Gyroscopes, tops, rotating eggs and other spinning objects

An article from my home page: www.olewitthansen.dk

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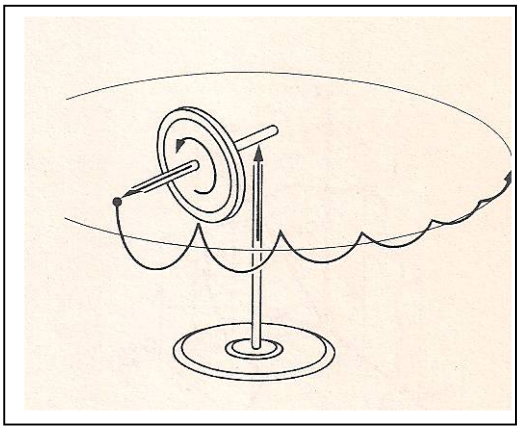
Preface

It is more than 35 years ago, that the concepts of torque, angular momentum an the conservation of angular momentum and the theory of rotation disappeared, from the science branch of the Danish 9-12 grade high school (Gymnasium). Until then rotation was a mandatory part of the curriculum at the highest level of physics.
In the textbooks on theoretical physics: Elementary Physics 1 - 3 that I wrote from 1976 to 1980 there was in the second volume a substantial chapter on rotation which went beyond the instructions from the department of education. A bit surprising, I found out that the chapter I wrote on rotation is in fact more comprehensive and a theoretically higher level than what has become a standard introductory book: Young and Freedman: University Physics, from Addison Wesley 2008.
The chapter on rotation has been translated into English, and can be found on my home page: www.olewitthansen.dk
But the chapter did not cover the theory of gyroscopes, which is the subject of this article.
True enough, the theory of rotation was already mathematically demanding in 1980'ties, and it would be utterly inconceivable to teach the physics of rotation at that level after 2005 in Denmark at least.

1. The physics of rotation

Why should web e concerned with rotation, when it is that mathematically demanding? One reason is that it is the area of Newtonian mechanics that has the most surprisingly consequences often contradictory to what one would intuitively expect from daily life experiences.
Gyroscopes have been traded as toys in about 100 years, where one end of the rotating axis rests, but not fixed op the top of a tower, and the gyro instead of falling down, as it would without a fast rotation around the axis, performs a vertical circular motion.



The figure shows such a gadget, where the massive wheel initially is put into rapid rotation and the axis is balanced vertically. If you let go the support in the free end, you would normally expect that the gyro would fall down, as would be the case, without the rotating disc.
Well, first the gyro actually begins to fall, but then, and that is the peculiar thing, it straightens up, and instead of falling through a horizontal axis it begins a precession about a vertical axis. Especially in the beginning the precession is accompanied by some irregular bumps, the so called nutation. In this article, however we shall only account for the precession itself.
As the rotational velocity of the disc decreases, the precession velocity becomes more rapid, and eventually the gyro will fall to the ground.
Applying Newtonian mechanics, it is not so hard to give an explanation of the behaviour of the precession motion of the gyroscope and calculate the angular velocity in the precession.
First it is however necessary to get some concepts clear concerning rotation, as well as the vector character of the equation of motion.
Rotation around an axis is mathematically equivalent to rectilinear motion about an axis, when you replace the position coordinate s with the angle of rotation θ. While the velocity is defined as

$$v = \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Then the angular velocity ω for rotation defined as:

(1.1) $$\omega = \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

For a rigid body rotation about a fixed axis the angular velocity is the same for all parts of the body.
The velocity of a particle having the distance r is from the axis is $v = \omega r$.

The torque H is defined as "lever times force". $H = rF_{\perp}$, where $F_{\perp}$ is the force acting perpendicular on r.

Writing the torque using, the cross product of two vectors, it is quite easy to show.

(1.2) $$\vec{H} = \vec{r} \times \vec{F}$$

From which follows, that if $\vec{r}$ is parallel to $\vec{F}$ then $\vec{H} = \vec{r} \times \vec{F} = 0$
In general the cross product $\vec{c}$ of two vectors $\vec{a}$ and $\vec{b}$ is written: $\vec{c} = \vec{a} \times \vec{b}$. $\vec{c}$ is a vector which is perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}, \vec{c}$ mentioned in this order compose a right hand screw: The length of the cross product $\vec{a} \times \vec{b}$, is written as: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\varphi$, where φ is the numerical least angle which moves $\vec{a}$ into $\vec{b}$.
Obviously there are two vectors of a given length, which are perpendicular on two non parallel vectors $\vec{a}$ and $\vec{b}$. The choice is determined by the so called "right hand screw" rule, which prescribe a twist performed with your right hand that leads $\vec{a}$ into $\vec{b}$, in the positive circular direction, where the direction of $\vec{c} = \vec{a} \times \vec{b}$ is the "direction" of your thumb.
The same movement as when you tighten a screw in the wall with a screwdriver.
An ordinary Cartesian 3 right angled coordinate system (*x, y, z*) forms a right hand screw.

(1.3) If $\vec{F} \perp \vec{r}$ then $H = r \cdot F$ else $H = rF\sin\varphi$

where φ is the numerical least angle between $\vec{r}$ and $\vec{F}$.

The direction of $\vec{H}$ corresponds to the axis in which the body is turned in the positive circular direction.
The angular momentum for a particle with respect to a point or an axis is quite generally defined as:

(1.4) $$\vec{L} = \vec{r} \times m\vec{v}$$

If $\vec{r}$ and $\vec{v}$ are parallel to each other, then the angular momentum is zero. For the magnitude of *L* we have.

(1.4) $$L = mrv\sin\varphi = mr^2\omega\sin\varphi$$

Differentiating $\vec{L} = \vec{r} \times m\vec{v}$ with respect to time gives:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt} = \vec{r} \times m\frac{d\vec{v}}{dt} = \vec{r} \times \vec{F} = \vec{H}$$

(1.5) $$\vec{H} = \frac{d\vec{L}}{dt}$$

To determine the torque and the angular momentum for an extended body it is necessary to divide the body into parts with masses m_i and add the contributions from the parts.
If we assume that the rotation is around an axis all parts of the body will have the same angular velocity ω.
The particle with mass m_i in the distance r_i from the axis of rotation has the velocity: $v_i = r_i\omega$.

The contribution to the angular momentum from that particle is therefore:
$L_i` = m_i r_i v_i = m_i r_i \omega r_i = m_i r_i^2 \omega$ The over all angular momentum is the found by adding the parts.

(1.6) $$L = \sum_i L_i = \sum_i m_i r_i^2 \omega = \omega \sum_i m_i r_i^2 = I\omega$$

(1.7) $$I = \sum_i m_i r_i^2$$

I is called the moment of inertia with respect to the axis of rotation..

For a solid body the moment of inertia is most often found by integration:

(1.8) $$I = \int r^2 dm$$

For example is the moment of inertia og a circular disc with respect to an axis perpendicular to the disc and through its centre.

(1.9) $$I_{disc} = \int_0^r r^2 dm = \int_0^r r^2 \rho dV = \frac{m}{\pi r^2} \int_0^r r^2 2\pi r dr = \frac{m}{r^2} \left[\tfrac{2}{4} r^4 \right]_0^r = \tfrac{1}{2} m r^2$$

At the same time, we have $H = \sum_i H_i$, so for the rotation of a body around an axis we get:

(1.10) $$H = \frac{dL}{dt} \qquad H = I\frac{d\omega}{dt} \qquad H = I\frac{d^2\theta}{dt^2}$$

Or when written with vectors for a system of particles or a rigid body:

(1.11) $$\vec{H} = \frac{d\vec{L}}{dt} \quad \text{or} \quad \vec{H} = \frac{\Delta\vec{L}}{\Delta t} \quad \Leftrightarrow \quad \vec{H} = \Delta\vec{L}\Delta t$$

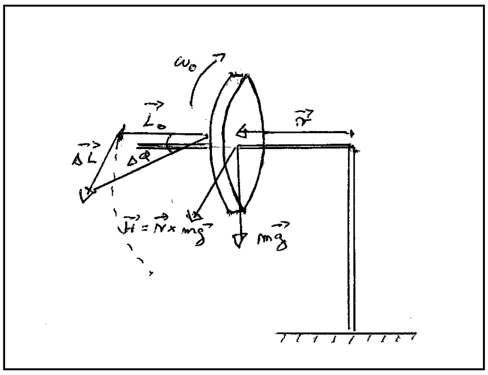
where

$$\vec{H} = \sum_i \vec{H}_i \quad \text{and} \quad \vec{L} = \sum_i \vec{L}_i$$

2. The Gyroscopes

To understand the following examples it is important to emphasize the vector character of the equation of motion. First of all they are vector equations, and must be treated as such.
The change of the angular momentum $\Delta\vec{L}$ has always the same direction as $\vec{H}$, that is, about the immediate axis of rotation. This is actually not very surprising, but if the body has already an angular momentum in another direction, then unexpected phenomena can occur.

On the figure below the gyro's disc has the angular velocity ω_0. The direction and the circular motion is shown in the figure, and the direction of the gyro's angular momentum L_0 is along the axis of rotation. The torque from gravity with respect to the point of support is $\vec{H} = \vec{r} \times m\vec{g}$, where m is the mass of the disc, neglecting the mass of the rod.



The magnitude of H is: $H = mgr$, since $\vec{r} \perp m\vec{g}$. Since the direction of $\vec{H}$ is perpendicular to $\vec{r}$ and $\vec{g}$ and therefore $\vec{H}$ is directed perpendicular to $\vec{L}_0$ in the horizontal plane. This means, however, that $\Delta\vec{L} = \vec{H}\Delta t$ has the same direction and the rod of the gyro will set for the direction $\vec{L}_0 + \Delta\vec{L}$, which is twisted an angle $\Delta\varphi$ from the direction of $\vec{L}_0$. From the figure is seen:

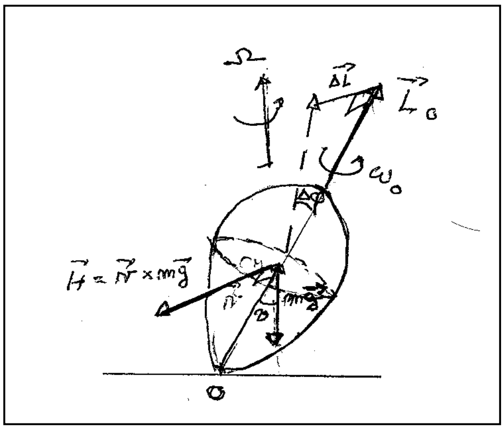
(2.1) $$\tan\Delta\varphi \approx \Delta\varphi = \frac{\Delta L}{L_0} = \frac{mgr\Delta t}{L_0} = \frac{mgr\Delta t}{I_0\omega_0}$$

From which we can find the angular velocity of the precession.

(2.1) $$\Omega = \frac{\Delta\varphi}{\Delta t} = \frac{mgr}{I_0\omega_0}$$

3. The Top

When it comes to precession, then the top is more common and looks less contradictory, when placed on a horizontal underlay. But the principles guiding the precession motion are the same. The torque with respect to the point of contact with the underlay, is calculated as before as $\vec{H} = \vec{r} \times m\vec{g}$, which gives $H = mgrsin\theta$, where θ is the angle between the axis of rotatiom and vertical. The mass of the top is m, while r is the distance from O, the supporting point of the underlay to G, the centre of mass of the top



The direction of $\Delta\vec{L} = \vec{H}\Delta t$, is the same as in the example with the gyroscope, and conversely to what you might think, the constant angular momentum $\vec{L}_0$ will get a horizontal twist $\Delta\varphi$ in the positive circular direction, where:

$$\tan\Delta\varphi \approx \Delta\varphi = \frac{\Delta L}{L_0} = \frac{mgr\sin\theta\Delta t}{L_0} = \frac{mgr\Delta t}{I_0\omega_0}$$

Resulting in the angular velocity of the precession.

(2.3) $$\Omega = \frac{\Delta\varphi}{\Delta t} = \frac{mgr\sin\theta}{I_0\omega_0}$$

4. Doing experiments with the gyroscope.

To the left is shown a gyroscope of the type that are traded in toy stores. We can only make a qualitative comparison, since I do not have the equipment necessary to measure the angular frequency of the rotating disc. The mass of the gyro is 128 *g*, and the mass of the disc is estimated to be 100 g. For this gyro most of the mass is placed at the border of the disc at a distance 2.6 *cm*. For the moment of inertia of the disc we therefore apply the formula for a circular ring $I = mr^2$ instead of the formula for the moment of inertia of a homogenous disc $I = \frac{1}{2}mr^2$ .The distance from the point of support to the centre of mass is 4.0 *cm*.

If we furthermore assume that the axis of the gyro has an angle of 30^0 with vertical, and if we tentatively put the circulation rate of the disc to 10 *Hz*, we may calculate the frequency in the precession from the formula (2.3).

(4.1)
$$\Omega = \frac{mgr\sin\theta}{I_0\omega_0} = \frac{0{,}128\cdot 9{,}82\cdot 0{,}040\cdot \sin 30}{0{,}10\cdot 0{,}026^2\cdot 2\pi\cdot 10}Hz = 5.9Hz$$

From which we get the period:

$$T = \frac{2\pi}{\Omega} = 1{,}1\ s$$

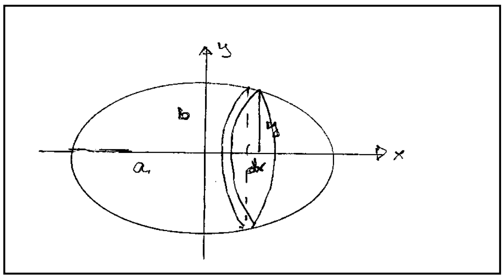
Making simple experiments with the toy gyro, one measures periods from 1.2 *s* to 2.1 *s* dependent of how much speed has been given to the rotating disc. So at least the experiments do not reject the theory.

5. The egg, which rises to stand upright

From the famous anecdote about Columbus, we know that it is impossible to let an egg stand upright without support, From my childhood I can remember that someone demonstrated, that you may take a hard boiled egg, and put it in rapid rotation on a table, it is (once in a while) possible that the egg will stand upright and keep rotating around its longitudinal axis for quite a while, and with a larger frequency than before.
This can not be explained by energetic considerations, since any system tend to a state of lower potential energy, whereas the egg, when it rises, increases its potential energy.
Also it is clear that the phenomena would not take place, if there was not a certain friction with the underlay.
It is actually a little easier to do the trick with a marble egg as shown below.
To the left is the egg, and to the right is the egg after it has been put into rapid rotation stands upright and rotates smoothly.

If the egg rotates about its minor axis, so that the axis intersects the point of contact, the friction with the underlay, cannot cause any torque that will rise the egg, but if the egg is dislocated a little bit from the axis of rotation, the frictional force might give a torque, that might have the direction to raise the egg. This we shall try to establish in the following.
As I have said, it is far from easy to make the egg stand upright. It requires a rapid rotation, and the rotating egg should stay on the spot.

From the conservation of energy it is possible to make a lower limit on the initial frequency of rotation to insure energetically that the egg rises an rotates on its high end. We assume that the egg is an ellipsoid with the semi axis *a* and *b*.



The volume of the ellipsoid can be found by by slicing it up in discs with radius *y* and thickness *dx*.
The equation of the ellipse in the coordinate system shown is

(5.1) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

For practical reasons we introduce $x'=\frac{x}{a}$ og $y'=\frac{y}{b}$ which makes equation easier to handle: $x'^2+y'^2=1$ The volume of the disc with thickness dx becomes $dV=\pi y^2dx$, and the volume is then found by integrating:

(5.2)
$$V=\int_{-a}^{a}dV=\int_{-a}^{a}\pi y^2dx=\pi b^2a\int_{-1}^{1}y'^2\,dx'$$
$$V=\pi b^2a\int_{-1}^{1}(1-x'^2)dx'=\left[x'-\tfrac{1}{3}x'^3\right]_{-1}^{1}=\tfrac{4}{3}\pi b^2a$$

In accordance with the volume of sphere $V_{sphere}=\frac{4}{3}\pi r^3$, since the ellipsoid becomes a sphere when $a=b=r$.

When determining the moment of inertia about one of the axes of the ellipsoid, we can apply the same drawing as above, since the moment of inertia *dI* of the disc with mass *dm* is:

(2.18) $$dI=\tfrac{1}{2}y^2dm\,,\ \text{where}\ dm=\rho dV=\rho\pi y^2dx$$

The density ρ of the ellipsoid is

$$\rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi ab^2}$$

(2.19)
$$I_a = \int_{-a}^{a} dI = \int_{-a}^{a} \tfrac{1}{2} y^2 dm = \int_{-a}^{a} \tfrac{1}{2} y^2 \rho\pi y^2 dx$$
$$I_a = \tfrac{1}{2}\rho\pi \int_{-a}^{a} y^4 dx = \tfrac{1}{2}\rho\pi b^4 a \int_{-1}^{1} y'^4\, dx'$$

$$I_a = \tfrac{1}{2}\rho\pi b^4 a \int_{-1}^{1} (1-x'^2)^2 dx' = \tfrac{1}{2}\rho\pi b^4 a \int_{-1}^{1} (1-2x'^2+x'^4)dx'$$
$$I_a = \tfrac{1}{2}\rho\pi b^4 a\left[x' - \tfrac{2}{3}x'^3 + \tfrac{1}{5}x'^5\right]_{-1}^{1} = \tfrac{1}{2}\rho\pi b^4 a(2 - \tfrac{4}{3} + \tfrac{2}{5}) = \tfrac{1}{2}\rho\pi b^4 a \tfrac{16}{15}$$
$$I_a = \tfrac{1}{2}\frac{m}{\frac{4}{3}\pi ab^2}\pi b^4 a \tfrac{16}{15}$$
$$I_a = \tfrac{2}{5} m b^2$$

You should notice that the moment of inertia with respect to the great axis, does not depend on the length of the great axis a. You should also notice, that we get the correct result for a sphere, $I_{sphere} = \frac{2}{5}mr^2$ if $b = a = r$. (radius of the sphere).

The moment of inertia around the minor axis b becomes correspondingly $I_b = \frac{2}{5}ma^2$.

We shall then apply energy conservation to find the connection between the angular frequencies if the egg rises from a rotation about the minor axis to a rotation about the major axis, and derive a condition on the frequencies that it is possible at all. This condition includes of course the potential energy mgh, which is larger, when the egg rotates on its end.

(2.20)
$$mgb + \tfrac{1}{2}I_b\omega_b^2 = mga + \tfrac{1}{2}I_a\omega_a^2$$
$$mgb + \tfrac{1}{2}\tfrac{2}{5}ma^2\omega_b^2 = mga + \tfrac{1}{2}\tfrac{2}{5}mb^2\omega_a^2$$
$$5gb + a^2\omega_b^2 = 5ga + b^2\omega_a^2$$
$$\omega_a^2 = \frac{5g(b-a)}{b^2} + \frac{a^2}{b^2}\omega_b^2$$

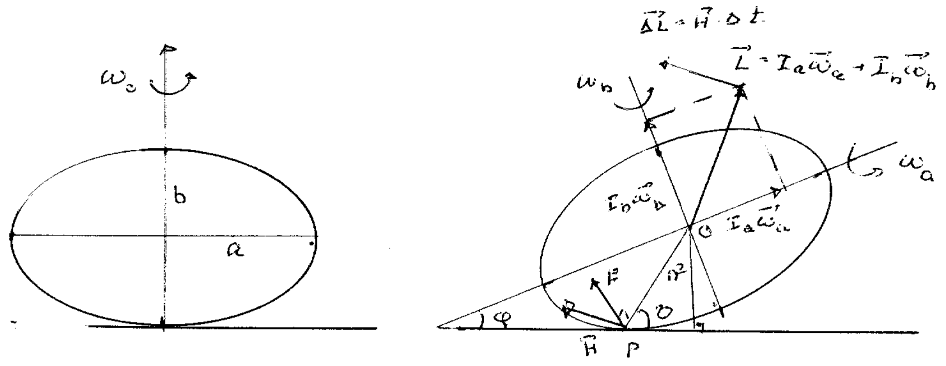
Since the potential energy grows linearly with distance, whereas the moment of inertia grows quadratic then despite the increase in potential energy $mg(b\text{-}a)$, the angular frequency must be larger, when the egg is rotating on its end. We therefore solve the inequality.

(2.21)
$$\omega_a^2 > \omega_b^2 \quad \Leftrightarrow \quad \frac{5g(b-a)}{b^2} + \frac{a^2}{b^2}\omega_b^2 > \omega_b^2$$
$$\omega_b^2\left(\frac{a^2}{b^2} - 1\right) > \frac{5g(a-b)}{b^2} \quad \Leftrightarrow \quad \omega_b^2(a^2 - b^2) > 5g(a-b)$$
$$\omega_b^2 > \frac{5g}{a+b} \quad \Rightarrow \quad \omega_b > \sqrt{\frac{5g}{a+b}}$$

For the marble egg in question: a =6.0 cm and b = 4.0 cm, and the energetic condition that the egg will rise and rotate o nits end is:

$$\omega_b = \sqrt{\frac{5\cdot 9.82}{0.10}} Hz = 22.2\ Hz \quad \text{corresponding to a frequency } \nu = \frac{\omega_b}{2\pi} = 3{,}5\ Hz$$

This is in accordance with the "empirical results" that the egg must have a substantial angular velocity, if it rises and rotates on the its end.
The energetic conditions, however, delivers no explanation why the egg stands upright, since it cannot do so without friction with the underlay, which may supply an external torque.
Without rotation the torque from gravity would certainly turn the egg back to its initial position, but as we have already seen from several examples, the torque from gravity will merely make the egg rotate (unevenly) about a vertical axis, so we do not need to be concerned about the torque from gravity. As is seen from the figure the torque around P from gravity $H_G = mgr\cos\theta$, θ -> 90 -> 74^0 -> 90^0 . So the torque from gravity is small in any case.



In the figure to the left is shown the initial position of the egg with a vertical rotation around the minor axis. To the right the egg has risen, while it now rotates both about the minor and the major axes. Because of the rotation there is a frictional force $\vec{F}$ against the underlay. In addition to that $\vec{F}$ will slow down the speed of rotation, it will produce a torque around the centre of gravity O.
According the elementary theory, the frictional force between two plane solid materials is always directed against the motion, and it has the value: $F = \mu F_N$ where F_N is the force acting perpendicular to the two plane surfaces, and μ is a constant that depends only on the nature of the two materials. The frictional force, however, depends neither on the speed nor on the size of the area in contact. Since the egg is supported only in P, then $F = \mu mg$.

On the figure $\vec{F}$ is perpendicular on $\vec{r} = \overrightarrow{OP}$ and directed opposite to $\vec{v}$ the speed of the egg at P, that is, a normal to the drawing. This gives rise to at torque around O: $\vec{H} = \vec{r}\times\vec{F}$. $\vec{H}$ lies in the plane of the paper, and it forms an angle $90^0 - \theta$ with the underlay, where θ is the angle between $\overrightarrow{OP}$ and the tangent i P. $H = \mu mgr$, in the position shown, but it will gradually disappear, when the egg approaches a rotation on one of its end points.
The egg rotates with an angular velocity $\vec{\omega}_a$ around the a-axis, and $\vec{\omega}_b$ around the b-axis.
For a rotation about an axis, we have $L = I\omega$, where L is the angular momentum, and I is the moment of inertia. For the instantaneous angular momentum of the egg, we therefore have:

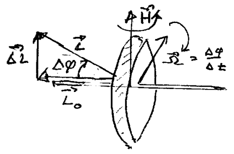
$\vec{L} = I_a\vec{\omega}_a + I_b\vec{\omega}_b$, as it is show in the figure above.

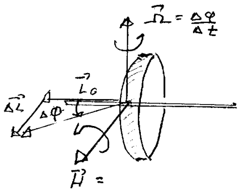
Within a small time Δt the torque $\vec{H}$ supplies a change in the angular momentum $\Delta\vec{L} = \vec{H}\Delta t$. From the figure to the right, it seems likely that this change in angular momentum will straighten up the angular momentum, so the egg will stand upright.
Within the present framework, it is not possible to com closer to an explanation of why a rotating egg stands upright.

6. The physics of rotating objects

If you don't have gravity as a direction to navigate with, then you may apply a gyroscope. If you have a massive disc rotating rapidly, which is not influenced by any torque, it will conserve its direction in space, as long as the rotation continues.

In my teaching in the 9 – 12 year high school (Gymnasium), we had in the 80'ties a heavy wheel mounted, so it could rotate about a rod through its centre.

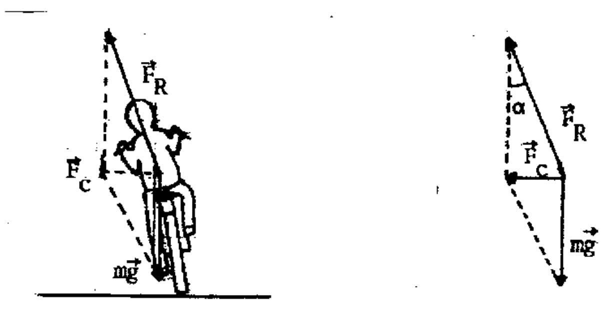




Keeping the axis horizontal and putting the wheel in rotation, then most people will find it "difficult" to move the axis to a vertical position. The reason is of course, that you naturally will attempt to twist it on horizontal axis. This is shown in the figure to the right. But this will cause a change in the angular momentum, with the same direction as $\vec{H}$. And therefore set the wheel in a rotation about a vertical axis.
What you should do is to rotate it gently about a vertical axis. Shown in the figure to the left
This will give a change in the angular momentum in the (initially vertical) direction, and make the wheel turn about a horizontal axis, eventually bringing the rod to a vertical position.

6.1 Cyclist turning a corner

The sometimes peculiar behaviour of rotating wheels, is actually well known when riding a bicycle. The figure below illustrates a cyclist turning.



The cyclist is influenced by gravity $\vec{F}_T$ and a reaction force $\vec{F}_R$ from the ground. If the turn has constant radius and is traversed with constant speed *v*, the cyclist is performing a uniform circular motion with speed *v* and radius *r*. The vector sum of $\vec{F}_T$ and $\vec{F}_R$ then deliver the centripetal force $\vec{F}_c$ necessary to uphold the circular motion.

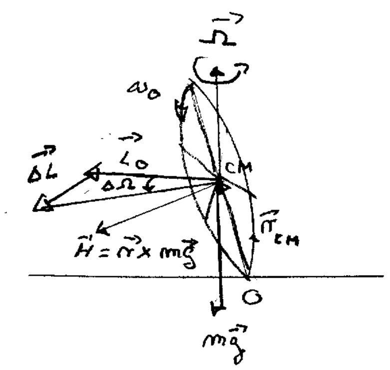
If α is the inclination angle between the cyclist and the vertical, then one can read from the figure above.

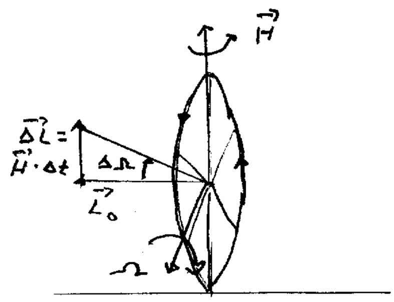
(2.22) $$\tan\alpha = \frac{F_c}{mg} = \frac{\frac{mv^2}{r}}{mg} = \frac{v^2}{rg}$$

As an example we may calculate the angle from the vertical in a circular curve with radius 10 *m,* where the speed is 20 *km/h* = 5.6 *m/s.*

$$\tan\alpha = \frac{(5.6\,m/s)^2}{10\,m\,9{,}82\,m/s^2} = 0{,}314 \quad \Rightarrow \quad \alpha = 17{,}4^0$$

We shall now analyze the same situation, a cyclist making a circular curve, but this time finding the angle of inclination from the theory of rotation.





When you drive a bike the rotating wheels have an angular momentum, where its direction together with the spinning direction forms a right hand screw.
It is the conservation of angular momentum, which causes that it is easy to keep balance on a bicycle having some speed. (It is almost impossible to keep balance on a bicycle when standing still).
If you, sitting on a bicycle, lean to one side and therefore enforce a horizontal torque $\vec{r}_{CM} \times \vec{F}_T$, the result is a horizontal change $\Delta\vec{L}$ in the angular momentum. (The figure to the left). This will result in a precession about a vertical axis, that is that the wheel will turn and the bicycle will follow a circular curve, without even the cyclist is trying to steer.
As we discussed it with spinning tops, we may write the relation for the angle of deflection as:

(2.23) $$\tan\Delta\varphi \approx \Delta\varphi = \frac{\Delta L}{L_{wheel}} = \frac{mgr\sin\theta\Delta t}{L_{wheel}} = \frac{mgr\Delta t}{I_{wheel}\omega_0}$$

And therefore the angular velocity in the precession becomes.

(2.24) $$\Omega = \frac{\Delta\varphi}{\Delta t} = \frac{mgr\sin\theta}{I_{hjul}\omega_0}$$

Assuming that the mass of the wheel is placed at the periphery of the wheel the moment of inertia becomes mr^2, and we find for the angular momentum.

(2.25) $$L_{wheel} = I_{wheel}\omega_0 = I_{wheel}\frac{v_0}{r} = mr^2\frac{v_0}{r} = mrv_0$$

If R is the radius in the circular motion that the bicycle performs, then the relation between the angular velocity Ω of the turning of the wheel, and the angular velocity Ω_{circle} in the circular motion of the bike given by:

(2.26) $$\Omega = \Omega_{circle}\cos\theta$$

From the uniform circular motion we have: $v = \omega r \quad \Rightarrow \quad v_0 = \Omega_{circle}R$ Using these two expressions in

$$\Omega = \frac{mgr\sin\theta}{I_{wheel}\omega_0}$$

We find:

$$\Omega_{circle}\cos\theta = \frac{mgr\sin\theta}{I_{wheel}\omega_0} \quad \Leftrightarrow \quad \frac{v_0}{R}\cos\theta = \frac{mgr\sin\theta}{mr^2\frac{v_0}{r}}$$

$$\tan\theta = \frac{v_0^{\ 2}}{Rg}$$

We thus find exactly the same relation, (and we ought to) as when we analyzed the situation using forces.

No one who has learned to ride a bike will try to turn the handlebars, when making a turn in moderate to high speed instead they lean into the direction of the bend as described above. And there is a good reason for that, since if you try to turn the front wheel by pulling the handlebars, you induce a vertical torque $\vec{H}$ that gives at vertical change in angular momentum $\Delta\vec{L} = \vec{H}\Delta t$. This will however generate a rotation around a vertical axis that will overthrow the bike to the opposite direction of the wanted bend. This is illustrated in the figures above to the right.

Explained mechanically the bike will be overthrown by the “centrifugal” force, which is not a force in itself, but reflects the lack of a centripetal force, that you get by leaning into the turn.

Most people know how to ride a bike, and they the lean themselves into the turn, when turning, but very few actually know why. This we have tried to make up for here.