(s) = W_1(s) - W_1(s_0)$, or written with the integral symbol:

(2.4) $$W = W_1(s) - W_1(s_0) = [W_1(s)]_{s_0}^{s} = \int_{s_0}^{s} F(s)ds$$

2.5 Example. The potential energy of a spring.
We are going to calculate the work done by a spring, when it moves a mass *m* from a stretched position x_0 to the un stretched position at $x = 0$.

Figur 2.5

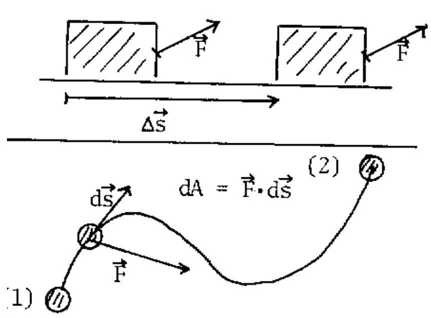


We have previously defined this work as the elastic potential energy of the spring. If *k* denotes the spring constant, the spring force is according to Hooke's law: $F = - kx$, when the spring is stretched by the length *x*. Therefore we have: $dW = Fdx = -kxdx$, and the work is found by integration:

(2.5.2) $$E_{pot} = W = \int_{x_0}^{0} -\tfrac{1}{2}kxdx = \left[-\tfrac{1}{2}kx^2\right]_{x_0}^{0} = \tfrac{1}{2}kx_0^{\,2}$$

3. Work done by a curvilinear motion

Figur 3.1



When the force and the displacement are not parallel to each other, the work ΔW that is done by the force $\vec{F}$ by the displacement $\Delta\vec{s}$ is calculated as the scalar product of $\vec{F}$ with $\Delta\vec{s}$.

(3.1) $$\Delta W = \vec{F}\cdot\Delta\vec{s}$$

If the trajectory is not linear, we must divide it in such tiny (infinitesimal) pieces $d\vec{s}$, each of which can be considered as linear. The work done on $d\vec{s}$, we shall then write:

(3.2) $$dW = \vec{F}\cdot d\vec{s}$$

And for the work done along a trajectory from a position (1) to a position (2), we shall symbolically write

(3.3) $$W = \int_{(1)}^{(2)} \vec{F}\cdot d\vec{s}$$

In this case the integral is a curve integral, what we express as: We integrate $\vec{F}$ along the trajectory from (1) to (2).

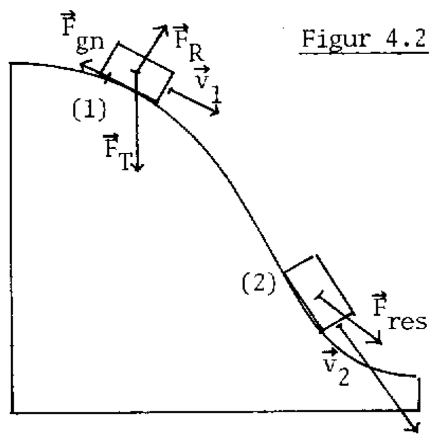
4. The power of a force. The work theorem

If we divide (3.2) by *dt*, we get:

(4.1) $$\frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{s}}{dt} = \vec{F}\cdot\vec{v}$$

dW/dt is equal to the done per unit of time, also called the *Power,* with SI unit *W.* The right hand side is often read as the power of the force.
The power of the force thus is calculated as the scalar product of the force with the velocity. The power of the force can be positive, that is, for an accelerating motion or negative for braking

motion, and especially the power of the force is zero, if the force is perpendicular to the velocity, in which case no work is done, albeit an accelerated motion is maintained.
The latter is for example the case of uniform circular motion, where the centripetal force is permanently perpendicular to the velocity.



Figur 4.2

We shall now calculate the work done by the resulting force $\vec{F}_{res}$, that is, the force that goes into Newton's 2. law: $\vec{F}_{res} = m\vec{a}$, and which is not necessarily a physical acting force, but rather the vector sum of all forces acting on the body.
So we imagine e.g. a body sliding down a slope affected by gravity and friction.
The resulting force $\vec{F}_{res}$ is the sum of gravity $\vec{F}_G$ and the reaction force from the underlay $\vec{F}_N$ (Newton's 3. law), and the force from the friction $\vec{F}_{fric}$.

$$\vec{F}_{res} = \vec{F}_G + \vec{F}_N + \vec{F}_{fric}\,.$$

From (4.1) we get:

(4.3) $$\frac{dW_{res}}{dt} = \vec{F}_{res} \cdot \frac{d\vec{s}}{dt} = \vec{F}_{res} \cdot \vec{v} = m\frac{d\vec{v}}{dt} \cdot \vec{v} = \tfrac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}(\tfrac{1}{2}mv^2)$$

From (4.3) we are then able quite generally to calculate the work done by the resulting force.

(4.4) $$W_{res} == \int_{(1)}^{(2)} \vec{F}_{res} \cdot d\vec{s} = \int_{(1)}^{(2)} \frac{d}{dt}(\tfrac{1}{2}mv^2) = \left[\tfrac{1}{2}mv^2\right]_1^2 = \tfrac{1}{2}mv_2^{\,2} - \tfrac{1}{2}mv_1^{\,2}$$

This is the very important work-theorem in mechanics. In generality this theorem can be compared to the conservation of energy. For example it is still valid in the special theory of relativity, although the expression for the kinetic energy are different, whereas Newton's 2. law, written in the form $F = ma$ is not.

The work done by the resulting force is equal to the change in kinetic energy

(4.4) $$W_{res} = \tfrac{1}{2}mv_2^{\,2} - \tfrac{1}{2}mv_1^{\,2} \quad \Leftrightarrow \quad \int_{(1)}^{(2)} \vec{F}_{res} \cdot d\vec{s} = \Delta E_{kin}$$

5. Conservative force-fields and potential energy

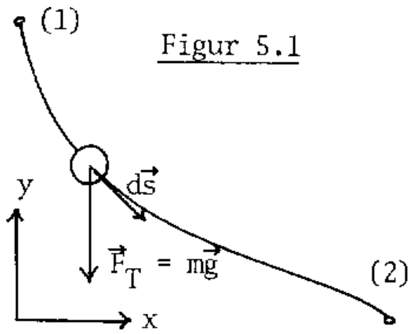
By a force field, we understand a situation, where the force acting on a body is determined in every point of space, every point in a plane or every point on a line.
The most common fields are electrical fields or the gravitational field, but also the force of a spring may be perceived as a force field in one dimension.
They share the common property, however, that when you calculate the work done by the field when moving a body from the position (1) to the position (2), there are usually an infinite number

of ways to reach the position (2) from position (1). In general the work done will depend on the path chosen. (E.g. when moving a piano). However:

The field of is called *conservative*, if the work done by the force, when moving a body from (1) to (2), only depends on the two positions, but not at the path chosen between the two positions.

The gravitational field, electric fields and force of a spring are all examples of conservative fields, while dissipate forces are not – evidently.

5.1 Example. The work done by gravity in a free fall.



As an example, we shall calculate the work that gravity performs when moving a body with mass *m* from a a position (1) to a position (2) along the path shown in the figure. The work can be calculated from the integral.

$$W = \int_{(1)}^{(2)} \vec{F}_G \cdot d\vec{s}$$

In the gravitational field $\vec{F}_G = (0, -mg)$ *and* $d\vec{s} = (dx, dy)$ with the coordinate system chosen, so in this case the integral is easily done.

(5.1.1) $$W = \int_{(1)}^{(2)} \begin{pmatrix} 0 \\ -mg \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_{(1)}^{(2)} -mgdy = -(mgy_2 - mgy_1) = -\Delta E_{pot}$$

The evaluated integral is obviously independent of the path chosen, and equals minus the change in potential energy, where $E_{pot} = mgy$.

Quite generally, we define in an arbitrary *conservative* force, a physical quantity, called the potential energy along the following guidelines:

The change in potential energy that a body acquires from moving from a position (1) to a position (2), is equal to minus the work done by the force, when it moves the body along an arbitrary path, meaning that the work done is independent of the path chosen.

(5.2) $$\Delta E_{pot} = E_{pot}(2) - E_{pot}(1) = -\int_{(1)}^{(2)} \vec{F}_{field} \cdot d\vec{s}$$

When the force is known, then (5.2) determines the potential energy completely, once the potential energy is fixed at some point. It is usually fixed at E_{pot}= 0.

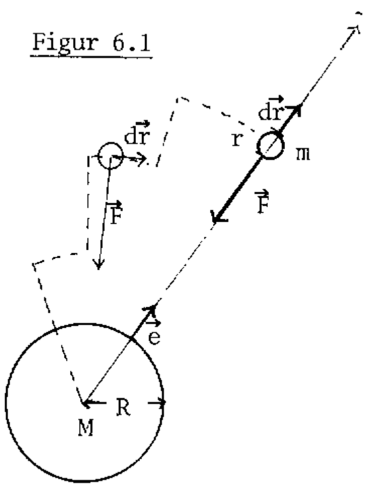
If the position (2) is chosen as zero point for E_{pot} , putting (2) = (0), we may write (5.2)

(5.3) $$E_{pot}(1) = \int_{(1)}^{(0)} \vec{F}_{field} \cdot d\vec{s}$$

We are then able to give a more precise definition of the concept of potential energy.

By the potential energy of a body in the position (1), in a conservative force-field, we understand the work performed by the force, when the body is moved from the position to the zero point for potential energy.

6. The potential energy in the gravitational field



From the definition (5.3), we are able to find an expression for the potential energy of a mass *m*, situated at the distance *r* from the centre of a central gravitational field, generated by a mass *M*.
It could for example be a satellite orbiting around the earth. The gravitational force is according to Newton's law of gravitation.

(6.2) $$\vec{F} = -G\frac{Mm}{r^2}\vec{e}$$

We shall not formally prove that a field of this kind is conservative, but merely observe that in a displacement perpendicular to a radial direction $\vec{F}\cdot d\vec{s} = 0$, and it will give no contribution when evaluation the integral (5.3).

Since any curvilinear path can be divided into an infinitesimal radial displacement and a displacement perpendicular to radial displacement, this concludes our argument that gravity is a conservative force.
We may thus evaluate the integral along a radial line. The zero point for potential energy we shall choose at infinity. The meaning of the vectors: $\vec{F}, \vec{e}, d\vec{r}$ is seen from the figure.
Then we get from (5.3).

(6.3) $$E_{pot}(r) = -\int_r^\infty G\frac{Mm}{r^2}\vec{e}\cdot d\vec{r} = -GMm\int_r^\infty \frac{1}{r^2}dr = -GMm\left[-\frac{1}{r}\right]_r^\infty = -G\frac{Mm}{r}$$

(6.4) $$E_{pot}(r) = -G\frac{Mm}{r}$$

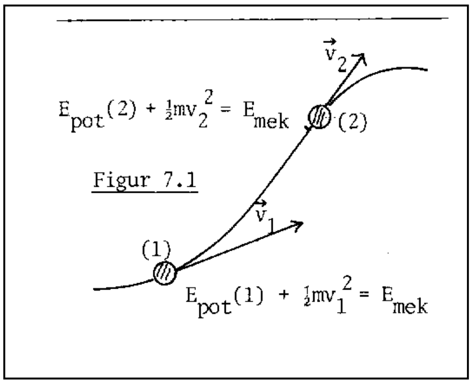
The expression (6.4) remains valid, whenever: $r \geq R$, where *R* is the radius of the central spherical body with mass *M*.
If we compare (6.4) with the well known expression of the potential energy of a mass *m*, near the surface of the earth: $E_{pot} = mgh$, then apparently there is little resemblance.

But the common expression is derived from the assumption, that the gravitational field is constant. $E_{pot} = mgh$ is therefore an approximation, valid only near the surface of the earth (actually onto a distance $h = R$).
To examine the agreement between the two expressions for the potential energy, we calculate from (6.4) the change in potential energy, when a body with mass *m* is lifted to the height *h* over the surface of the earth. The radius of the earth is *R*.

(6.5)
$$\Delta E_{pot} = E_{pot}(R+h) - E_{pot}(R) = -G\frac{Mm}{R+h} - (-G\frac{Mm}{R})$$
$$= -GMm\left(\frac{1}{R+h} - \frac{1}{R}\right) = -GMm(\frac{-h}{(R+h)R}) \approx G\frac{Mm}{R^2}h$$

In the last approximation we have put $R+h \approx R$, being valid when $h << R$. (R =6.370 10^6 m)
If we put the gravitational force at the surface of the earth equal to mg, we find:
$G\frac{Mm}{R^2} = mg$, and we see, that $g = G\frac{M}{R^2}$, so that ΔE_{pot} in (6.5) is actually equal to mgh.
So in the approximation $h << R$, we retrieve the familiar formula: $\Delta E_{pot} = mgh$.

7. Energy conservation in a conservative force field



We shall now consider the situation, where a body with mass m moves freely in a conservative force field, such that the resulting force on the body equals the field force. If we compare the work theorem (4.5): $W_{res} = \Delta E_{kin}$ with (5.2) we have $W_{field\ force} = -\Delta E_{pot}$, and it follows from:
$W_{field\ force} = W_{res} \quad \Leftrightarrow \quad \Delta E_{kin} = -\Delta E_{pot}$

(7.2) $$\Delta E_{kin} + \Delta E_{pot} = 0$$

The equation (7.2) expresses one of the most fundamental theoretical results in theoretical physics.

(7.2)	In a conservative force field the mechanical energy is conserved

If we write it out: $\Delta E_{kin} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$ and $\Delta E_{pot} = E_{pot}(2) - E_{pot}(1)$ and inserting in (7.2), rearranging the terms, we have:

(7.3)	$E_{pot}(2) + \frac{1}{2}mv_2^2 = E_{pot}(1) + \frac{1}{2}mv_1^2$

The equation (7.3) yields that, when a body moves in a conservative force field, the sum of the kinetic and potential energy remains constant.

Conservation of energy in a conservative force field goes beyond Newtonian mechanics.
With minor modification of the expression of the kinetic energy, it still holds good in special relativity and quantum physics, and is therefore considered as the most important theorem in theoretical physics.
In elementary mechanics, we are familiar with the concept of conservation of energy from the free fall of a body near the surface of the earth, a oblique throw of a ball and a mass performing oscillations in a ideal spring.

Typical non conservative forces (dissipative forces) are forces of friction, which are inevitable when performing experiments on the earth, and only the genius of Newton, with his law of inertia was able to separate dissipative forces from conservative forces.

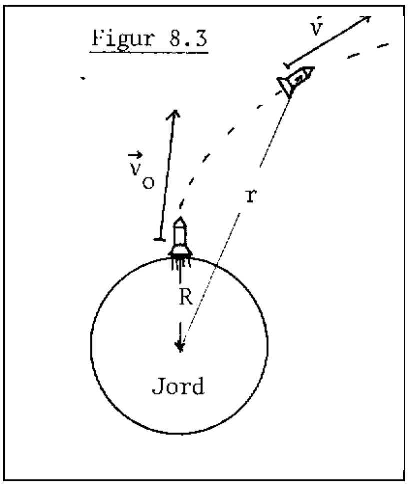
If dissipative forces are present, we must modify (7.2) to include the loss of mechanical energy. The work done by these forces is equal to loss in mechanical energy:

(7.4) $$W_{dissipative} = \Delta E_{mek} = \Delta E_{kin} + \Delta E_{pot}$$

8. Conservation of energy in the earth's gravitational field

For the motion of a satellite in the earth's gravitational field, the law of conservation of energy applies, since the gravitational field is conservative.
When it occasionally happens that a satellite in orbit falls down and crashes on the surface of the earth, the cause is the weak but persistent dissipative force from the thin atmosphere.
The energy of a satellite moving in the gravitational field of the earth may be expressed as:

(8.1) $$E_{mek} = E_{kin} + E_{pot} = \tfrac{1}{2}mv^2 - G\frac{Mm}{r}$$



The conservation of energy in a gravitational field may be applied to find the escape velocity for a body launched with a speed *v*, from the earth. The escape velocity is the velocity needed to escape the gravitational field of the earth.
If a rocket with mass *m* and speed *v* is launched from the earth, having mass *M* and radius *R*, the energy is:

(8.2) $$E = \tfrac{1}{2}mv^2 - G\frac{Mm}{R}$$

In the limit, where the body precisely escapes, it will have zero speed at infinity, where the potential energy is also zero.
To find this speed we put (8.2) to zero.

(8.4) $$\tfrac{1}{2}mv_{escape}^2 - G\frac{Mm}{R} = 0 \quad \Leftrightarrow \quad v_{escape} = \sqrt{\frac{2GM}{R}}$$

Inserting the values of the constants: $G = 6.67\ 10^{-11}\ N\,m^2/kg^2$, $M = 5.98\ 10^{24}\ kg$, $R = 6.370\ 10^6\ m$, we find the escape velocity from the earth.

(8.5) $$v_{escape} = 11.2\ km/s$$

If we use the same formula to find the escape velocity from the sun (and the sun actually sheds a large amount of particles), we should apply: $M_{sun} = 2.0\ 10^{30}\ kg$, $R_{sun} = 6.96\ 10^8\ kg$. We then find: $v_{escape} = 619\ km/s$.

It is a curiosum (but an exception) that even if the formula (8.2) is non relativistic it gives the correct result – derived from general relativity – for the radius of a black hole, where the escape velocity is the speed of light, (which is the reason that it is black).

So if we put $v_{escape} = c$ (the speed of light) we find:

$$c = \sqrt{\frac{2GM_{black\ hole}}{R_{black\ hole}}} ,$$

It is usually solved with respect to $R_{black\ hole}$ to give:

(8.6) $$R_{black\ hole} = \frac{2GM_{black\ hole}}{c^2} \quad \text{Or just} \quad R = \frac{2GM}{c^2}$$

For a black hole having a mass of 10 sun masses, we find for example.

$$R = \frac{2 \cdot 6.67 10^{-11} \cdot 2.0 10^{31}}{9 10^{16}} = 3.0 10^{4} m = 30\ km .$$