Hydrodynamics
is
complex mathematics

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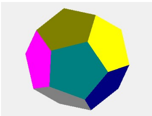

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1. Vector analysis

Within the classical physics, hydrodynamics is probably one of the areas, which from a mathematical point of view, belongs the most complex, but together with electrodynamics also the most impressive and rewarding applications of vector analysis.

The theoretical hydrodynamics is fundamentally simple (and mathematically beautiful, if you have such inclinations). It is based on Newtonian dynamics, but most of the partial differential equations, which are a consequence of the theory, are non linear, and therefore they cannot be solved, except in cases of high symmetry.

Thus hydrodynamics is a beautiful rounded theory, which unfortunately only to a modest extend may be applied to nature. What the theory cannot explain from first principles is the occurrence of turbulence.

When dealing with hydrodynamics in a non phenomenological way it is, however, imperative to master parts of the mathematical formalism vector analysis of fields, outlined below

A **scalar field** is a function of time and position: $\varphi = \varphi(x,y,z,t)$.

A **vector field** i.e. $\boldsymbol{v} = (v_x, v_y, v_z)$ consists of three spatial components each being a function of position and time. The vector analysis uses the following mathematical symbols:

The **gradient** of a scalar field φ:

$$\vec{\nabla}\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \qquad \text{(Also written as } grad\ \varphi\text{)}$$

Divergence of a vector field $\boldsymbol{v}$:

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad \text{(also written as } div\ \boldsymbol{v}\text{)}$$

The **Laplace operator**:

$$\nabla^2\varphi = \vec{\nabla}\cdot\vec{\nabla}\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \qquad \text{(Sometimes also written as } \Delta\varphi\text{)}$$

Curl of a vector field:

$$\vec{\nabla}\times\vec{v} = (\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}) \qquad \text{(Also written as rot } \boldsymbol{v}\text{)}$$

For any vector field the following equations must be valid:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0 \qquad \text{and} \qquad \vec{\nabla}\times(\vec{\nabla}\varphi) = \vec{0}$$

This may be "relatively" easy proven, by writing the expressions in coordinates, but we only settle for proving the first:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = \frac{\partial}{\partial x}(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})$$

The two terms : $\frac{\partial}{\partial x}\frac{\partial v_z}{\partial y}$ *and* $-\frac{\partial}{\partial y}\frac{\partial v_z}{\partial x}$ will cancel, and similarly for the other two pairs

2. The continuity equation

This equation expresses the fact that the amount of fluid that flows from a closed surface per unit of time is equal to the rate of change of the amount of fluid within the closed surface.

If the fluid is incompressible, then the change in the amount of fluid comes from sources or drains within the closed surface.

$\vec{v}$ is the velocity vector of a particle element with density ρ.
The surface area element $d\vec{A}$ is a vector, which is a outward normal to the surface, having the size dA, and $\rho\vec{v}\cdot d\vec{A}$ is then the flux of the liquid. In this case, it is the amount of liquid, which flows through the surface area dA per unit of time. The equation of continuity may therefore be written.

$$\int_{surface} \rho\vec{v}\cdot d\vec{A} = \frac{\partial}{\partial t}\int_{volume} \rho dV$$

3. Stokes theorems and Gauss' lov

Stokes theorems are valid (under some general conditions) for an arbitrary vector field, but in hydrodynamics we state them in terms of the vector field $\rho\vec{v}$, where $\vec{v}$ is the velocity of a particle having density ρ.

3.1 Stokes first theorem

The flux of a vector field through a closed surface is equal to of the divergence of the vector field integrated over the volume within the surface.

$$\int_{surface\ A} \rho\vec{v}\cdot d\vec{A} = \int_{volume\ V} \vec{\nabla}\cdot(\rho\vec{v})dV$$

The left hand integral is, through the continuity equation, equal to $\frac{\partial}{\partial t}\int_{volumen}\rho dV$, and thus we find:

$$\int \vec{\nabla}\cdot(\rho\vec{v})dV = \frac{\partial}{\partial t}\int_{volume}\rho dV$$

Since this equation must be valid for every volume, the two integrands must be equal, so the following equation must be valid.

$$\vec{\nabla} \cdot (\rho \vec{v}) = \frac{\partial \rho}{\partial t}$$

In this case the equation represents the continuity equation in differential form.

If ρ is constant in space and time $\rho(x,y,z,t) = \rho$, then the equation is reduced to $\vec{\nabla} \cdot \vec{v} = 0$, which is the case of an incompressible liquid.

3.2 Stoke's second theorem

The curve integral of a vector field along a closed curve is equal to the surface integral of the curl of any open surface which has the curve as its border curve.

$$\oint_{\text{closed curve}} \vec{v} \cdot d\vec{s} = \int_{\text{surface}} \vec{\nabla} \times \vec{v} \cdot d\vec{A}$$

It is not entirely uncomplicated to prove Stokes theorems, but a proof "for physicists" may be found in The Feynman Lectures II from 1963.

3.3 Stokes theorems in electrodynamics

The most well known applications of the theorems of Stoke is probably in electrodynamics. Thus the Maxwell equations for the electric and magnetic fields $\vec{E}$ *and* $\vec{B}$, when they are written in differential form:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}$$

Having a surface, $d\vec{A}$ denote the outward normal to the differential area dA, and $\vec{E} \cdot d\vec{A}$ is the electric flux through the surface area $d\vec{A}$.

Applying Stokes first theorem, we shall now calculate the overall flux through a closed surface.

$$\int_{\text{closed surfaces } A} \vec{E} \cdot d\vec{A} = \int_{\text{volume } V} \vec{\nabla} \cdot \vec{E}\, dV = \int_{\text{volume } V} \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

Which is Gauss' law (Maxwell's 1. equation on integral form), expressing that the electric flux through a closed surface is equal to the signed collected charged within the surface divided by ε_0.

In a similar manner, one may apply Stokes second theorem to derive Ampere's in integral form from the Maxwell equations.

$$\oint_{closed\ \ curve} \vec{B} \cdot d\vec{s} = \int_{open\,surface} \vec{\nabla} \times \vec{B} \cdot d\vec{A} = \frac{1}{\varepsilon_0 c^2} \int_{open\,surface} \vec{j} \cdot d\vec{A}$$

For static magnetic fields, it then it applies that the curve integral of the B-field along a closed curve is equal to the surface integral of the current density through the surface, which has the curve as its border curve.

For an infinite conductor with a running current I, one may choose a circular curve with radius r with the conductor perpendicular to the circle, through its centre.
For symmetry reasons the B-field must have the direction along the tangent to the circle, and it must have the same value everywhere on the circle. Therefore:

$$\oint_{circular\ curve} \vec{B} \cdot d\vec{s} = 2\pi r B \qquad \text{and} \qquad \frac{1}{\varepsilon_0 c^2} \int_{surface} \vec{j} \cdot d\vec{A} = \frac{1}{\varepsilon_0 c^2} I = \mu_0 I$$

From this follows Ampere's law: $B = \frac{\mu_0}{2\pi} \frac{I}{r}$

4. The hydrodynamic equations of motion

In a fluid it is valid that the force per unit volume is equal the gradient of the pressure p: $\vec{F} = \vec{\nabla} p$ plus the actions of external forces. Newton's 2. law for a volume element is thus.

$$\rho \frac{d\vec{v}}{dt} = \vec{\nabla} p + \vec{F}_{ext}$$

If there are viscous forces in the fluid, one should add a term F_{visc} , but even if it might be possible to establish an expression for the viscous force, the are no solution to the resulting differential equations except in very special cases.

Hydrodynamics becomes mathematically complex for several reasons, but especially because when evaluating the material differential quotient

$$\frac{d\vec{v}}{dt}$$

The components of the velocity, apart from being explicitly dependent of time also depend on $x, y, z,$ which are time dependent, because the liquid element moves.
Namely, because the liquid element, which at time t has the position (x,y,z) it has the position at time $t + \Delta t$ $(x + \Delta x,\ y + \Delta y, z + \Delta z)$

For that reason we must differentiate through *x, y, z*, to form the material differential quotient. Below we show the calculation of dv_x/dt in detail, but the same procedure applies for the differentiation of v_y and v_z.

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial v_x}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial v_x}{\partial t} = v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial t}$$

This can be written in the symbolic form.

$$\frac{dv_x}{dt} = \vec{v}\cdot(\vec{\nabla} v_x) + \frac{\partial v_x}{\partial t}$$

If we merge the equations for the three coordinates, we may then write:

$$\frac{d\vec{v}}{dt} = \vec{v}\cdot(\vec{\nabla}\vec{v}) + \frac{\partial \vec{v}}{\partial t}$$

The cross product of two vectors $\vec{a}$ *and* $\vec{b}$ is defined as:

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y\ ,\ a_z b_x - a_x b_z\ ,\ a_x b_y - a_y b_x)$$

And one can show that:

$$\vec{v}\cdot(\vec{\nabla}\vec{v}) = (\vec{\nabla}\times\vec{v})\times\vec{v} + \tfrac{1}{2}\vec{\nabla}v^2$$

We settle for calculating the *x*-component of the right hand side, and we do it in a series of steps. First we evaluate the curl of ***v***.

$$(\vec{\nabla}\times\vec{v}) = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\ ,\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\ ,\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

We then evaluate cross product with ***v***.

$$((\vec{\nabla}\times\vec{v})\times\vec{v})_x = \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)v_z - \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)v_y$$

If we to this expression add:

$$\tfrac{1}{2}\vec{\nabla}_x({v_x}^2 + {v_y}^2 + {v_z}^2) = \tfrac{1}{2}\frac{\partial}{\partial x}({v_x}^2 + {v_y}^2 + {v_z}^2) = v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial x} + v_z\frac{\partial v_z}{\partial x}$$

Then the terms equal to $v_y\frac{\partial v_y}{\partial x}$ *and* $v_z\frac{\partial v_z}{\partial x}$ cancel each other, and it leaves us with

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}$$

as asserted.

The equations of motion for a liquid in motion then become:

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p + \vec{F}_{ydre} \quad \Leftrightarrow \quad \rho(\vec{\nabla}\times\vec{v})\times\vec{v} + \tfrac{1}{2}\rho\vec{\nabla} v^2 + \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p + \vec{F}_{ext}$$

The external force is most frequently the gravity, but in any case if the force is conservative i.e. $\vec{\nabla}\times\vec{F} = 0$. (This means, according to Stokes law that the integral of ***F*** along a closed curve is zero, or that the curve integral of ***F*** between two points is independent of the path chosen), which again means that ***F*** can be written as the gradient of some potential function $U = U(x,y,z)$

$$\vec{F} = -\vec{\nabla} U$$

Hemce the equations of motion become:

$$\rho(\vec{\nabla}\times\vec{v})\times\vec{v} + \tfrac{1}{2}\rho\vec{\nabla} v^2 + \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p - \vec{\nabla} U$$

This equation is usually referred to as Navier Stokes equation

5. Rotational free flow

If the liquid is overall rotational free, that is, if $\vec{\nabla}\times\vec{v} = 0$, then the first rather troublesome term in Navier Stokes equation most conveniently disappears, and we get, when collecting all terms on the left side.

$$\vec{\nabla} p + \vec{\nabla} U + \tfrac{1}{2}\rho\vec{\nabla} v^2 + \rho\frac{\partial \vec{v}}{\partial t} = 0$$

Furthermore if the flow is laminar, so that the velocity with which the liquid flows is unchanged in time, so $\frac{\partial \vec{v}}{\partial t} = 0$, we can in fact move the gradient operator outside a parenthesis, which simplifies the equation considerably.

$$\vec{\nabla}(p + U + \tfrac{1}{2}\rho v^2) = 0$$

Resulting in:

$$p + U + \tfrac{1}{2}\rho v^2 = const$$

Finally if the external force is gravity, and the potential energy therefore is $U{=}\rho gh,$ we arrive at Bernoulli's law.

$$p + \rho gh + \tfrac{1}{2}\rho v^2 = const \quad \text{(along a streamline).}$$

Bernoulli's law is actually nothing but a formulation of the conservation of energy, as a consequence of Newtonian mechanics, and it can relatively easy be derived, without resorting to vector analysis.

6. Vortices

We shall base our considerations on the Navier-Stokes equation.

$$\rho(\vec{\nabla}\times\vec{v})\times\vec{v}+\tfrac{1}{2}\rho\vec{\nabla}v^2+\frac{\partial\vec{v}}{\partial t}=-\vec{\nabla}p-\vec{\nabla}U$$

We collect all terms which are prefixed by a gradient on the left side, and put the gradient outside a parenthesis.

$$\rho\frac{\partial\vec{v}}{\partial t}+\rho(\vec{\nabla}\times\vec{v})\times\vec{v}=-\vec{\nabla}(p+U+\tfrac{1}{2}\rho v^2)$$

Then we define the *vorticity,*(the vorticity vector), as the curl of the velocity vector.

$$\vec{\Omega}=\vec{\nabla}\times\vec{v}$$

Here after the equation of motion reads:

$$\rho\frac{\partial\vec{v}}{\partial t}+\rho\vec{\Omega}\times\vec{v}=-\vec{\nabla}(p+U+\tfrac{1}{2}\rho v^2)$$

If we then take the curl on both sides, the right side vanishes (because $\vec{\nabla}\times(\vec{\nabla}\varphi)=\vec{0}$)

$$\rho\frac{\partial\vec{\nabla}\times\vec{v}}{\partial t}+\rho\vec{\nabla}\times(\vec{\Omega}\times\vec{v})=-\vec{\nabla}\times\vec{\nabla}(p+U+\tfrac{1}{2}\rho v^2)$$

$$\frac{\partial\vec{\Omega}}{\partial t}+\vec{\nabla}\times(\vec{\Omega}\times\vec{v})=0$$

The three equations

$$\frac{\partial\vec{\Omega}}{\partial t}+\vec{\nabla}\times(\vec{\Omega}\times\vec{v})=0 \qquad \vec{\Omega}=\vec{\nabla}\times\vec{v} \qquad \vec{\nabla}\cdot\vec{v}=0$$

Are the dynamic equations of motions for vortices, but they are certainly not easily solved!

Although it requires a more formal proof, the equations strongly indicate that, if the vorticity vector $\boldsymbol{\Omega}=\mathbf{0}$ at some instant, it will remain zero at all time, and if the vorticity $\boldsymbol{\Omega}\neq\mathbf{0}$, then it will remain non zero forever
The classical hydrodynamic equations of motion, may thus describe both laminar flow and turbulent flow (vortices), but it cannot explain how or why turbulent flow arises from laminar flow. In fact the equations tell us that it can’t.

Since almost all flows in water or air even at moderate speeds demonstrates vortices, we are in the rather unsatisfactory situation that we have a simple and mathematically beautiful theory, derived from first principles, but it has only little to do with the real world.

7. Examples of applying hydrodynamics to the (almost) real world:

Bernoulli's law

$$p + \rho g h + \tfrac{1}{2}\rho v^2 = const$$

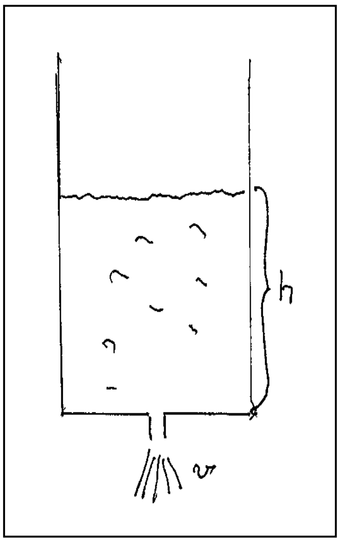
Can account for many daily life experiences (that you whish you should never experience yourself). Most people have seen movies, where a window has been broken in an airplane, and passengers near the window hold on to everything to avoid being sucked out of the window. Since there is no difference in height between inside and outside Bernoulli's law becomes: $p + \tfrac{1}{2}\rho v^2 = const$.

If we put the pressure to 1 atm = 1.013 10^5 N/m^2 inside the plane, and if the plane has a speed of 800 km/h = 222 m/s and ρ_{air} =1.0 kg/m^3 , then we may calculate the difference in pressure from the inside to the outside. $p_{inside} + 0 = p_{outside} + \tfrac{1}{2}\rho_{air} v_{air}{}^2$

$\Delta p = \tfrac{1}{2}\rho_{air} v_{air}{}^2 = 2.5\, 10^4 \;\; N/m^2 = 0.25 \; atm$ which is in fact quite significant.

Another scaring example, which you hopefully only have experienced watching a movie, is the dragging of everything including people up into a tornado. The huge velocity of the circulating air in a tornado makes the pressure very low, and hereby sucking everything up not bolted from the ground.

7.1 Emptying of a container from a tub at the bottom:



Next we shall consider emptying a container, using a tub at the bottom. Assuming that we have a container filled to the height h with liquid, free from viscosity, and equipped with a tub at the bottom.
The liquid has density ρ, and $p = p(y)$ denotes the pressure in the height $y,$ measured from the bottom. We the apply Bernoulli's law to find the velocities at different levels.

$$p_1 \;+ \tfrac{1}{2}\rho\, v_1{}^2 + \rho\, g\, y_1 \;= p_2 + \tfrac{1}{2}\rho\, v_2{}^2 + \rho\, g\, y_2 \,.$$

We shall omit the (external) pressure, since it does not change in the extension of the container. Replacing y with h (the height of the water level), we have the equation:

$$\tfrac{1}{2}\rho\, v_1{}^2 + \rho\, g\, h_1 \;=\; \tfrac{1}{2}\rho\, v_2{}^2 + \rho\, g\, h_2$$

At $h = 0$, $v = 0$, and in the depth h the velocity is v.

Thus Bernoulli's law gives the same result as a free fall at the surface of the earth.

$$\tfrac{1}{2}\rho\, v^2 + \rho\, g(-h) = 0 \quad \Leftrightarrow \quad v = \sqrt{2gh}$$

Where v is the speed, with which the water leaves the tub.

If $m = m(t)$ is the mass of the water in the container, and if the tub has the cross section D, then the continuity equation for the mass dm, that in the time interval dt flows from the tub, gives:

$$\frac{dm}{dt} = -\rho D v. \text{ (Minus because } m \text{ is decreasing)}$$

If the cross section of the container is A, then the mass $m = \rho A h$, so that

$$\frac{dm}{dt} = \rho A \frac{dh}{dt}$$

If we put the two expressions for $\frac{dm}{dt}$ together we get: $\rho A \frac{dh}{dt} = -\rho D v$, and inserting the expression for the velocity $v = \sqrt{2gh}$, we find a differential equation for h.

$$\rho A \frac{dh}{dt} = -\rho D \sqrt{2gh} \quad \Leftrightarrow \quad \frac{dh}{dt} = -\sqrt{2g}\frac{D}{A}\sqrt{h}$$

This equation can readily be solved by separation of the variables.

$$\frac{dh}{\sqrt{h}} = -\sqrt{2g}\frac{D}{A}dt \quad \Leftrightarrow \quad \int_{h_0}^{h}\frac{dh}{\sqrt{h}} = -\sqrt{2g}\frac{D}{A}\int_0^t dt \quad \Leftrightarrow$$

$$2\sqrt{h} - 2\sqrt{h_0} = -\sqrt{2g}\frac{D}{A}t \quad \Leftrightarrow \quad \sqrt{h} = \sqrt{h_0} - \sqrt{2g}\frac{D}{2A}t \quad \Leftrightarrow$$

$$h = \left(\sqrt{h_0} - \frac{D\sqrt{2g}}{2A}t\right)^2$$

The container is empty, when $h = 0$. This happens according to the equation above at the time:

$$t = \frac{A}{D}\sqrt{\frac{2h_0}{g}}$$

Having a container with cross section A = 50 x 50 cm^2 and h_0 =1.0 m, D=2.0 cm^2, it gives the duration 564 s.

7.2 Rotating liquid with constant angular speed.

If a liquid is rotating with constant angular speed ω: the position of a liquid element is

$$\vec{r} = (x, y, z) = r(\cos\omega t, \sin\omega t, z)$$

$$\vec{v} = \frac{\partial \vec{r}}{\partial t} = r(-\omega \sin\omega t, \omega \cos\omega t, 0) \qquad \vec{v} = \omega(-y, x, 0)$$

We then calculate the vorticity $\vec{\Omega} = \vec{\nabla} \times \vec{v}$. The x and y components are zero, because the x and y components of the velocity do not depend on z, and the z component of the velocity vector does not depend on x and y.

$$\vec{\Omega} = \vec{\nabla} \times \vec{v} = \omega\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) = (0, 0, 2\omega)$$

Thus $\vec{\Omega}$ is constant in magnitude and direction, and $\Omega = 2\omega$ (The double of the angular velocity)

Until about twenty years ago one could frequently experience vortices made by cigar smokers as smoke rings.

A smoke ring could cross a library without diluting, beautifully illustrating the constancy of $\vec{\Omega}$.
If we have a smoke ring (a torus) with radius R is situated in the x-y plane , and the circular cross section has radius r, then a parameter representation for a particle which moves in a uniform circular motion, can be written as:

$$(x, y, z) = ((R + r\cos\omega t)\cos\varphi, (R + r\cos\omega t)\sin\varphi, r\sin\omega t)$$

To evaluate $\vec{\Omega} = \vec{\nabla} \times \vec{v}$ directly from the general expression is not worth while, so we restrict ourselves to the case where $\varphi = 0$, so that $(x, y, z) = ((R + r\cos\omega t), 0, r\sin\omega t)$

This corresponds to uniform circular motion in the x-z plane $\vec{\Omega} = \vec{\nabla} \times \vec{v} = (0, 2\omega, 0)$

If one draws the field lines of $\mathbf{\Omega}$, then such lines are closed. The reason for this is that the divergence $\vec{\nabla} \cdot \vec{\Omega} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$. This is analogous to the magnetic field lines, which are always closed, since $\vec{\nabla} \cdot \vec{B} = 0$, because there are no magnetic monopoles (I was taught at the university).

Helmholz has formulated the theorem that the integral of $\mathbf{\Omega}$ over a surface perpendicular to $\mathbf{\Omega}$ is constant. The vorticity flux through a surface that follows the movement of the liquid is constant.

A more rigorous proof of this assertion is somewhat complicated, but one may show that Helmholtz' presumption has the conservation of angular momentum as a consequence, which tend to straighten the argument

We look at the vorticity in a circular pipe, which follows the flow. Let the cross sections of the pipe be A_1 and A_2 at the times t_1 og t_2.

The mass of the liquid (or gas) in a disc that follows the flow must be the same, $M_1 = M_2 = M$.
Helmholtz assertion is then:

$$A_1\Omega_1 = A_1\Omega_1$$

Multiplying by the mass, and inserting $A_1 = \pi r_1^2$ and $A_2 = \pi r_2^2$, one finds: $M\pi r_1^2\, \Omega_1 = M\pi r_2\, \Omega_2$,
But Ω is proportional to the angular velocity ω, so each of the two expressions are proportional to the moment of inertia $I = Mr^2$ times ω, which is equal to the angular momentum!