# Magnetism

## Magnetic field, Biot and Savart’s law, Laplace law, Ampere’s law, Paramagnetism

Chapter 2 of the textbook
Elementary Physics 3

This is an article from my home page: www.olewitthansen.dk

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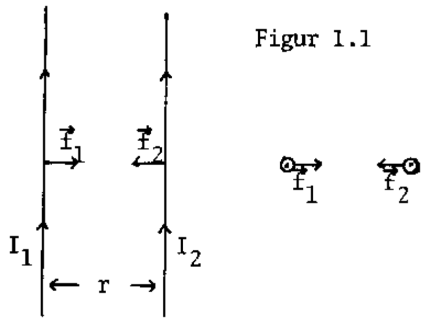
# 1. Magnetism

It is a fairly old experience that two conductors carrying electric currents affect each other with forces. However, an electric charge $Q$ at rest is not affected by a force from currents.

The forces with which currents affect each other are called *magnetic forces*.

We shall here establish, and later justify, that all magnetic forces are caused by electric currents. In this connection, electric currents must be understood in a very broad sense as charges in motion.

It is somewhat complicated to write down a general expression for the force between two conductors of arbitrary design carrying currents, so we shall initially investigate the forces between two infinite long parallel wires carrying the currents $I_1$ and $I_2$.




Figur 1.1

Let the two parallel conductors be placed at a distance $r$ from each other as illustrated in figure (1.1) .
We define *the pull* $f$ on the wire as the force per unit length. If the length $L$ of the conductor is affected by a force $F$, then *the pull* is:

$$f = \frac{F}{L}$$

If the currents in the two conductors are $I_1$ and $I_2$ , then experiments show that the pull is given by the expression.

(1.2) $$f_1 = f_2 = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r}$$

*The pull is directly proportional to each of the currents, and inversely proportional to the distance between them. It should be noticed that the force is an attraction for aligned currents and the force is repulsive for opposite directed currents.*

The empirical law (1.2) is the foundation of the SI-unit ampere definition for electric currents:

The current one *ampere*, is the common current that runs in two infinite long and parallel wires, which are placed at a distance of one meter from each other, and that acts on each other with a force per unit length which is $2.000^{-7}$ $m$.
$\mu_0$ is a constant of nature and its value is determined by the Ampere definition.
If we insert $I_1 = I_2 = 1\ A$ , $r = 1\ m$, and $f$ =2.000 $10^{-7}$ $m$ in (1.2), we find:

(1.3) $$\frac{\mu_0}{4\pi} = 2.000 10^{-7}\ N/A^2$$

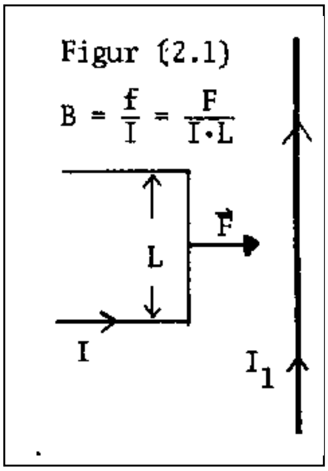
The constant $\mu_0$ belongs (like the vacuum permittivity $\varepsilon_0$) to the fundamental constants of nature, and $\mu_0$ it establishes the "strength" of the magnetic forces in the same manner as $\varepsilon_0$ establishes the "strength" of the electric forces.

$\mu_0$ is called the *vacuum permeability*, because the fundamental experiment defining the Ampere definition, strictly speaking should be performed in vacuum.

The difference, when performing the experiment in air is, however, immensely small, but some "magnetic" materials like Iron and Nickel have a permeability, which is several hundred times greater than the vacuum permeability.

The Ampere unit is, as you know, one of the five fundamental SI-units, from which all other units may be derived. From the Ampere unit is derived the unit 1 Coulomb: As the charge $Q$ that passes through a conductor per second, when the current is 1 Ampere. 1 Coulomb = 1 ampere·second.

## 2. Magnetic field




In the same manner as we introduced the electric field $E$, we shall introduce the *magnetic induction* (or the *magnetic field*) $B$ from the ampere definition. This is illustrated in figure (2.1). The *pull* (the force per unit length) in the conductor to the left is $f$, and according to (1.2) given by:

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I}{r}$$

The pull is proportional to the current $I$, but the pull divided by $I$ is independent of $I$, and reflects "the magnetic situation" at the place of the current in the conductor. In the same manner as we defined the electric field as force on a charge divided by that charge, we define the magnetic field $B$ as *the pull* on a current divided by that current.

(2.1) $$B = \frac{f}{I} \quad or \quad B = \frac{F}{I \cdot L} \quad or \quad F = B \cdot I \cdot L$$ (Definition of magnetic field $B$)

The conducting wire carrying a current $I$ has the length $L$, is and is affected by a magnetic force $F$.

From the defining equation (2.1) it is seen that the SI-unit for magnetic induction is $N/(A\,m)$. This unit is, however, rewritten as follows:

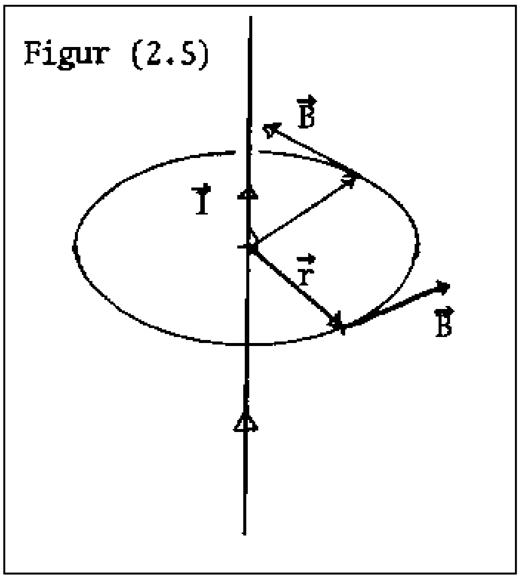
(2.2) $$\frac{N}{Am} = \frac{Nm}{Am^2} = \frac{N\,m\,s}{C\,m^2} = \frac{J\,s}{C\,m^2} = \frac{V\,s}{m^2} = \frac{Wb}{m^2} = T$$

We have in (2.2) defined two new units. 1 *Weber* = 1 *Volt* 1 *Second*. 1 $W$ = 1 $V\,s$, and 1 *Tesla* = 1 $T$ = 1 $Wb/m^2$. The SI-unit for the magnetic induction is hereafter 1 *Tesla* = 1 $Wb/m^2$.

Comparing (1.2) with (2.1) it leads to an expression for the magnetic induction $B$ at the distance $r$ from a straight line conductor with the current $I$. (We put $I_1 = I$).

(2.3) $$f = \frac{\mu_0}{2\pi}\frac{I\,I_2}{r} \quad \wedge \quad B = \frac{f}{I_2} \quad \Rightarrow \quad B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

Since the magnetic field is derived from a force is has a direction aligned to or opposite to the force.




We may therefore define a magnetic field vector $\vec{B}$.
For a long straight conductor, the direction of the *B*-field is defined as shown in figure (2.5).
Let $\vec{I}$ be a vector in the direction of the current and let $I = |\vec{I}|$. $\vec{r}$ is a radially directed vector from the conductor to the point where we want to find the *B*-field.
The field $\vec{B}$ is then given by:

(2.4) $$\vec{B} = \frac{\mu_0}{2\pi}\frac{\vec{I}\times\vec{r}}{r^2}$$

According to the definition of the cross product, $\vec{B}$ is perpendicular to $\vec{I}$ as well as $\vec{r}$.
The direction of $\vec{B}$ is illustrated in the figure (2.5).
Finding the magnetic induction *B*, we take the length of (2.4) remembering that $\vec{I} \perp \vec{r}$:

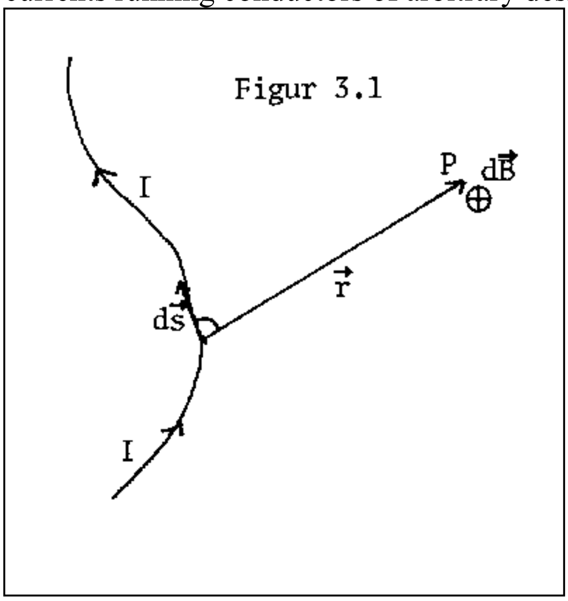
$$B = |\vec{B}| = \frac{\mu_0}{2\pi}\frac{|\vec{I}\times\vec{r}|}{r^2} = \frac{\mu_0}{2\pi}\frac{I}{r}$$

We notice that we regain the expression (2.3).

From the figure, we also see that the magnetic field lines are concentric circles having the conductor at its centre. Notice also that in contrast to the electric field lines, the magnetic field lines are always *closed curves* . The reason for this is that there exists no “magnetic charges”.

## 3. Biot and Savart’s law

Up till now we have only given a formula for the magnetic field from an infinite long straight conductor. The Frenchmen Biot and Savart delivered a general formula based on their experimental work, from which it is (theoretically) possible to establish the magnetic field from currents running conductors of arbitrary design.




In figure (3.1) is shown a conductor carrying the current *I*. And below is shown a formula, which can be applied to calculate the contribution to the magnetic field $d\vec{B}$ from the section $d\vec{s}$ of the conductor, at the point *P*.
$\vec{r}$ is the position vector to the point *P* from the small conductor element $d\vec{s}$.

According to Biot and Savart we have:

(3.1) $$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{s}\times\vec{r}}{r^3}$$

And written in scalar form the formula becomes:

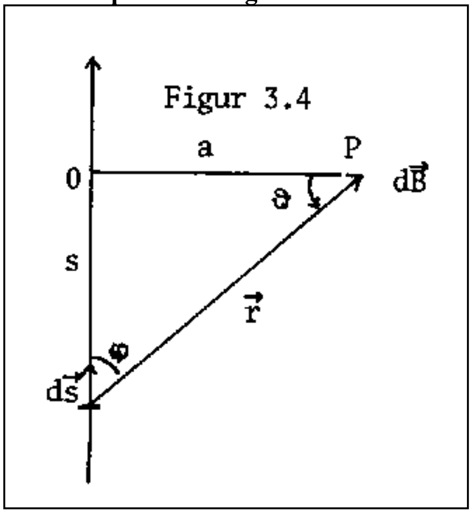
(3.2) $$dB = \frac{\mu_0}{4\pi}\frac{Ids\sin\varphi}{r^2}$$

In the last equation, φ is the angle between $d\vec{s}$ and $\vec{r}$. Notice from the figure that $d\vec{B}$ has direction into the paper, a direction which is symbolized with $\oplus$.

If the field in some point is to be determined from all parts of the conductor, it is done by integration along the conductor. (Integration with respect to $d\vec{s}$ ).
With the last remark, we have emphasized the vector character of the magnetic field.

This means also that the *principle of superposition* is valid for the magnetic field.

**3.4 Example. The magnetic field from a long straight conductor**




We shall now calculate the B-field from an infinite long straight conductor, using Biot and Savart's law. If we obtain the same result as (2.3), it should be taken as a confirmation of Biot and Savart's law. From the figure we can convince ourselves that the contributions to the *B*-field in *P* from all parts of the conductor have the same direction, namely into the paper.
This follows from the direction of the cross product $d\vec{s}\times\vec{r}$ in the formula (3.1). So in this case we may replace the vector addition by scalar integration.
If figure (3.4) we have invented an oriented *s*-axis so that the zero point on *s* is the projection of *P* on the *s* axis. Furthermore we have invented an oriented angle θ as shown in the figure. The distance from *P* to the conductor is *a*. According to Biot and Savarts law we have:

$$dB = \frac{\mu_0}{4\pi}\frac{Ids\sin\varphi}{r^2}$$

To perform the integration, it is necessary to express all the variables by one variable. We therefore express *r,* sinφ, and *ds* by with the help of the angle θ.

From the figure one can see that: sinφ = cos θ, and $r = \frac{a}{\cos\theta}$. From $s = a\tan\theta$, it follows that: $ds = \frac{a}{\cos^2\theta}d\theta$ :

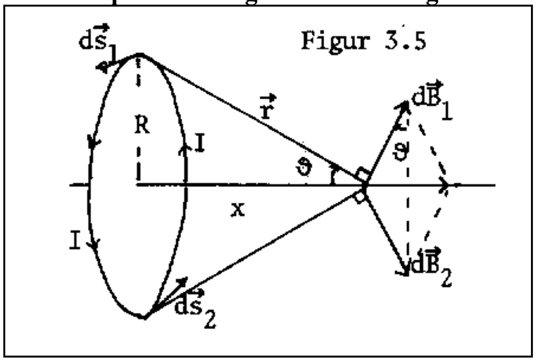
Which gives:

$$dB = \frac{\mu_0}{4\pi}\frac{I}{(\frac{a}{\cos\theta})^2}\cos\theta\frac{a}{\cos^2\theta}d\theta = \frac{\mu_0}{4\pi}\frac{I}{a}\cos\theta d\theta$$

(3.4a) $$B = \frac{\mu_0}{4\pi}\frac{I}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta = \frac{\mu_0}{4\pi}\frac{I}{a}[\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\mu_0}{2\pi}\frac{I}{a}$$

We notice that (3.4.a) is in accordance with (2.3)

**3.5 Example. The magnetic field along the axis of a circular conductor.**




The figure shows a circular conductor with a current *I*.
We shall use Biot and Savart's law to calculate the magnetic field at a point lying on the axis through the centre of the circular conductor.

From the figure we see that the contributions $d\vec{B}_1$ and $d\vec{B}_2$ from the two diametrically opposite pieces of the conductor lie symmetric with respect to the axis. If the contributions are added pair wise, then the sum will be directed along the axis, and the vector addition of the contributions can be replaced by scalar addition. The sum of the two contributions is therefore.

$$| d\vec{B}_1 + d\vec{B}_2 | = 2B_1 \sin\theta$$

The meaning of $R$ and $r$ can be read from the figure. We shall evaluate the magnetic field at a point having the coordinate $x$.
The magnetic field is then determined by integrating Biot and Savart's law for $dB$ on half of the circumference.

$$dB_1 = \frac{\mu_0}{2\pi}\frac{I}{r^2}ds_1 \quad \wedge \quad B(x) = \int_0^{\pi R} 2dB_1 \sin\theta$$

$$B(x) = \frac{\mu_0}{2\pi}\frac{I}{r^2}\int_0^{\pi R} ds_1 = \mu_0 \frac{I\,R}{2r^2}\sin\theta$$

Inserting $r = \sqrt{R^2 + x^2}$ and $\sin\theta = \frac{R}{r}$ we find an expression for $B(x)$.

(3.5b) $$B(x) = \mu_0 I \frac{R^2}{2(R^2 + x^2)^{\frac{3}{2}}}$$

Especially we may evaluate the magnetic field in the centre of a circular conductor with radius $R$, by setting $x = 0$.

(3.5c) $$B(x) = \frac{\mu_0 I}{2R}$$ (The magnetic field in the centre of a circular conductor)

**3.6 Exercises**

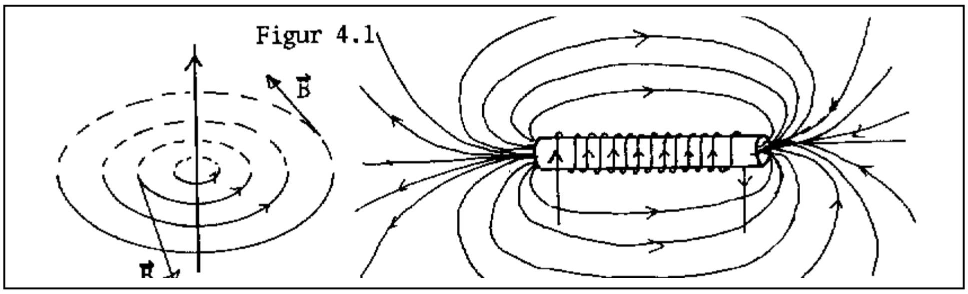
1) Two long straight wires are placed perpendicular to a plane at the points (-0.50, 0) and (0.50, 0).
They both carry a current 5.0 *A* in the same direction.
a) Determine the strength and the direction of the magnetic field in the points (0, 0), (0.25, 0), (0, 025) and (0.25, 0.25). (All coordinates are in meters).
b) Answer the same questions above, when one of the two currents is reversed.

2. Two high voltage wires apt for 50 *A*, are placed in the air at a distance 1.0 *m*. Calculate the pull (force per meter) between the wires, and calculate the force between two such wires, when they are hung in wire masts, when the masts are separated by 100 *m*.

## 4. Magnetic Field lines

As it is the case in electrostatics one may draw field lines to illustrate the magnetic field around wires carrying a current.
The magnetic field lines are defined as curves that overall has the magnetic field vector as its tangent vector.
The crucial difference to electricity is that the magnetic field lines are always closed curves.
The reason for this is that there do not exists "magnetic charges" (magnetic monopoles), where the field lines begin or end.

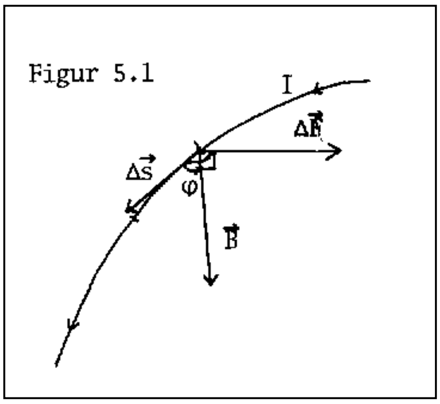
The pattern of the field lines may be investigated by placing small compass needles in vicinity of the currents that create the magnetic field. (Later we shall explain the laws that make a compass needle align to the magnetic field).




Above are shown the magnetic field lines from a long straight wire, and the field lines from a coil with an iron core. The field lines of the latter are the same as is found from a permanent iron rod magnet.

## 5. Laplace' law

Laplace' law is a formula for calculating the force, of which a conducting wire is affected, when placed in a magnetic field.



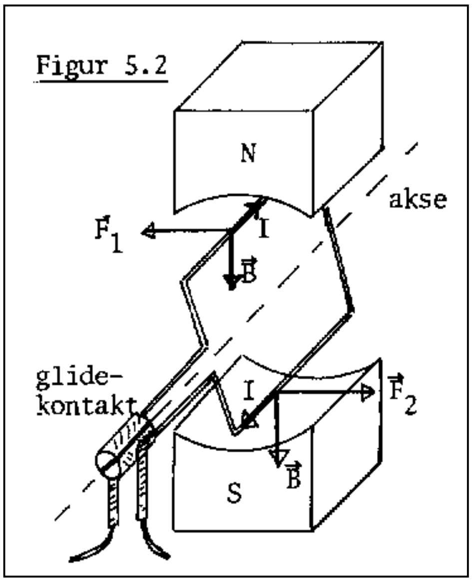

In the figure is shown a conducting wire placed in a magnetic field. If the magnetic field is at right angle to the wire, then the expression for the force is given by the equation (2.1). In the opposite case, we shall designate a small straight part of the wire $\Delta\vec{s}$ , which is aligned with the direction of the current in the wire.
We assume that $\vec{B}$ is constant on the piece of wire.

According to Laplace' law, the piece of wire will be affected by a force $\Delta\vec{F}$ given by:

(5.1) $\Delta\vec{F} = I\Delta s \times \vec{B}$ (or in scalar form) $\Delta F = I\Delta s B \sin\varphi$

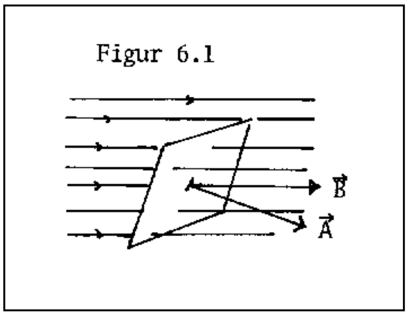
The angle $\varphi$ is the angle between $\Delta\vec{s}$ and $\vec{B}$ , (the wire and the *B*-field). One should notice that the force $\Delta\vec{F}$ is perpendicular to the *B*-field as well as to the wire. And that $\Delta\vec{F} = 0$, if $\Delta\vec{s} \parallel \vec{B}$ .

**5.3 Example. The electric engine**



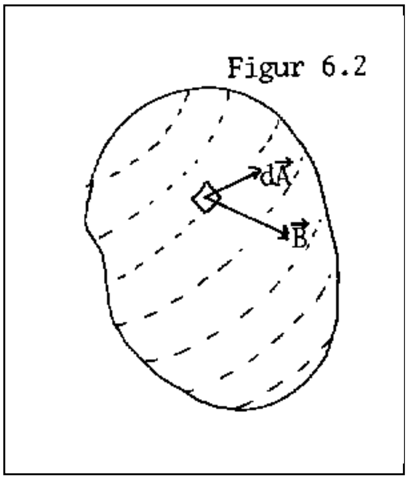

The discovery of the Laplace force on a conducting wire has had an enormous technological impact on the industrial development in the Western World.
More specifically, by the invention of the electric motor.
Figure (5.2) shows schematically the principles involved in a primitive specimen of an electric motor.
The anchor (here illustrated by only one turn) is placed in a magnetic field, and it receives a torque from the Laplace forces $F_1$ and $F_2$.
The current $I$ comes from the sliding contacts, which causes the current to switch direction for every half turn of the anchor.
In this manner the anchor will be affected by a torque persistently turning it in the same direction.
The sliding contacts are such a simple but ingenious invention, which has brought the electric engine to be a part of all households and transportation in the modern world.
The wonderful thing about the electrical engine is that in principle it can also be applied as a dynamo generating a current if you turn the anchor around, something that we shall learn more about in the section on induction.

## 6. Magnetic flux. Amperes law




Everything we know about static magnetic fields (called magnetostatics) can be derived from two fundamental laws.
We have already discussed Biot and Savart's law.
Here we shall introduce Amperes law, but that requires the introduction of the concept of magnetic flux.
Magnetic flux is defined in the same manner as electric flux, and the only difference is that the electric field is replaced by the magnetic field.
The magnetic flux through a flat piece with flat normal vector $\vec{A}$, where $|\vec{A}| = A$,( the area of the flat piece), is defined by:

(6.2) $$\Phi_B = \vec{B} \cdot \vec{A}$$




The formula requires that the field $\vec{B}$ is constant over the area $A$, and that the area is flat.
If the B-field is not constant in strength or direction, or if the area is not flat, the surface must be divided into such small pieces that $\vec{B}$ can be considered constant and the area $d\vec{A}$ can be considered flat. The flux through the entire surface $f$ can then be found by integration.

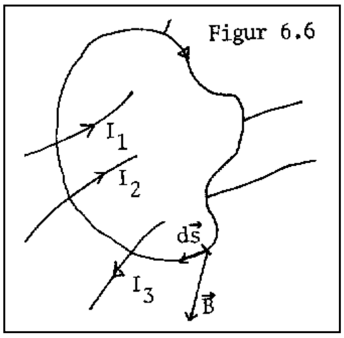
(6.3) $$d\Phi = \vec{B} \cdot d\vec{A} \quad \text{and} \quad \Phi = \int_f \vec{B} \cdot d\vec{A}$$

After having defined the concept of magnetic flux, we are then able to formulate the two main theorems of magnetostatics.

The magnetic field lines are always closed curves. There do not exist (as it is the case of electric charges) magnetic "charges" or "monopoles" where the field lines begin and end.
As a consequence, there must always enter and leave the same number of field lines from a closed surface.
Or said in other words: The magnetic flux through any closed surface is *always* zero.

(6.4) $$\Phi = \int_f \vec{B} \cdot d\vec{A} = 0$$

**Amperes law**:



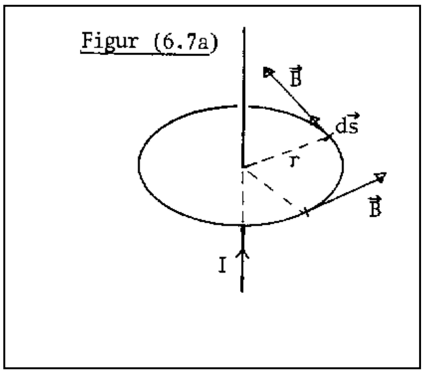

The curve integral of the magnetic field along a closed curve is equal to $\mu_0$ times the signed sum of currents that passes through a surface that has the curve as its border curve.

(6.5) $$\oint_{closed\ curve} \vec{B} \cdot d\vec{s} = \mu_0 I \quad where \quad I = \sum \pm I_k$$

Amperes law requires that a convention of signs is specified for the currents passing the surface. Choosing a positive orientation along the closed curve as shown in figure (6.6), then a current should be counted Positive, if the orientation of the border curve together with the direction of the current forms a right hand screw. In the figure, the positive currents go into the paper, such that $I_1>0$, $I_2>0$ and $I_3<0$).

From the two laws (6.4) and (6.5) it is in principle possible to calculate the magnetic field from an arbitrary distribution of stationary currents.
For example: Biot and Savart's law is a consequence of these two laws. We are not going to derive Biot and Savart's law, however, since it requires that we rewrite the two laws in differential form, that is, Maxwell's third and forth equation.
Instead we shall apply Amperes law to some simple cases, and show that they deliver the same results as Biot and Sawart's law.

**6.7 Example. Application of Amperes law to a long straight conducting wire with a current *I*.**



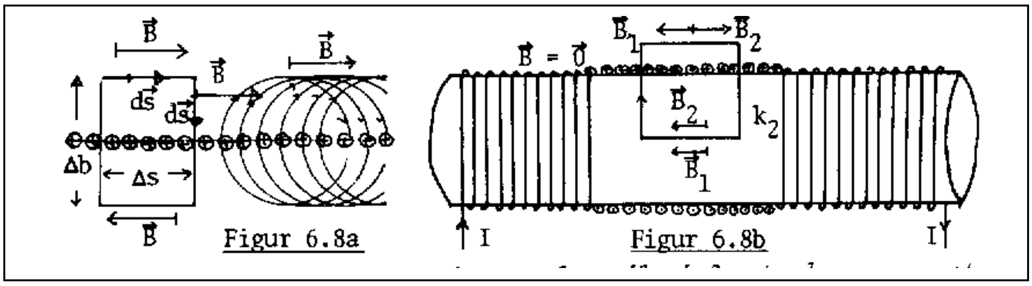

Since the field lines are closed curves they must for symmetry reasons be concentric circle with the wire going through their centres. For the same reasons the magnetic field must have the same strength on the periphery of the circles.
To apply Amperes law we integrate along a circle with radius $r$

$$\oint_{circle} \vec{B} \cdot d\vec{s} = \mu_0 I \quad \Leftrightarrow \quad B\oint_c ds = \mu_0 I \quad \Leftrightarrow \quad B2\pi r = \mu_0 I$$

(6.7a) $$B = \frac{\mu_0 I}{2\pi r}$$ (The field from a straight wire)

We notice that it is the same result as we obtained from Biot and Sawart's law (by using substantially more effort). The derivation is made to demonstrate that often it is much easier it is to apply Amperes law, (at least in configurations having a large degree of symmetry).

**6.6 Example. The magnetic field inside a long coil (A solenoid)**




To calculate the $B$-field inside a (infinite) long coil with many turns, we shall first calculate the $B$-field from a row of conducting wires as shown in figure to the left (6.8a). For reasons of symmetry the field must be directed as illustrated in figure (6.8a). We therefore choose a rectangular shaped curve with sides $\Delta s$ and $\Delta b$ as shown in the figure. Through the curve runs $\Delta N = n\ \Delta s$ wires, where $n$ is the number of wires per unit length.

Applying Amperes law to the rectangle, we notice that $\vec{B} \perp d\vec{s}$ on the vertical sides, so the contribution to the integral is zero on these sides and that $\vec{B} \parallel d\vec{s}$ on the horizontal pieces, so $\vec{B} \cdot d\vec{s} = Bds$ there.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \sum I_k \quad \Rightarrow \quad 2B\Delta s = \mu_0 \Delta NI \quad \Rightarrow \quad 2B\Delta s = \mu_0 n \Delta s I \quad \Rightarrow \quad B = \tfrac{1}{2}\mu_0 nI$$

One should notice that the $B$-field does not depend on the distance from the (infinite) row of conducting wires.
(It can be compared with the electric field from an (infinitely big) charged plate)
We shall then use this result to calculate the magnetic field inside an (infinite) long coil having $n$ turns per unit length. This is shown in figure (6.8b).
What we see is that outside the solenoid the $B$-field from the lower and upper row of turns will cancel each other, since the field is independent of the distance from the row of turns.
On the other hand, inside the solenoid we have: $B = \frac{1}{2}\mu_0\, nI + \frac{1}{2}\mu_0\, nI = \mu_0\, nI$.
The result may also be obtained directly from Amperes law, integrating along the curve $k_2$

$$\text{(6.8b)} \quad \oint_{k_2} \vec{B} \cdot d\vec{s} = \mu_0 \sum I_k \quad \Rightarrow B\Delta s = \mu_0 \Delta NI \quad \Rightarrow \quad B = \mu_0 \frac{\Delta N}{\Delta s} I \quad \Rightarrow \quad B = \mu_0 nI$$

Where $n$ has the same meaning as before: The number of turns per unit length. If a solenoid has $N$ turns and the length $L$, then: $n = N/L$.

**6.9 Exercise**
Determine, using Amperes law the magnetic field inside a torus (a doughnut shape), having radius $R$ with a circular cross section with radius $r$ and having $N$ turns of conducting wire.

## 7. Magnetic Dipole. Magnetic Moment.

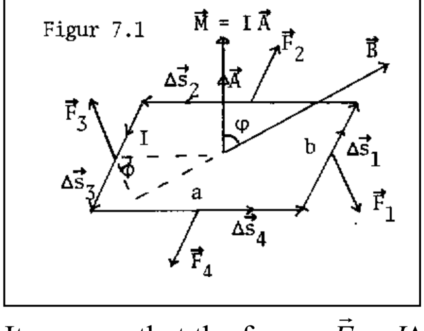



Figure (7.1) shows a small rectangular circuit, which is placed in a homogenous magnetic field $\vec{B}$. The sides of the rectangle are denoted $\Delta\vec{s}_1$, $\Delta\vec{s}_2$, $\Delta\vec{s}_3$, $\Delta\vec{s}_4$. They are all oriented in the direction of the current, and they have the lengths $a$ and $b$, the sides in the rectangle.
The magnetic forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \vec{F}_4$,which act on the sides of the rectangle can be found using Laplace' law, and in the figure is sketched their direction.

It appears that the forces $\vec{F}_2 = I\Delta\vec{s}_2 \times \vec{B}$ and $\vec{F}_4 = I\Delta\vec{s}_4 \times \vec{B}$ are equal, but opposite to each other, and since they lie in the same plane acting on the same circuit they cancel each other.

The forces: $\vec{F}_1 = I\Delta\vec{s}_1 \times \vec{B}$ and $\vec{F}_3 = I\Delta\vec{s}_3 \times \vec{B}$ are also equal but opposite to each other, but together they will exert a torque on the rectangular circuit, which tends to turn the circuit around an axis along $\vec{F}_2$ and $\vec{F}_4$ .
So we conclude that when a circuit is placed in a magnetic field, the field exert a torque on the circuit (even if the resulting force on the circuit is zero), which seeks to turn the circuit into a direction at right angle to the magnetic field. You should notice that when the *B*-field is perpendicular to the circuit then all the forces $\vec{F}_2, \vec{F}_2, \vec{F}_3, \vec{F}_4$ all lie in the same plane as the circuit and therefore they cancel each other pair wise, so there is no torque on the circuit.

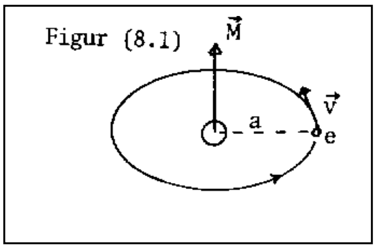
As we have done before, we introduce a flat normal vector $\vec{A}$, where its direction is perpendicular to the circuit, and its size: $A = ab$, is equal to the area of the circuit.
From the figure we can see that the lever of the torque is $\frac{1}{2}a\sin\varphi$, where $\varphi$ is the same angle as the angle between $\vec{B}$ and $\vec{A}$. From this we get: $H = F_1\frac{1}{2}a\sin\varphi + F_3\frac{1}{2}a\sin\varphi$.
But since $F_1 = F_3$ we find: $H = F_1 a\sin\varphi = 2I\,|\Delta\vec{s}_1|\,B\frac{1}{2}a\sin\varphi = IabB\sin\varphi$.
To express the torque on vector form, we introduce a new vector: $\vec{M} = I\vec{A}$, where $\vec{A}$ is the flat normal vector and *I* is the current in the circuit.

$\vec{M} = I\vec{A}$ is defined as the magnetic moment of the circuit, and the circuit is often referred to as being a *magnetic dipole*.
The direction of $\vec{M}$ is the direction from the ”south pole” of the *magnetic dipole* (the magnet) to its ”north pole”.
If we take the direction and size of $\vec{B}$, $\vec{H}$ and $\vec{M}$ in consideration, we may establish the following vector equation.
(7.2) $\vec{H} = \vec{M} \times \vec{B}$ (Where $\vec{M} = I\vec{A}$ is the magnetic moment)

From (7.2) we can see that the torque $\vec{H}$ tends to turn the magnetic moment so that $\vec{M}$ is aligned to $\vec{B}$ ,( where the torque becomes zero).
For the sake of argument, we have confined ourselves to analyze a plane rectangular circuit, but the results will equally apply for any plane circuit, where *I* and $\vec{A}$ have the same significance as above.
The magnetic moment is also defined for a solenoid with *N* turns as: $\vec{M} = NI\vec{A}$, where the direction of the magnetic moment is along the axis of the solenoid. This is so, because a solenoid may be perceived as *N* turns stacked on top of each other.
Any system, where one can define a magnetic moment vector $\vec{M}$ is called a magnetic dipole, and one can show that whatever the design is of the circuit that generates the magnetic moment, it will be affected by a torque given by (7.2) when placed in a magnetic field $\vec{B}$ .

## 8. The magnetic moment of an atom. Permanent magnets

A model of the atom can crudely be described by an extremely small positively charged nucleus surrounded by negative electrons. The atom as a whole is neutral, since the charge in the nucleus exactly corresponds to the charges of the electrons.




The orbit of an electron around the nucleus corresponds to an electric current and therefore the atom may be viewed as a small magnetic dipole with a magnetic moment.
To elaborate on this, we shall make a crude calculation of the magnitude of the magnetic moment created by the electron in the hydrogen atom.
The electron is assumed to perform a uniform circular motion, with a radius equal to the radius $a$ of the atom.
The frequency in the electrons motion is denoted υ. The current in the electrons circuit is then: $I = \upsilon e$, where $e$ is the (numeric) charge of the electron.
The strength of magnetic moment then becomes: $M = IA = \upsilon e\pi a^2$
If the electron performs a uniform circular motion, the frequency can be determined by holding together the expressions for the centripetal force with the Coulomb force

$$F_C = F_e \quad \Leftrightarrow \quad \frac{1}{4\pi\varepsilon_0}\frac{e^2}{a^2} = m_e(2\pi\nu)^2 a \quad \Rightarrow \quad \nu = \frac{1}{4\pi}\frac{e}{a}\frac{1}{\sqrt{\pi\varepsilon_0 m_e a}}$$

This expression can then be used to calculate the magnetic moment. If we put $a = 0.53\ 10^{-10}\ m$ (The Bohr radius) and the mass of the electron $m_e = 9.11\ 10^{-31}\ kg$, we find:

(8.2) $$M = \nu e\pi a^2 = \frac{e^2}{4}\sqrt{\frac{a}{\pi\varepsilon_0 m_e}} = 9.3 10^{-24}\ Am^2$$

It appears that this calculation of the magnetic moment cannot be maintained because of some atomic physics conditions, among others the spin of the electron, but we have nevertheless hit the right magnitude of the magnetic moment.

All atoms have a magnetic moment, but its calculation is normally difficult (or not possible).
To get an impression of the magnitude of the magnetic moments of the atoms, we mention that a circuit passed by an electric current of 1 Ampere and having a circular cross section of 1 $m^2$ has a magnetic moment of 1 $Am^2$.
So although the magnetic moment of an atom as calculated above seems infinitely small, we should recall that there are 6.0 $10^{23}$ atoms in a mole, so if all the magnetic moments point in the same direction (they seldom do), a material can in principle have a substantial magnetic moment. Furthermore it should be mentioned that some atoms have a much lager magnetic moment than the one calculated above.

Normally the atoms, and thereby the magnetic moments, will be randomly orientated and the collected magnetic moment will be averaged to zero.
Therefore a material does not in general have a magnetic moment.

## 8.1 Magnetization of materials

If a material where the atoms have magnetic moments is placed in a magnetic $B$ -field, then the field, (according to (7.2) $\vec{H} = \vec{M} \times \vec{B}$ ), will seek to align the magnetic moments of the atoms to the direction of the $B$-field, and this will therefore reinforce the $B$-field. The material as a whole has got a magnetic moment. It has become a *magnet*.

How strong the alignment will be, depends on several circumstances.

1) The strength of the external $B$-field.
2) The temperature.
3) The properties of the atoms in question.

1) It should be obvious that the magnetization increases with the external field, since the torque on a magnetic moment according to (7.2) is directly proportional to the *B-field.*
2) We know that the mean kinetic energy of an atom $<\frac{1}{2}mv^2> = \frac{3}{2}kT$ is proportional to the temperature *T.* At a higher temperature an atom will be subject to collisions with the other atoms, and that may bring the magnetic moments out of alignment.
3) When the magnetic moments of the atoms align with an external magnetic field it will reinforce the field, leading to a further alignment. The atomic dipoles themselves create a magnetic field, when they are aligned, and this field contributes to keep the alignment of the atoms, and in some cases even when the external field is removed the alignment persists. What is left is a *permanent magnet*. Especially the structure of the atoms in Iron and Nickel make them suitable for permanent magnets. For this reason the phenomenon of permanent magnetization also bear the name *ferromagnetism*.

Permanent magnets are either designed as bar magnets, where the magnetic field is identical to that of a solenoid, or as a horseshoe, where both poles are aligned and therefore can attract magnetic materials, mainly iron simultaneously.
A permanent magnet can, as you know, be demagnetized when heated. There exists a critical temperature called the *Curie temperature*, above which a permanent magnet can not exist.
This is because the thermal motion becomes too violent to sustain the alignments of the magnetic moments of the atoms. The magnetic moments of the atoms will hereafter again have an arbitrary orientation with respect to each other.

The fact, (earlier well known by almost every child) that that two “north poles” or two “south poles” repel each other, while a “north pole” and a “south pole” attract each other may easily be explained from the fundamental law of attraction and repulsion of electric currents.

In a bar magnet, the dipole currents (when summed up) effectively correspond to the currents in a solenoid, and therefore two bar magnets will attract or repel each other in the same manner as two solenoids.
This is an elementary consequence from the fact that aligned currents attract each other, and opposite currents repel each other. It is sought illustrated in the figure below.

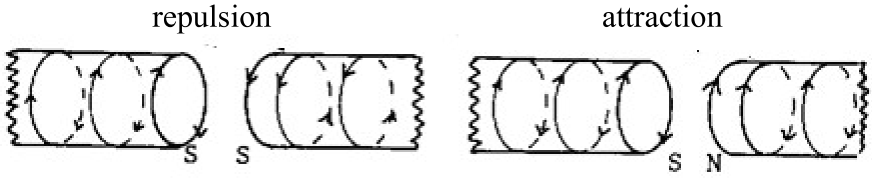



Figure 8.3

When a piece of iron is attracted to a magnet it is due to the fact that the field from the magnet creates a magnetic moment in the iron with its “north pole” nearest to the magnets “south pole” and vice versa.

## 9. Magnetic moment. Hysteresis curve for a bar magnet

As mentioned earlier, the magnetic field from a bar magnet is identical to that of a solenoid. This may be used to derive a relation between the magnetic moment $M$ of a bar magnet and the magnetic flux $\Phi_B = BA$ through the magnet, where $A$ is the cross section of the magnet.
For a solenoid, having length $L$, cross section $A$, and $N$ turns, it therefore follows:

$$B = \mu_0 \frac{NI}{L} \quad \text{(According to 6.8b)} \quad \text{and} \quad M = NIA \quad \text{(According to § 7)}$$

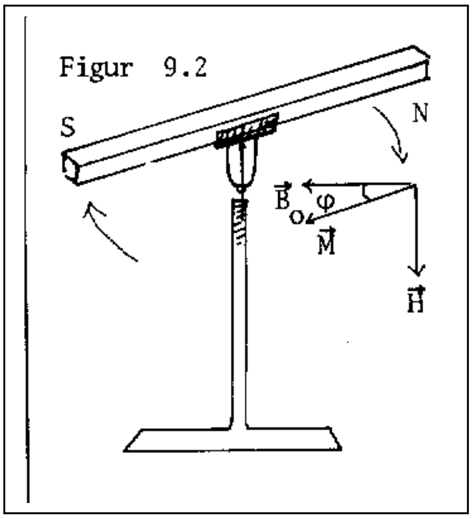
Eliminating the current $I$ from the two equations, we find:

$$\text{(9.1)} \qquad M = \frac{LBA}{\mu_0} \quad \Leftrightarrow \quad M = \frac{L}{\mu_0}\Phi_B \quad \Leftrightarrow \quad M = \frac{V}{\mu_0}B$$

Where $V = AL$ is the “volume” of the solenoid, and $\Phi_B = BA$ is the magnetic flux through the solenoid. As we remarked above, the relation (9.1) must also be true for a bar magnet, since the $B$-field, the magnetic moment and the geometry are quite analogous to that of a solenoid.

It is relatively easy to do an experimental determination of the magnetic moment of a bar magnet (apart from measuring it with an electronic device). From (9.1) one may hereafter find the magnetic flux and the $B$-field of the magnet.

(In the experiment 37, which belong to the Danish edition of this book, the flux is determined by letting a magnet be lowered through a coil and plotting the *emk* as a function of time, which confirms the theory).




The magnetic moment of a bar magnet can be determined by letting it perform horizontal harmonic oscillations in the magnetic field of the earth.
As illustrated in the figure, this can be accomplished by suspending the magnet horizontally in a needle bearing, where it may swing freely. Figure (9.2).
If the horizontal terrestrial component of the magnetic field is $B_0$, then the magnet will be acted by a torque (moment of force) $\vec{H} = \vec{M} \times \vec{B}_0$ , with the size $H = -MB_0 \sin\varphi$ .
For small oscillations we may put $\sin\varphi \approx \varphi$, and thus $H = -MB_0\varphi$ . If $I$ is the moment of inertia of the magnet, then from the torque equation: $H = I\dfrac{d^2\varphi}{dt^2}$, it follows:

$$I\frac{d^2\varphi}{dt^2} = -MB_0\varphi \quad \Leftrightarrow \quad \frac{d^2\varphi}{dt^2} = -\frac{MB_0}{I}\varphi$$

The last equation is the differential equation for a harmonic oscillation, which has the solution:

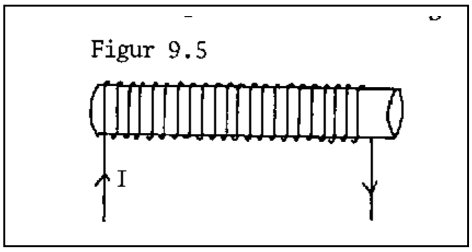
$$\varphi = \varphi_0 \cos(\omega t + \delta),$$

where

(9.4) $$\omega^2 = \frac{MB_0}{I} \quad and \quad \omega = \frac{2\pi}{T} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{I}{MB_0}}$$

The moment of inertia can be calculated, knowing the mass and the dimensions of a rod, and if the horizontal component of the terrestrial magnetic field is known then the magnetic moment can be calculated from (9.4) if the period *T* is measured with a stopwatch.

## 9.1 Magnetic permeability




If an iron core is wrapped with several turns of a conducting cobber wire we have a device, which is known as an electromagnet. The magnetic field coming from an electromagnet is many times stronger than the field generated by the coil, without the iron core.
The strength of the *B*-field may then be determined from Amperes law together with the knowledge of irons ability to become magnetized.
The magnetic field inside the iron core consists of two contributions:
One contribution comes from the magnetic field from the external electric currents and can be evaluated from Amperes law. This field we call *H* (according to an old tradition, and should not be confused with the torque).
The *H*-field from the currents will, however, generate a magnetic field *S,* coming from the magnetic moments of the atoms in the iron core, which is aligned with *H.*

The measured field *B* should then be calculated as the sum of *H* and *S*: $B = H + S$.
If the *H*-field is calculated from Amperes law, we have according to (6.8b) : $H=\mu_0 nI$, where $\mu_0$ is the permeability of vacuum, *n* is the number of turns per unit length and *I* is the current.

In the first approximation, we shall assume that the magnetic field coming from the magnetization of the iron core is proportional to the field *H* generated by the current in the coil. (The hysteresis curve illustrated below shows that this is only the correct for moderate currents).
Within the interval, where the proportionality is approximately true we can therefore write:
$S = \kappa H$. Then we may establish an expression for the magnetic *B*-field that we measure.

(9.5) $$B = H + S \quad \text{and} \quad S = \kappa H \quad => \quad B = (1 + \kappa)H = (1+ \kappa)\, \mu_0 nI = \mu nI$$

Which we write as: $$B = \mu_r H$$

What we see is that we get the same expression for the magnetic field as without the iron core, if only we substitute $\mu_0$ , (the permeability of the vacuum), with $\mu = \mu_r\mu_0$ (the permeability of iron).

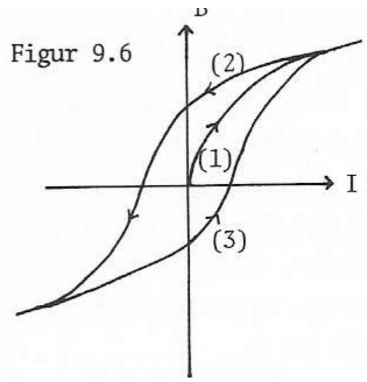
$\mu_r$ is called the *relative permeability,* and it can be very large (several thousands), and there exist alloys with a relative permeability of about a million. For larger currents the relative permeability $\mu_r$ depends on the strength of the current.

However, there also exist materials (e.g. copper) having a relative permeability less than one.
We shall not go further into that, but the phenomenon is called *diamagnetism*.

The relation $B = \mu_r H$ is quite general, and it is not only valid for a coil with an iron core.
If we want to find the magnetic field in any other material than vacuum, it can in general be accomplished in exactly the same manner as in vacuum, (Amperes law, Biot and Savarts law), if only we substitute $\mu_0$ with $\mu = \mu_r \mu_0$.

Quite often you will encounter Amperes law written, using the $H$-field instead of the $B$-field.
The reason for this is that on the right side of Amperes law are only included the external currents.
The $B$-field may then found using the relation $B = \mu_r H$.

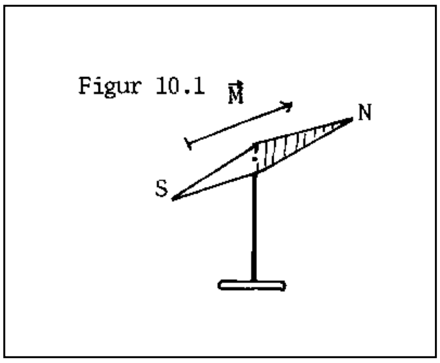
Figur 9.6




The ability of a material to be magnetized is more precisely described by its hysteresis curve. The hysteresis curve is a mapping of the magnetic $B$-field from an electromagnet versus the current in the coil. If the starting point is a completely non magnetized rod of iron, then the magnetic field will grow as shown as the curve (1) in figure (9.6). We can see that for moderate currents, we may speak of a proportionality, as given by the equation: $B = \mu_r H$.
For larger currents the curve will flatten out ending again with a seemingly linearity but with a much lower slope than in the beginning. The point where the curve becomes linear is the point of *saturation*.
At the saturation point the iron cannot be further magnetized (at that temperature), and from that point the slope of the magnetization curve is the same as for a coil without iron core, but displaced upwards by a constant magnetic field corresponding to a completely magnetized iron core.

If we then turn down for the current but continue with a reversed current after the current has reached zero, we get the curve (2), and we can se that there remains a $B$-field even when the current is zero. The iron core has been permanently magnetized.
As the reversed current becomes stronger, the magnetic field will go to zero for subsequently to be established in the opposite direction, as a mirror image of the upper curve (2). If the current again is reversed after saturation, we have the curve (3). The closed curves (2) plus (3) are called the hysteresis curve for the material (iron).
For most metals (except iron and nickel) the hysteresis curve is a straight line, that is, the same line as without a core.
The hysteresis curve gives an accurate picture of the ability of a material to be magnetized.
To predict the magnetic field from an electromagnet, it is actually necessary to have knowledge of its prehistory, since a permanent magnet can in principle be anywhere on the hysteresis curve.

## 10. Terrestrial Magnetism



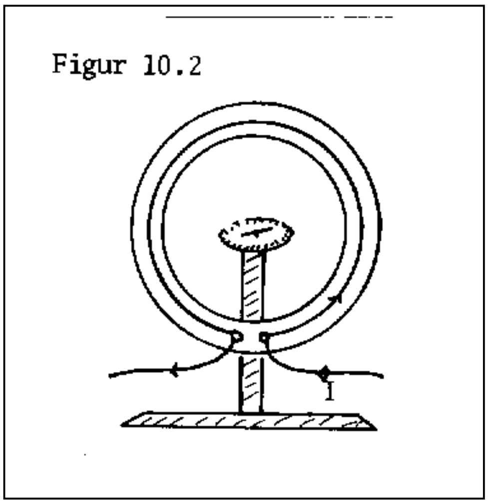

To a crude determination of the direction of (especially the terrestrial magnetic field) one uses small permanent magnets, the so called magnet needles or compass needles. They are suspended on the pin of a needle, so they may move freely horizontally or vertically.

The compass needle has a magnetic moment $\vec{M}$, which is aligned with the needle, from the south pole to the north pole of the magnet. If the compass needle is placed in a magnetic field $\vec{B}$, the compass will align with the field, such that its magnetic moment $\vec{M}$ points in the same direction as $\vec{B}$.

This follows from the fact that the torque on a magnetic moment is given by: $\vec{H} = \vec{M} \times \vec{B}$, and therefore: $\vec{H} = 0 \quad \Leftrightarrow \quad \vec{M} \,||\, \vec{B}$ . (The position, where $\vec{M}$ is opposite $\vec{B}$ is not a stable equilibrium).

If you place a compass needle so it can turn freely in the horizontal plane, it will direct itself so it points toward the magnetic north pole (the deviation from the direction to the geographic north pole is called the variation). From this, one may conclude that the earth is surrounded by a magnetic field with its magnetic north pole near the geographic south pole and vice versa.

The terrestrial magnetic field varies somewhat with the geographic position. Before the appearance of modern electronic devices apt to determine the terrestrial magnetic field, it could be experimentally determined in class, with a device called a *tangent boussole*, shown in the figure to the left.

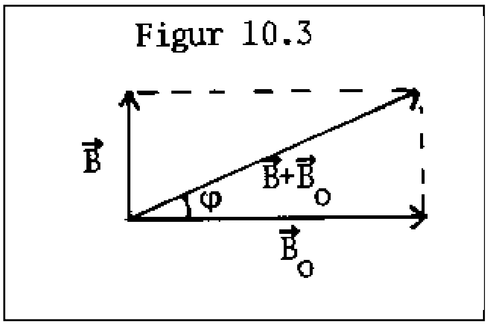



The tangent boussole is built from a flat vertical coil, in the centre of which is place a compass needle, which can turn freely and its position read off on a scale of degrees. The magnetic field in the centre of a flat coil with radius *R*, and having *N* turns is according to (3.5c):

(10.2) $$B = \frac{\mu_0 NI}{2R}$$

We denote the horizontal component of the terrestrial magnetic field $\vec{B}_0$

When there is no current in the coil the compass needle will align itself with $\vec{B}_0$.




When a current is induced in the coil, creating a magnetic field $\vec{B}$ at the position of the compass needle, it will align to the vector sum of the two magnetic fields.

The deflection of the needle we denote φ.

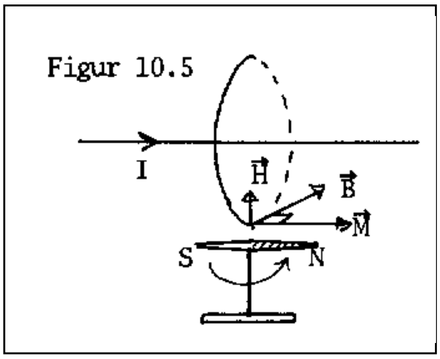
From figure (10.3) we can see that.

(10.4) $$\tan\varphi = \frac{B}{B_0} \quad \Rightarrow \quad B_0 = \frac{B}{\tan\varphi}$$

When the magnetic field *B* from the coil is calculated from (10.2), one may also calculate $B_0$.

## 10.1 The experiments of H.C. Oersted

Historically the magnetism from iron has been discovered long before the magnetism created from electric currents. It was the Danish physicist H. C. Oersted who was the first to recognize, and describe the connection between electric currents and magnetism.

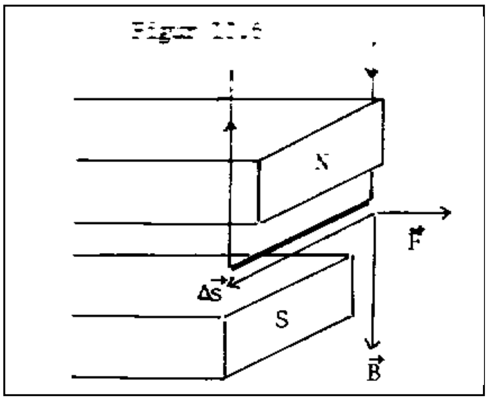



In the famous, so called Oersted experiment a compass needle is placed below a straight conducting wire.
If a current $I$ is induced in the wire, the needle will be deflected. This we understand today, since the compass needle having a magnetic moment $\vec{M}$ will be affected by a torque $\vec{H} = \vec{M} \times \vec{B}$. From the direction of the vectors, we can see that the north pole of the compass needle will swing into the plane of the paper.
This is in accordance with Oersted's "right hand thumb rule" (that all children in Danish schools had to learn by heart until the mid sixties):

"*Place your right hand with your fingertips in the direction of the current and the palm turned to the north pole of the compass needle. Then the north pole will turn in the direction of your thumb*"

Which is nothing but an outdated way to indicate the direction of a cross product of two vectors.

Oersted's experiment demonstrates that a current affects a magnet, but According to Newton's 3. law the reverse must also be the case. This was earlier demonstrated in class rooms with a device shown below in figure (10.6)




The setup of the appliance consists of a horseshoe magnet, with a strong magnetic field between the poles. The conducting rod between the poles of the magnet is designed as a swing, which is suspended in a pair of thin conducting wires.
If a current is sent through the circuit, the rod will be affected by a force.
The strength, and the direction of the magnetic force is then given by Laplace' law.
The orientation of the conductor is along the current and given by the vector $\Delta\vec{s}$, and assuming that the magnetic field between the poles to be homogenous, the force on the conducting rod given by Laplace' law (5.1).

$$\vec{F} = I\Delta\vec{s} \times \vec{B}$$

The directions of the three vectors $\Delta\vec{s}$, $\vec{B}$ and $\vec{F}$ are indicated in figure (10.6), and one may convince oneself, that the direction of the force is in accordance with Oersted's "little finger rule".

*"Put your right hand with the fingertips in the direction of the current and the palm towards the north pole of the magnet. The conductor will then swing towards the direction of your little finger".*