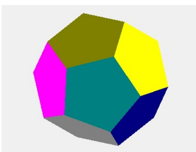
The physics of
Air Mirroring and Mirages

This is an article from my home-page: www.olewitthansen.dk



Ole Witt-Hansen 1976 (2017)

Contents

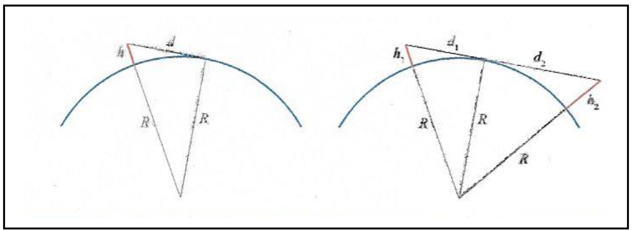
1. A view to Christiansoe. The distance to the horizon

Denmark has 5 major islands. The smallest of these is Bornholm and is situated about 100 km east from the rest of Denmark. I have been so fortunate to have a summer cottage on the northeast coast. The house lies less than 50 *m* from the sea, and estimated 5 *m* above sea level.
Christiansoe is a very small island some 0.39 km^2, the most eastern point of Denmark, and it used to be a fortress built in 1639.
The position of Christiansoe, as seen from my cottage is 70^0 east and at a distance of 23.5 *km*.

Generally, you can not spot Christiansoe, when sitting at the terrace. But on the other hand you can if you in clear weather move some 7 meters vertically up in the terrain.

This is of course because of the curvature of the earth, already noticed by Columbus in 1490, and is mathematically explained by the two figures below. In the first figure there is evaluated the distance to the horizon, and in the second figure the largest line of sight between two "towers".

The formula for the distance to the horizon, as seen from a height *h* is well known, but its derivation does not appear very often. I have, however, used it for many years as an application of Pythagoras theorem for the first year in high school.



In the figure to the left, we shall evaluate the distance to the horizon from a height *h*. What is below the horizon, you cannot see, because of the curvature of the earth. The earth has the radius *R* = 6.370 *km*.
The right angle triangle has the sides *R* and *d*, and the hypotenuse *R* + *h*, so according to Pythagoras theorem we have:

$$(R+h)^2 = R^2 + d^2 \quad \Rightarrow \quad R^2 + h^2 + 2Rh = R^2 + d^2$$

Since $h/R \approx 10^{-5}$, we may ignore h^2, and isolating *d* gives: $d = \sqrt{2Rh}$

Using this formula it is customary to measure the height *h* in meters, and the distance in *km*. If we therefore insert *h*/1000 instead of *h,* and the radius of the earth in *km*, we obtain a well known formula.

$$d = 3.57\sqrt{h} \;\; [km]$$

In the second figure we want to find out weather an observer positioned in the height h_1 can see a tower, having the height h_2. This is however accomplished by adding the two distances d_1 and d_2 calculated from the formula above.

We now return to the view of Christiansoe from our summer cottage 5 *m* above sea level, aiming at a point which lies 10 meters above the water, we can find the largest distance between the two points where they can see each other.

$$d_1 + d_2 = 3.57(\sqrt{h_1} + \sqrt{h_2})\; km = 3.57(\sqrt{5} + \sqrt{10})\; km = 19.3\; km$$

Since Christiansoe lies 23.5 *km* away, one cannot, according to the calculation above, see the top of a 10 *m* high building from our terrace elevated 5 *m* over see level.

However, if you are positioned 15 *m* over sea level, then you will be able to see a 10 *m* high object at a distance at about 25 *km*. The distance 23.5 *km* corresponds to a height 12 *m* over sea level.

In nights, however, it is possible to see the lighthouse at Christiansoe. Assuming that the lighthouse is 25 meters, one can from a height of 5 *m* see objects at a distance.

$$d_1 + d_2 = 3.57(\sqrt{h_1} + \sqrt{h_2})\ km = 3.57(\sqrt{5} + \sqrt{25})\ km = 25.8\ km$$

Under normal circumstances, these elementary calculations are in full accordance with experience.

2. Air mirroring

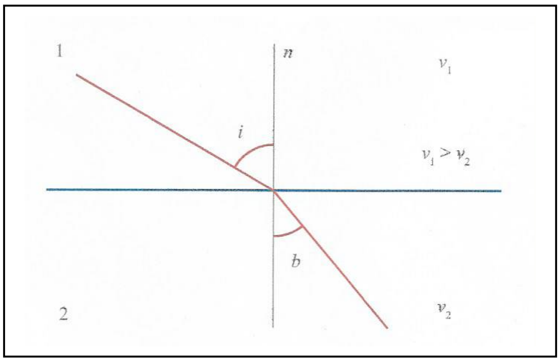
The strange phenomena, however, is that sometimes it is possible to spot Christiansoe from the terrace at our summer cottage, which according the presentation above should be impossible.
And more strangely, but very rarely, one can see some contours in the direction of in Christiansoe.
I have studied the phenomena with a binocular, but have been unable to interpret it.
I have ascribed it to formations of clouds and haze, until I ran into a photo taken, and presented in Wikipedia in an article of air reflections (or mirages).

Air mirroring of Christiansoe . 24. juli 2012

The photo shows an inverse air mirroring of Christiansoe

3. The law of refraction on differential form

To explain air mirroring and mirages, it is necessary to enlarge the law of refraction, so it does not only describe the refraction of light between the intersection of two plane homogenous materials but also to encompass light which passes through a gas with a continuous changing index of refraction. The generalization is in fact trivial, (if you consider it), but it formally follows from the law of refraction presented in differential form.



For the transitions of light between two plane transparent materials where the light propagates with velocities v_1 and v_2, respectively, and where the angles i and b are the incoming and outgoing angles with the normal to their intersection we have the law of refraction:

$$\frac{\sin i}{\sin b} = \frac{v_1}{v_2} = \frac{\frac{v_1}{c}}{\frac{v_2}{c}} = \frac{n_1}{n_2} = n_{12}$$

n_1 and n_2 are the indices of refraction for the two materials (1) and (2). n_{12} is called the relative index of refraction on the transition of the light from (1) to (2), and obviously $n_{21} = 1/n_{12}$.

We shall now generalize this law to the case, where the index of refraction changes continuously in the y - direction, that is, along the normal to the surface of the materials. We shall calculate the infinitesimal deviation, corresponding to an infinitesimal change of the index of refraction $n = n(y)$. The angle that the light ray forms with the normal is denoted θ. So the law of refraction writes.

$$\frac{\sin\theta}{\sin(\theta+d\theta)} = \frac{n(y)}{n(y+dy)} \quad \Leftrightarrow \quad n(y+dy)\sin\theta = n(y)\sin(\theta+d\theta)$$

By subtracting $n(y)\sin\theta$ on both sides and putting outside a parenthesis, we have:

$$\sin\theta(n(y+dy)-n(y)) = n(y)(\sin(\theta+d\theta)-\sin\theta) \quad \Leftrightarrow \quad \sin\theta\, dn = n(y)d\sin\theta$$

or

$$\frac{d\sin\theta}{\sin\theta} = \frac{dn}{n(y)}$$

This equation can immediately be integrated.

$$\int_{\theta_1}^{\theta_2}\frac{d\sin\theta}{\sin\theta} = \int_{n_1}^{n_2}\frac{dn}{n(y)} \quad \Leftrightarrow \quad \ln\sin\theta_2 - \ln\sin\theta_1 = \ln n(y_2) - \ln n(y_1) \quad \Leftrightarrow \quad \frac{\sin\theta_2}{\sin\theta_1} = \frac{n(y_2)}{n(y_1)}$$

Thus, we recover the law of refraction unchanged. The refraction angle depends only on the index of refraction in the upper layer and the lower layer. This means that refraction through a layer with a thickness of hundreds of meters can be treated as refraction in a single borderline.
This has the important consequence that total reflection is possible in layers of air many meters thick, and this is the key to explain the phenomena of air mirroring and mirages.

Total reflection may occur in the case of a decreasing index of refraction, so that the *relative* index of refraction becomes less than one, such that the light is bended away from the normal. (The angle of refraction is greater than the angle of approach).
Total reflection occurs when the angle of refraction exceeds 90^0.

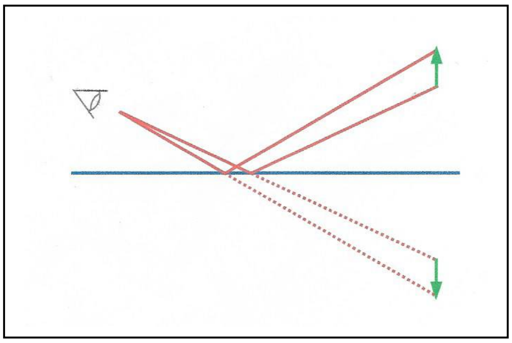
$$\frac{\sin i}{\sin b} = n_{12} \quad \Leftrightarrow \quad \sin b = \frac{\sin i}{n_{12}}$$

If $n_{12} < 1$ and inserting $b = 90^0$ we get the limiting angle i_g for total reflection: $\sin i_g = n_{12}$.
For the transition water/air, the index of refraction is about 1/1.33, which gives a limit angle of 49^0.

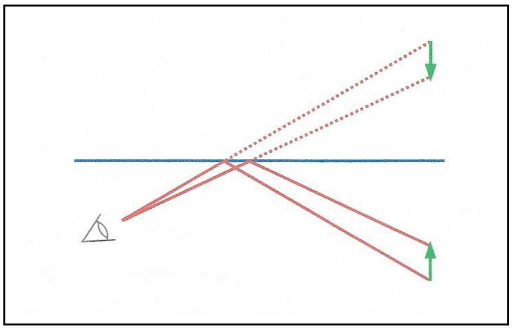
In air the index of refraction is much less. At an index of refraction 1/1.001, you will find a limiting angle about 87^0, which corresponds to a grazing angle of 3^0.

The absolute index of refraction for air is 1.000292, so the quick conclusion would be that the limiting angle for total reflection in air can at most be $\sin i_g$ =1/1000292, which gives i_g =88.6^0, which corresponds to a grazing angle of 1,4^0. Total reflection at a grazing angle 1^0 corresponds to a refractive index 0.9998 with the reciprocal value 1.0002.

Lower air mirroring.



Upper air mirroring



Above is very schematic illustrated the light path in a lower and upper air mirroring.

The lower air mirroring is most common, since it can be experienced on sonny days, where it appears that there is water on the asphalt road ahead a vision which disappears as soon as you get nearer.
The explanation is straightforward. Because the air gets heated near the black asphalt lane, this gives it a smaller density, and consequently a less index of refraction.

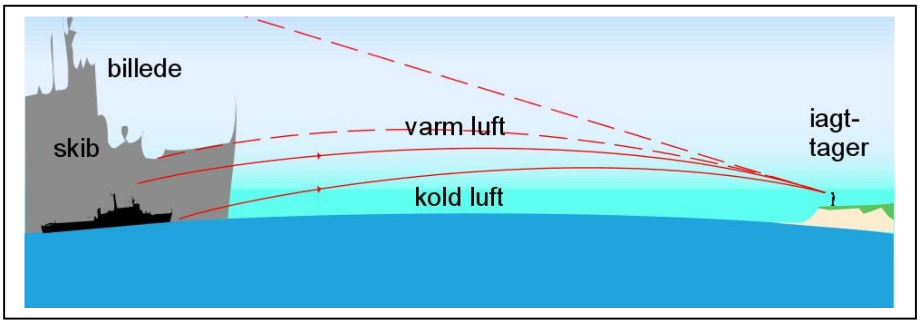
As emphasized above, the phenomena of total reflection is independent of whether the reflection takes place at a borderline or it takes place in a gradual change of the index of refraction through an arbitrary thick layer.

As demonstrated in the example above, the difference in the index of refraction above and below is of the same magnitude as the refractive index in air 1.000292, and that the angle of grazing should be of the magnitude of 1^0.
However, it is not very relevant at this point to discuss whether it is possible or not, since experience shows us that it happens.
The puddles that you can see on a black road in sonny weather is in reality an (inverse) lower mirroring of the sky.

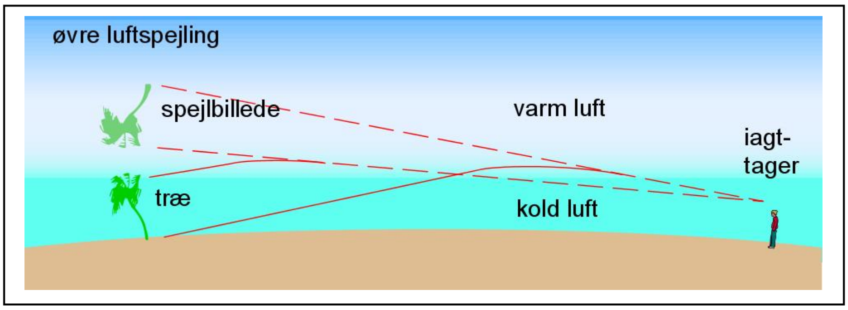
The upper mirroring is the same phenomena, the only difference being that the air near the ground is cooler than the air higher up. The upper mirrorings are much more rare at our latitudes, and surprisingly enough they appear mostly in polar areas.

Common for the two kinds of mirroring is that they produce an inverse image of the object.
Concerning the mirages (fata morgana), it is not a mirroring, but instead it is caused by light that is refracted in the same direction as the curvature of the earth.
For this reason it is possible to see objects that otherwise are below the horizon. The figures below show the principles of mirages, upper and lower air mirroring. They are collected from the Danish Encyclopedia, so the text is in Danish, but the drawing should speak for themselves. ("Varm/kold luft" means Warm/cold air).

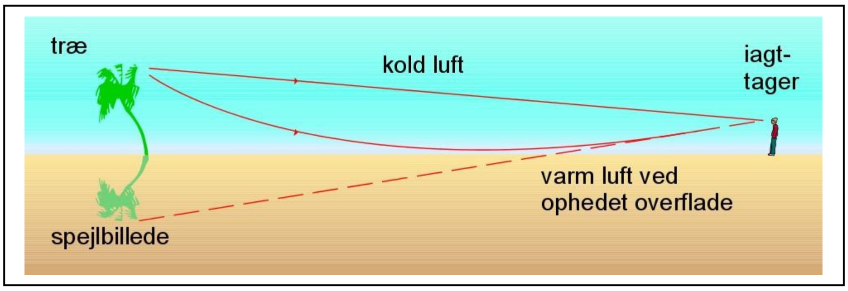
A *mirage*. The light ray is bended towards the curvature of the earth. The image has the right side up, but it is usually distorted.



Upper air mirroring. The image is upside down, but the geometry is unaltered

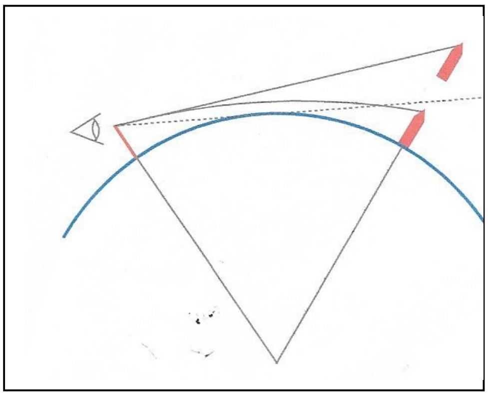


Lower air mirroring. The image is upside down, but the geometry is unaltered



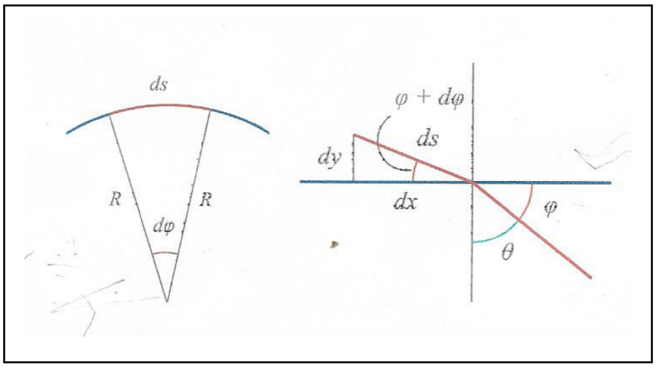
4. Mirages

We shall discuss in somewhat more detail the phenomena of mirages (fata morgana), which occurs when a light ray is refracted along the curvature of the earth.



The figure to the left shows a tower, which lie below the horizon, which means that it cannot be seen, if the light followed a straight line. If the light ray is bended when passing the atmosphere, however, the tower will be seen in a direction following the tangent to the ray at the point of observation, as "floating" in the air. The refraction is due to an abnormal temperature gradient, that is, the temperature rises with the height from the ground. The cooler air has a higher density, and therefore a higher index of refraction.

A light ray which moves slightly upwards will move towards a slightly less (relative) index of refraction, and consequently be refracted away from the normal to the layer, that is, downwards. The next step is to establish a relation between the differential bending $d\varphi$ of the beam and the change dn/dy in the index of refraction per unit length along the normal to the layer.



The figure above to the left depicts a section of the surface of the earth, where the connection between the arc ds, the radius of the earth R and the angle $d\varphi$ is $ds = R \cdot d\varphi$

The second figure shows a light ray on its way up through the atmosphere getting an infinitesimal deflection $d\varphi$ at the distance ds.

φ is the grazing angle between "the borderlines", and $\varphi = 90 - \theta$, where θ is the incoming angle with the normal to the layer.

The law of refraction in differential form reads: $\frac{d\sin\theta}{\sin\theta} = \frac{dn}{n}$ where $d\sin\theta = \cos\theta \cdot d\theta$, and if we use $\varphi = 90 - \theta$ we get $d\varphi = -\, d\theta$.

We drop the sign and get: $\frac{d\sin\theta}{\sin\theta} = \frac{dn}{n} \Leftrightarrow \cos\theta\, d\theta = \frac{\sin\theta}{n} dn$

Since $\cos\varphi = \cos(90-\theta) = \sin\theta$ *og* $\sin\varphi = \sin(90-\theta) = \cos\theta$ we rewrite it as $d\varphi = \frac{\cos\varphi}{n\sin\varphi} dn$

We then use $dn = \frac{dn}{dy} dy$, and since $dy = \sin\varphi\, ds$ then $dn = \frac{dn}{dy} \sin\varphi\, ds$, we finally have:

$$d\varphi = \frac{\cos\varphi}{n\sin\varphi} \sin\varphi \frac{dn}{dy} ds \quad \Leftrightarrow \quad d\varphi = \frac{\cos\varphi}{n} \frac{dn}{dy} ds$$

If a light ray should follow the curvature of the earth, we may determine the gradient of the index of refraction. Comparing this with $ds = R \cdot d\varphi \Leftrightarrow d\varphi = 1/R\, ds$ we have:

$$d\varphi = \frac{\cos\varphi}{n} \frac{dn}{dy} ds \quad \text{and} \quad d\varphi = \frac{1}{R} ds$$

gives:

$$\frac{\cos\varphi}{n} \frac{dn}{dy} = \frac{1}{R} \quad \Leftrightarrow \quad \frac{dn}{dy} = \frac{n}{R\cos\varphi}$$

Assuming that $\varphi = 0.5^0$, and we put n=1 and R= 6.37 10^6 m, we obtain the value:

$$\frac{dn}{dy} = 1.6 \cdot 10^{-7}\ m^{-1}$$

According to the references the gradient will seldom have such a large value.
The necessary gradient of the index of refraction must for small angles grow in proportion to the grazing angle.
Assuming a gradient being 1,6 10^{-8} m^{-1}, that is, one tenth of the value above, then the deviation on the distance 23.5 km from Christiansoe to the summer cottage can be calculated from:

$$\Delta\varphi = \frac{\cos\varphi}{n}\frac{dn}{dy}\Delta s$$

Under these assumptions, it will cause a deviation in the line of sight of 0.00038 radian.
Multiplying with the distance 23.5 *km*, we find a difference in height of 8.8 m, of what it should be without deflection.
This calculation thus delivers a (also quantitative) explanation of the phenomena that Christiansoe in clear weather can be viewed from a height of 10 – 15 meters above sea level from the eat coast of Bornholm, also occasionally can be viewed rather clearly from a height of only 5 meters.

The photo presented in the beginning of this article is an upper air mirroring of Christiansoe. Although that I have spent holidays in the summer cottage for more than 50 years, I have only observed the phenomena once in the summer 2013. My interpretation then was special clouds, mist or haze until I began to investigate it further. The result was this article.
The image is upside down, which we have explained earlier. There is no mystery. Some rays move in the same height as the ground, where the index of refraction is constant, Others have a tiny deflection upwards towards a lover index of refraction, so small that it is able to cause a total reflection in an upper layer.
That the air on a hot summer day can be warmer up in the air than close to the chilly sea water does not sound improbable at all.