Geometrical optics

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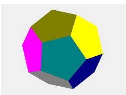


Ole Witt-Hansen 2011

Contents

Ch 1. Geometrical Optics

1. The path of light rays in materials

Geometrical optics deals with the reflection of light and diffraction of light rays in transition from one material to another. Geometrical optics is based on three fundamental laws of physics.

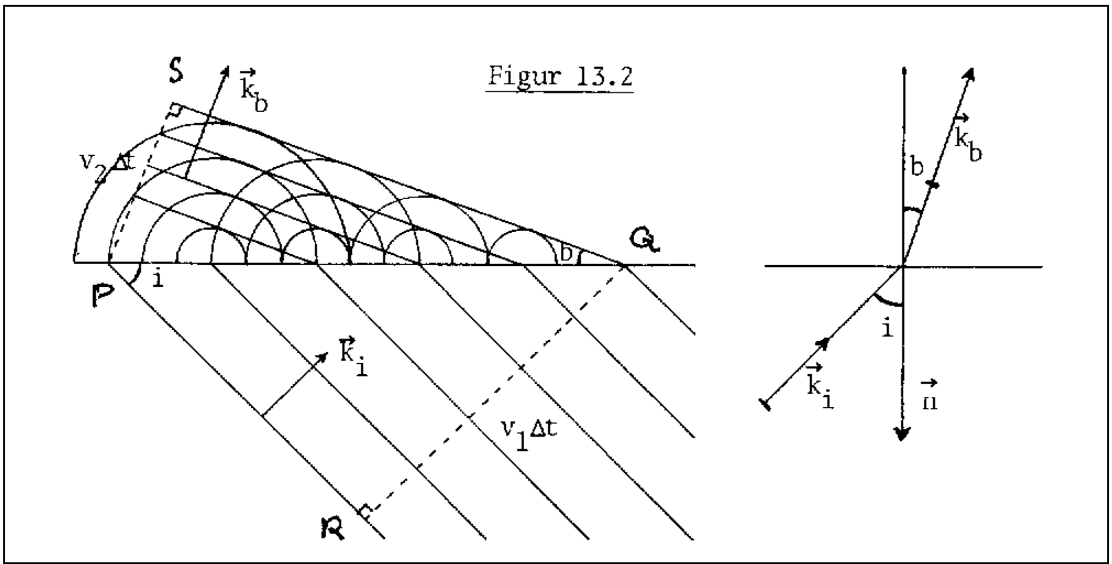
1. For a given material, the light always propagates along straight lines.
2. *The law of reflection*: By the reflection of a light from a plane surface, the incoming and the reflected light lie in the same plane together with the normal to the surface at the point of reflection. The entrance angle and reflection angle are defined as the angle between the light and the normal in the point of reflection.
3. The law of refraction: By refraction of a beam by the transition from one material to another, the incoming and the refracted ray lie in the same plane together with the normal to the surface at the point of refraction. The relation between the angle of entrance i and the angle of refraction b is given by *the law of refraction*:

(1.1) $$\frac{\sin i}{\sin b} = n$$

n is called the relative refraction index between the two materials.

1.1 Derivation of the law of refraction

Below is given a derivation of the law of refraction taken from: Elementary Physics 2."The propagation of waves" www.olewitthansen.dk.



The wave fronts for the refracted wave are, according to Huygens principle, constructed as the resulting wave front from the ring waves created at the borderline. The refracted wave is also a plane wave, but with a minor wavelength and having a change of direction, as the figure above indicates. The entry angle i is the angle between the incoming wave front and the border line.

The refraction angle b is the angle between the refracted wave front and the border line. Both angles are marked in the figure. From the triangles PQS and PQR we can see:

$$\sin i = \frac{v_1 \Delta t}{|PQ|} \qquad and \qquad \sin b = \frac{v_2 \Delta t}{|PQ|}$$

And by division of the two equations:

(1.3) $$\frac{\sin i}{\sin b} = \frac{v_1}{v_2} \quad (= n_{12})$$ (The law of refraction)

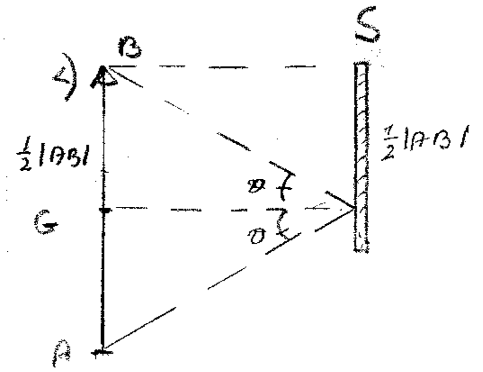
This famous relation is called *the law of refraction*. Although is best known for light rays, where it is the foundation of all geometrical optics, lenses, glasses, binoculars, microscopes etc. it is likewise applicable for sound waves, sea waves etc.
It expresses that independently of the entry angle, then *sine* to the entry angle divided with *sine* to the refraction angle is constant, and their ratio is called the *index of refraction*.

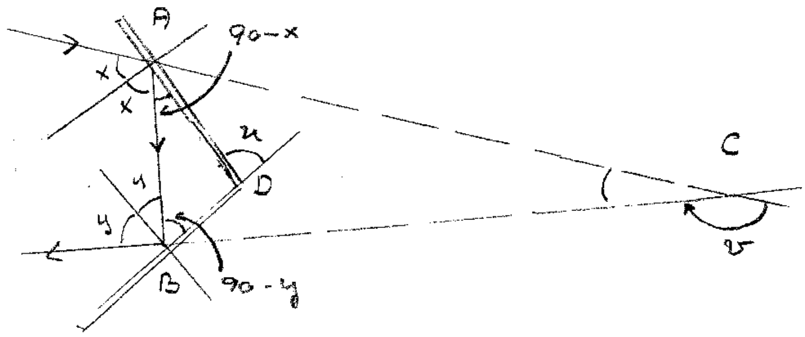
If the refractive index is greater than 1, there will always be a refracted wave, since the equation: $\frac{\sin i}{\sin b} = n \quad \Leftrightarrow \quad \sin b = \frac{\sin i}{n}$ will always have a solution for *b*. On the other hand if $n < 1$, as it is the case when light passes from glass to air, the condition for having a refracted ray is: $\frac{\sin i}{n} < 1 \quad \Leftrightarrow \quad \sin i < n$, if sin $i > n$, there is no refracted beam, but instead a phenomenon called total reflection from the surface. Depending on the sort of glass the refractive index is 1.3 – 1.6. If we choose n = 1.5 the limiting angle of the incoming beam will be: $\sin i = \frac{1}{1.5} \quad \Leftrightarrow \quad i = 50.28$.
An incoming beam with a larger angle of entrance will be totally reflected.
It should however be noticed, that there for minor angles always will be a refracted beam and a refracted beam.

2. Reflection from a plane mirror

In the figures above it is demonstrated how the point *Q* of an object *y* is reflected in a mirror.
Since the eye always will perceive the light as propagating along straight lines, then the image is perceived as situated at *Q*'. The image *Q*' lies where the extension of the two rays from *Q* meet.
The image is *virtual*, (imaginary) meaning that there exists no image behind the mirror.
It follows trivially from the geometry that the height of the image *y*' is the same as the height of the object *y*.
In the second figure you can see that the image is inverted, since the image of a left hand becomes a right hand. Reading a text in a mirror, (and it is certainly difficult) you must read from right to left.

In the figure below to the left is demonstrated the fact that to have a full image of an object the mirror must have at least half the size of the object, independently of the distance from the object to the mirror.
The eye situated in *B* is in line with the upper edge of the mirror.
A ray coming from the object at *A*, and is seen at *B*, is reflected from a point in the mirror, namely the normal in which is the bisector to *AB*. To be able to see the mirror image of *A* from *B,* the mirror must accordingly have at least the height ½|*AB*|.





In the figure to the right we have two mirrors, which have an inclination u with each other.
The mirrors are fixed at D and they can turn around an axis a D changing the angle u.
The line of the incoming and the reflected ray are prolonged meeting at the point C, so they form the triangle ABC. The angle that the incoming ray is deflected is v.
We shall apply the (trivial) geometrical fact that the supplement angle ($180^0 - v$) to an angle in a triangle is equal to the sum of the other two angles. When we look at the triangles ABC and ADB, we can see that:

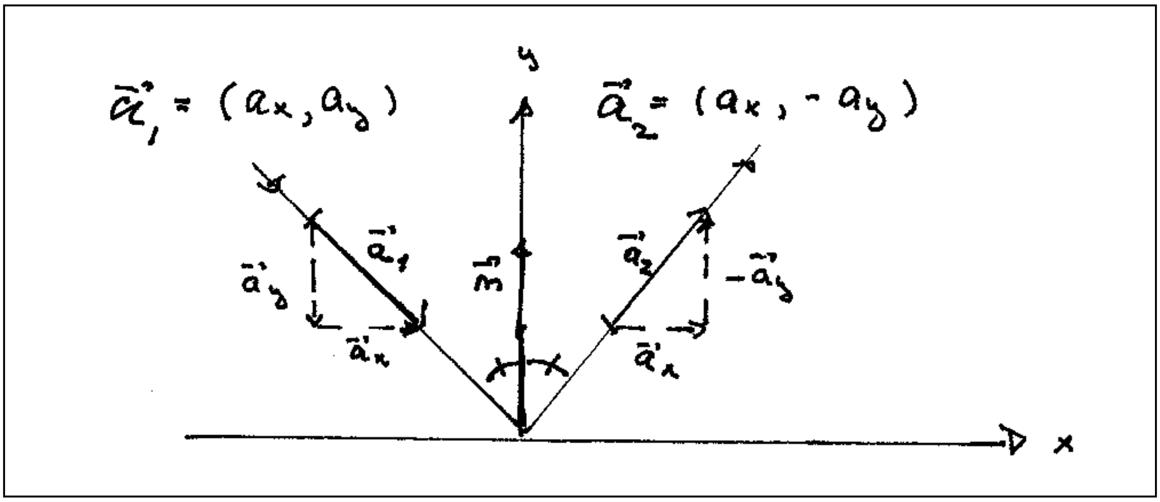
$$v = 180 - 2x + 180 - 2y = 360 - 2(x + y) \quad \text{and} \quad u = 90 - x + 90 - y = 180 - (x + y)$$

It then follows that $v = 2u$, and therefore $\angle ACB = 180 - 2u$. By the reflection in two mirrors to ray is turned the double of the angle between the two mirrors. This is for example used when navigating with a sextant.

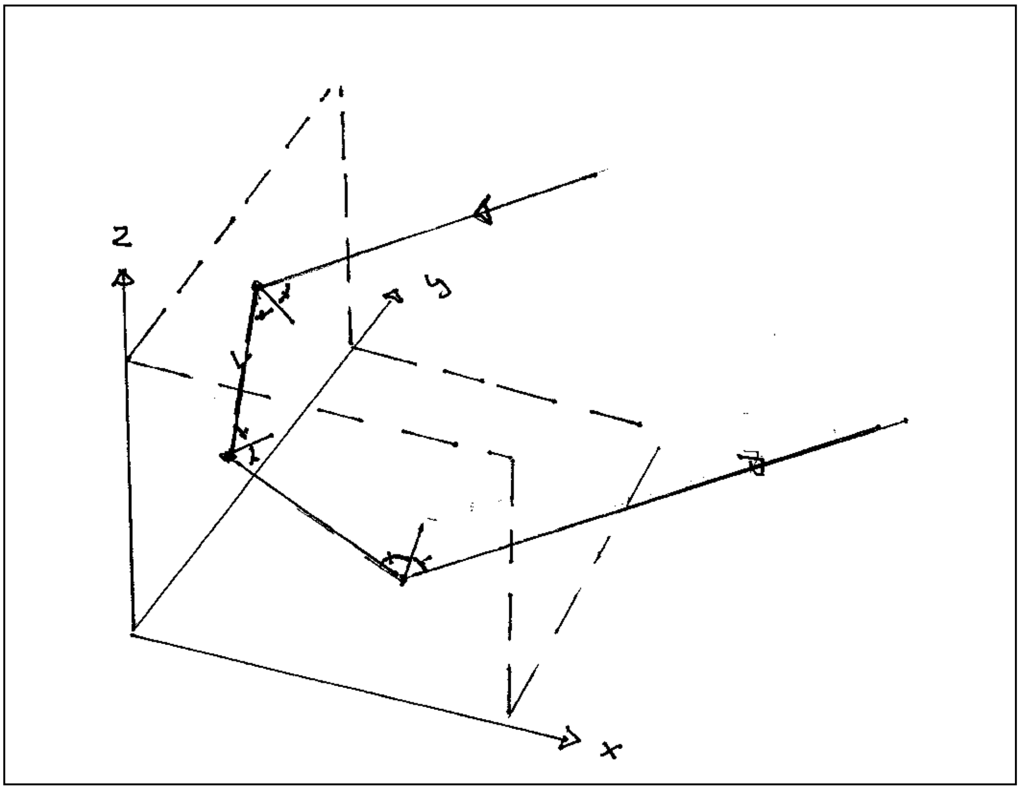
2.1 The reflective piece. The three right angled reflective corner

Especially if the two mirrors are placed perpendicular to each other, so that $u = 90^0$, then $v = 180^0$, which means that the ray is reflected in exactly the opposite direction of the incoming ray.
The same principle applies to a reflective piece. Also called a three right angled corner, that is, three reflecting surfaces placed at right angles to each other.

The reflective piece is characterized by that no matter what the angle of incoming ray is on the reflective piece, the ray will be reflected in the opposite direction. The reflective piece should be seen from all directions, where light hits its surface.
To prove that this is the case, is not very easy using analytical geometry in space, however using vectors it becomes almost evident.
In the figure below we have illustrated the law of reflection using vectors.
The incoming ray is along the vector $\vec{a}_1 = (a_x, a_y)$ and the outgoing ray is along the vector $\vec{a}_2$.
Since the incoming and the outgoing ray together with the normal to the point of reflection are in the same plane, it is obvious from the geometry that $\vec{a}_1 = (a_x, -a_y)$. What happens is that the coordinate perpendicular to the mirror changes sign, and that's all.



In the figure below is shown how this simple principle can be applied on a three dimensional reflective piece.

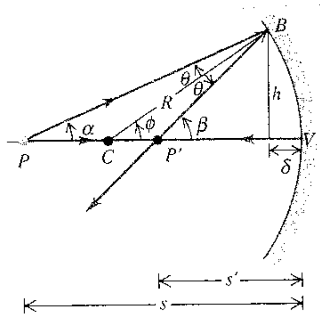


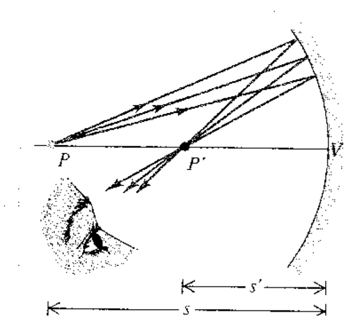
If the direction of the incoming ray is represented by the vector $\boldsymbol{s} = (s_1, s_2, s_3)$, then the ray first hits the $y - z$ plane. Hereby the x –coordinate changes sign (and nothing else), so the direction of the ray now is $\boldsymbol{s} = (-s_1, s_2, s_3)$. Next the ray hits the $x - z$ plane. The y coordinate changes sign and the direction of the ray is hereafter: $\boldsymbol{s} = (-s_1, -s_2, s_3)$. Finally the ray hits the $x - y$ plane, and the direction will be $\boldsymbol{s}_r = (-s_1, -s_2, -s_3) = -\boldsymbol{s}$ opposite to $\boldsymbol{s}$.

3. *Reflection in a concave mirror*

For all the concave and convex mirrors we shall look at in the following, we shall assume that the surface is part of a sphere, and also that the radii in the corresponding spheres are much lager than the diameter of the mirrors and lenses. The angles we shall consider are therefore small.
When we derive the formulas, we shall therefore (almost) consequently replace sin *v* and tan *v*, with the angle *v* measured in radians. For angles less than 10^0 it is a very good approximation.
In the figures, however the angles are not that small, taking into account the understanding of the understanding of the illustrations.

In the English literature there is a tradition for designating the distance from the mirror/lens with *s* and *s'*. I intend to follow this convention, even if has both advantages and disadvantages.





Above is shown a concave mirror. That means that the reflecting mirror is on the inside of the sphere that the concave mirror is a part of.

The centre of the sphere having radius *R* is at *C*. The object is placed at the point *P*.
The distance to the centre of the concave mirror is *s*.

A ray coming from *P* having the inclination *α* with the symmetry axis of the mirror is reflected so it hits the axis in *P'*. The distance from *P*' to the centre of the mirror is *s'*. The ray is reflected from the mirror in *B*, at the distance *h* from the axis. The normal to the mirror in *B* passes through the centre of the sphere.

The significance of *δ,* and the angles *θ, β* and *φ* should be apparent from the figure above.
If the angles are small then *δ* is vanishing compared to *s* and *s'*.
From the triangles *ΔPBC* and *ΔCBP'* it is seen (using the rule about the supplement angle to an angle in a triangle) that:

(2.1) $$\beta = \phi + \vartheta \quad and \quad \phi = \theta + \alpha \quad \Rightarrow \quad \beta = 2\varphi - \alpha$$

Furthermore it is seen from the three right angled triangles:

(3.2) $$\tan\alpha = \frac{h}{s-\delta}, \quad \tan\varphi = \frac{h}{R-\delta}, \quad \tan\beta = \frac{h}{s'-\delta}$$

If the angles are small and *δ* is vanishing small compared to *R, s* and *s'*, then we may discard *δ*, and by the same token replace tangent to the angles by the angles themselves. Thus we find:

$$\alpha = \frac{h}{s} \quad \varphi = \frac{h}{R} \quad og \quad \beta = \frac{h}{s'}.$$

Applying (2.1) $\beta = 2\varphi - \alpha \quad \Leftrightarrow \quad \beta + \alpha = 2\varphi$, we obtain the relation:

(3.3) $$\frac{h}{s'} + \frac{h}{s} = 2\frac{h}{R} \quad \Leftrightarrow \quad \frac{1}{s'} + \frac{1}{s} = \frac{2}{R}$$

Or, as it is most often written:

(3.4) $$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f} \quad where \quad f = \frac{R}{2}$$ is called the focal length.

(3.4) is called the *image equation*. It is an equation which ties together the distance to the object *s*, and the distance to the image *s'* with the focal length *f*.

It shows up, remarkably, that this equation with minor changes appears throughout the entire area of geometric optics.

Some points are worth noticing:

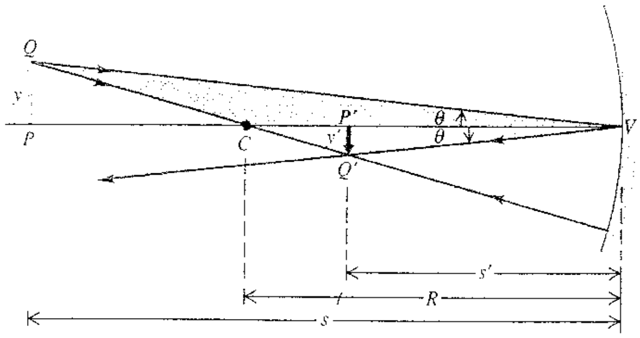
- The equation is independent of the angle α between the incoming ray and the symmetry axis. This means, however, that all rays coming from *P* (and are reflected in the concave mirror) will meet in the image point *P'*. This we have tried to illustrate in the figure above to the right.
- If $s \to \infty$ *then* $s' = \frac{R}{2} = f$. An incoming beam of light parallel to the symmetry axis will be collected in a point *F* at the distance *f* = ½*R* from the mirror. *F* is called the focal point, and *f* is, as mentioned earlier, called the focal length.
- The image is real, as long as *P* is beyond the focal length. Real means that it can be fetched on a screen of paper or a photographic plate.

If *s* is less than the focal length, then *s'* becomes negative according to the image equation (3.2). In this context negative distances does not make any sense, but physically it manifests itself as a virtual image at the distance *s'*. This is another general aspect of the image equation.

If s' becomes negative a physical image is replaced by a virtual image.

The relation between the height of the image and the height of the object appears from the illustration below, using the following guidelines.
Determining the image of a point on the object, you should bare in mind that according to the analyzing above all the rays coming from the same point on the object will gather in the same point after the reflection. It is therefore only necessary to determine the intersection of two out of four possible rays.

- A ray parallel to the axis will after reflection pass through the focal point.
- A ray which passes through the focal point will after the refection be parallel to the axis
- A ray passing through the centre of the sphere will be reflected through the centre, (because the radius is perpendicular to the surface of the sphere).
- A ray which hits the centre of the mirror, will be reflected symmetrically with respect the axis.



In the figure above: *ΔQVP ~ ΔQ'P'V'* (The triangles are even angled), so the ratio between the height of image and object is.

$$\frac{y'}{y} = \frac{s'}{s}$$

From the image equation: $\frac{1}{s'} + \frac{1}{s} = \frac{1}{f}$ it then follows:

$$sf + s'f = ss' \quad \Rightarrow \quad s'(s - f) = sf \quad \Rightarrow \quad \frac{s'}{s} = \frac{f}{s - f}$$

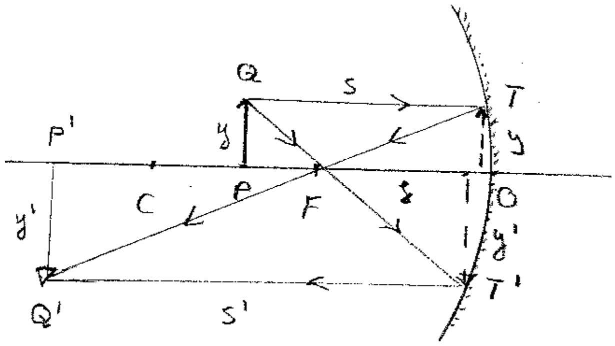
From the last equation, we may the find the ratio of the height of the image y' and the height of the object y, which is called the magnification.

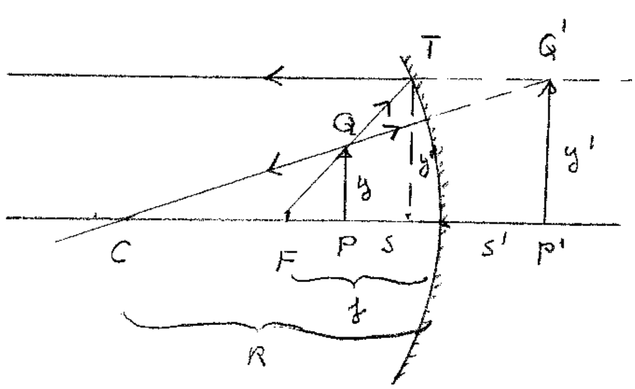
(3.5) $$\frac{y'}{y} = \frac{f}{s - f}$$

The image is physical, but upside down, as it also appears from the figure.
Mathematically this is reflected by that y' has the opposite sign of y.

When y is farther out than $R = 2f$, that is, when $s > 2f$, then the image is less than the object.
When $f < s < 2f$ it is relatively easy to show that (3.5) still applies. The image is however now bigger than the object. If the object is placed at (or near to) the focal point, then the image becomes (infinitely) large. This is well known, when a concave mirror is used to study "irregularities" in the skin of the face, or when visiting a mirror hall in a Tivoli.

Below is illustrated the two situations, where the object is beyond the focal length and where it is not. When the object is within the focal length, then (3.5) is still operative, but the image equation takes a slightly different form.





In the figure to the left it follows from the two even-angled triangles:

$$\Delta QPF \sim \Delta T'OF \text{ and } \Delta Q'P'F \sim \Delta TOF$$

$$\frac{y'}{y} = \frac{s'-f}{f} \quad \wedge \quad \frac{y'}{y} = \frac{f}{s-f} \quad \Rightarrow \quad f^2 = (s'-f)(s-f) \quad \Leftrightarrow$$

$$f^2 = s's - s'f - fs + f^2 \quad \Leftrightarrow \quad s's = s'f + fs$$

By division by $ss'f$, we find again the image-equation.

(3.4) $$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f}$$

And the magnification: $\dfrac{y'}{y} = \dfrac{f}{s-f}$ can be read from the first and the second equation.

In the figure to the right, the object is placed within the focal length. And from the figure it turns out that the image is situated behind the mirror. From the even angled triangles:

$$\Delta CQP \sim \Delta CQ'P' \quad \text{and} \quad \Delta FQP \sim \Delta FTO$$

We find:

$$\frac{y'}{y} = \frac{s'+2f}{2f-s} \quad \wedge \quad \frac{y'}{y} = \frac{f}{s-f} \quad \Rightarrow$$

$$f(2f-s) = (f-s)(s'+2f) \quad \Leftrightarrow$$

$$2f^2 - fs = fs' + 2f^2 - s's - 2fs \quad \Leftrightarrow$$

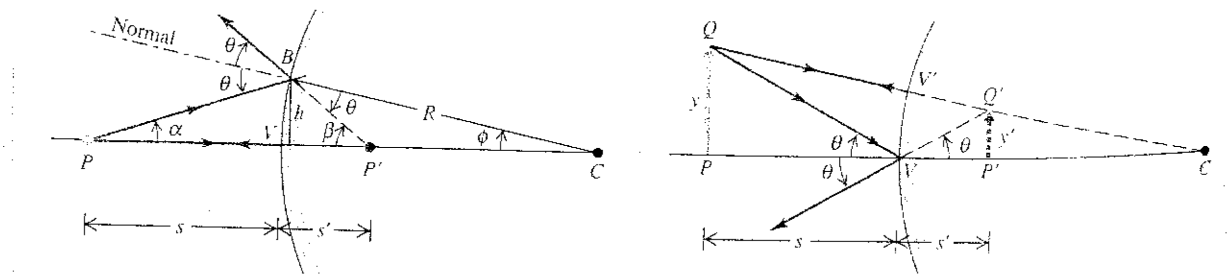
$$s's = s'f - fs$$

By division with $ss'f$, we find again the image-equation, but now with the difference that *s'* is negative. Physically this is interpreted so that the image is virtual.

(3.5) $$\frac{1}{s} - \frac{1}{s'} = \frac{1}{f}$$

The expression for the magnification is, however, the same as before. When the object is within the focal length, a physical image is not created, but rather a virtual image, which is lager than the object.

4. Reflection in a convex mirror



In the figure above is illustrated a convex mirror, where the reflecting surface is on the outside. Also for a convex mirror a variant of the image equation is applicable, since from the figures to the left it is seen that for small angles apply:

$$\alpha \approx \tan\alpha = \frac{h}{s} \quad \beta \approx \tan\beta = \frac{h}{s'} \quad \varphi \approx \tan\varphi = \frac{h}{R} \quad and \quad \beta = \varphi + \theta \quad 2\theta = \alpha + \beta$$

From the last two equations follow:

$$\beta = \alpha + 2\varphi \quad \Rightarrow \quad \frac{h}{s'} = \frac{h}{s} + 2\frac{h}{R} \quad \Rightarrow$$

(4.1)
$$\frac{1}{\frac{R}{2}} = \frac{1}{s'} - \frac{1}{s} \quad \Rightarrow$$

$$\frac{1}{f} = \frac{1}{s'} - \frac{1}{s}$$

f and s' have the same sign, since they both are situated in the same side of the mirror, but the opposite sign of s. When s and s' have opposite sign, then the image is *virtual.*
From the figure to the right can be seen that the magnification is:

(4.2)
$$\frac{y'}{y} = \frac{R - s'}{R + s} = \frac{2f - s'}{2f + s}.$$

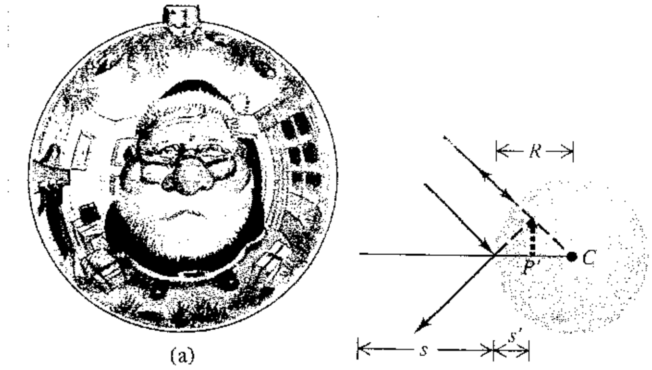
Isolating s' from the image equation, we have: $s' = \frac{fs}{s + f}$.

When inserted in (4.2) and some reduction we then have:

(4.3)
$$\frac{y'}{y} = \frac{f}{s + f}$$

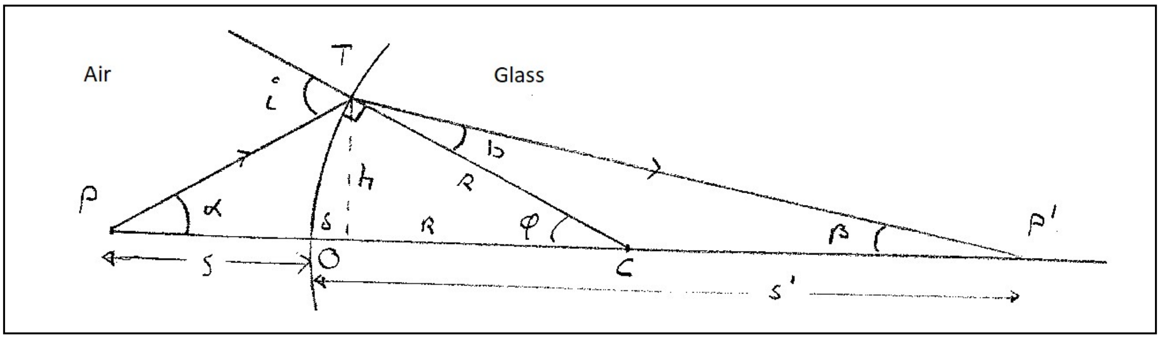
This happens to be the same expression that we obtained for the concave mirror, apart from the sign of f.

When mirroring in a convex mirror, the image is always virtual and always less than the object. This is probably well known and illustrated below by a Christmas bulb.



(a)

5. Refraction in the surface of a sphere



In the figure is shown the path of a light ray emerging from *P*, that is, refracted in the surface of a transparent sphere and is mapped in *P'*. The distance *s* is as before the distance from the object at *P*, *s* = |*OP*| and *s'* is the distance from the image at *P'* , *s'* = |*OP'*|. *C* is the centre of the sphere with radius *R*. The refractive index from air to glass (or any other transparent material) is denoted *n*. The law of refraction states that:

(5.1) $$\frac{\sin i}{\sin b} = n \quad \text{and for small angles we write this as: } i = n \cdot b$$

Again we apply the rule of the supplement angle in a triangle is equal to the two other angles. Using the designations in the figure.

$$\varphi = b + \beta \;\wedge\; i = \alpha + \varphi \quad \Rightarrow \quad i = \alpha + b + \beta$$

(5.2) $$\alpha \approx \tan\alpha = \frac{h}{s} \;, \quad \beta \approx \tan\beta = \frac{h}{s'} \;, \quad \varphi \approx \tan\varphi = \frac{h}{R}$$

$$i = \alpha + b + \beta \;\wedge\; i = n \cdot b \quad \Rightarrow \quad (n-1)b = \alpha + \beta \quad \Rightarrow$$
$$(n-1)(\varphi - \beta) = \alpha + \beta \quad \Rightarrow$$
$$(n-1)\varphi = \alpha + n\beta$$

Inserting the approximate expressions for α, β *and* φ from (5.2), we find:

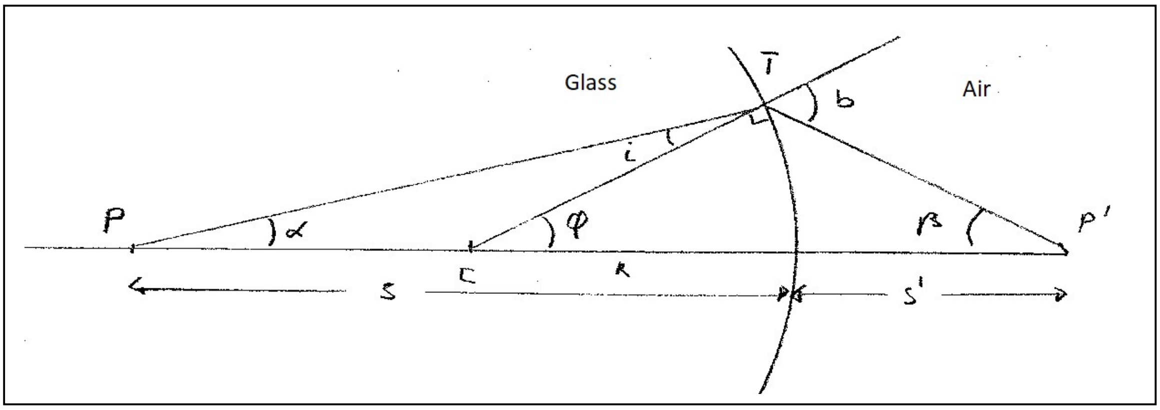
(5.3) $$\frac{n-1}{R}=\frac{1}{s}+\frac{n}{s'}$$

Notice that the equation is independent of α, and the other angles. This implies that every ray coming from P, will be mapped in P'. For an incoming beam parallel to the axis, which correspond to $s\to\infty$, we find the distance to the point of the image from the equation: $\frac{n-1}{R}=\frac{n}{s'}$, and from this the focal width:

$$\frac{1}{f}=\frac{1}{s'}=\frac{n-1}{nR}\Rightarrow f=\frac{nR}{n-1}$$

This formula is, however, rarely used (for obvious reasons)

A lens is in general composed of two spherical surfaces, a convex one and a concave one, but it may also be two convex lenses or two concave lenses. For this reason we shall also analyze the path of a ray going in the opposite direction.



The index of refraction is now $1/n$, and the law of refraction becomes then: $i=\frac{1}{n}b \Leftrightarrow b=n\cdot i$.

Using the designation from the figure above, and the rule about the supplement angle, we have:

$$\varphi=\alpha+i \;\wedge\; b=\varphi+\beta \quad\Rightarrow\quad b=\alpha+i+\beta$$

$$\alpha\approx\tan\alpha=\frac{h}{s}\;,\quad \beta\approx\tan\beta=\frac{h}{s'}\;,\quad \varphi\approx\tan\varphi=\frac{h}{R}$$

$$b=\varphi+\beta \quad\wedge\quad b=n\cdot i \quad\wedge\quad i=\varphi-\alpha \quad\Rightarrow\quad (n-1)\varphi=n\alpha+\beta$$

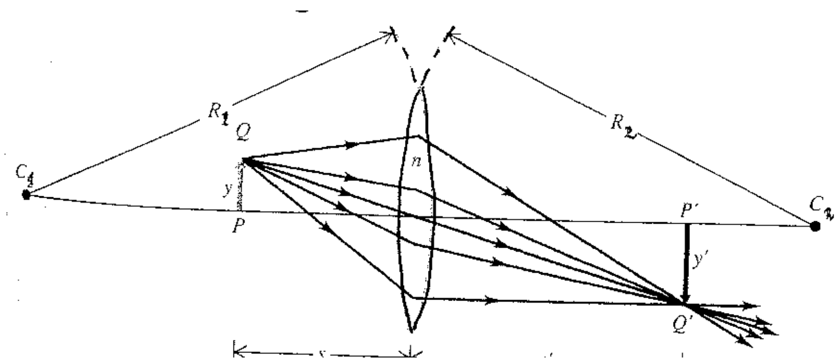
Inserting the approximate expressions for α,β *and* φ, given above, we find:

(5.4) $$\frac{n-1}{R}=\frac{n}{s}+\frac{1}{s'}$$

6. Then condenser lens. The Lensmakers formula.

In the figure below is shown a condensor, where the object *y* is mapped into the image *y'*. Dealing with a condenser lens, we have a combination of the two cases mentioned above, a concave and a convex spherical surface.

Designating the incoming ray with index (1) and the outgoing ray with index (2), we shall again write the equations for the two paths of the rays.



$$\frac{n-1}{R_1} = \frac{n}{s_1'} + \frac{1}{s_1} \quad \text{and} \quad \frac{n-1}{R_2} = \frac{n}{s_2} + \frac{1}{s_2'} \quad \Rightarrow$$

$$\frac{n-1}{R_1} - \frac{1}{s_1} = \frac{n}{s_1'} \quad \text{and} \quad \frac{n-1}{R_2} - \frac{1}{s_2'} = \frac{n}{s_2}$$

Since it is the same ray, we must, however, have $s_2 = -s_1'$. The minus sign because s_2 and s_1' lie on opposite sides of the lens. By addition of the two equations, we therefore find:

$$\frac{n-1}{R_1} - \frac{1}{s_1} + \frac{n-1}{R_2} - \frac{1}{s_2'} = 0 \quad \Leftrightarrow$$

$$\frac{n-1}{R_1} + \frac{n-1}{R_2} = \frac{1}{s_1} + \frac{1}{s_2'} \quad \Leftrightarrow$$

$$(n-1)(\frac{1}{R_1} + \frac{1}{R_2}) = \frac{1}{s_1} + \frac{1}{s_2'}$$

If we drop the indexes (1) and (2) on the *s*'es (since it is the same ray)we have the formula:

(6.1) $$(n-1)(\frac{1}{R_1} + \frac{1}{R_2}) = \frac{1}{s} + \frac{1}{s'}$$

For an incoming beam, which corresponds to letting $s \to \infty$ we will find the distance to the image point *s'*, equal to the focal length *f*.

(6.2) $$\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$$

The equation (6.2) is often called *the lens makers formula* because the knowledge of the radii in the two spheres and the refractive index, allows you to calculate the focal length of the lens.

From (6.1) and (6.2) follows immediately the image equation for lenses. As before s is the distance from the lens to the object and s' is the distance from the image to the lens. f is the focal length.

The image is real, but upside down. Applying (6.2) in (6.1), we again find the image equation.

(6.3) $$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$$

From the figure above with the condenser lens, we immediately see that the magnification, that is, the ratio between the height of the image and the height of the object is: $\frac{y'}{y}=\frac{s'}{s}$.

Using the image equation $\frac{1}{s'}+\frac{1}{s}=\frac{1}{f}$, it then follows:

$$sf+s'f=ss'\Rightarrow s'(s-f)=sf\Rightarrow\frac{s'}{s}=\frac{f}{s-f}$$

So that the magnification is:

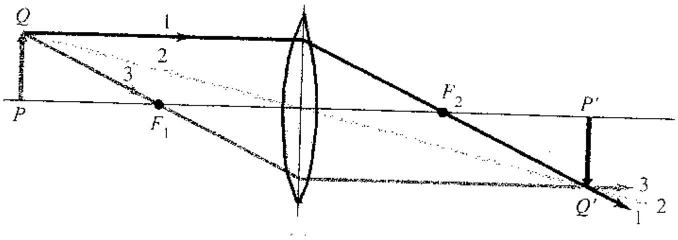
(6.4) $$\frac{y'}{y}=\frac{f}{s-f}$$

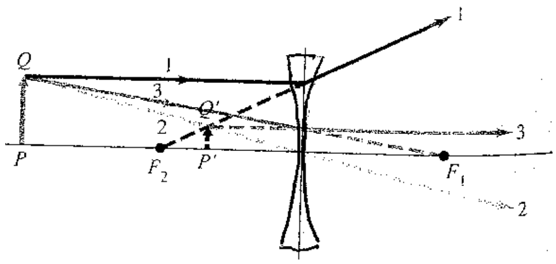
From (6.4) it is also seen that, if the object is beyond two focal lengths: $s > 2f$, then the size of the image is less than the size of the object. If $f < s < 2f$, then the image is larger than the object.

If the object is within the focal lengths: $0 < s < f$, then the fraction describing the magnification becomes negative. This means that the image is no longer real, but virtual. Applied in this manner the condenser lens becomes a magnifying glass.

Besides a “magnifying glass” a condenser lens was earlier called a ”burning glass” . Namely, if the lens is held toward the sun, and if you place some flammable material at the focal length, then all of the sunlight coming into the lens will pass the lens and gather in the focal point, creating enough heat to ignite paper or wood.

7. Diverging lenses





In the figures above is shown examples of the light path through a condensor lens and through a diverging lens.

One may derive a formula for the diverging lens in the same manner as for the condensor lens, but it is easier to notice that the index of refraction $n_{air,glass}=1/n_{glass,air}$, so by inverting the light path in the two spherical surfaces in a condenser lens and replacing the index of refraction n by $1/n$, the you should get the *lens makers formula* for a diverging lens. Not surprisingly it turns out to be the same formula, apart from the sign of focal length, and where the refractive index n is replaced with $1/n$.

(7.1) $$\frac{1}{f} = (n_{glass,air} - 1)(\frac{1}{R_1} + \frac{1}{R_2})$$

You may also derive the image equation for diverging lenses, by applying the two single angled triangles ΔQPF_1 and $\Delta Q'P'F_2$ in the figure above to the right. As usual we put the height of the object to $y = |QP|$ and the height of the image to $y' = |Q'P'|$. If s is the distance to the object, s' is the distance to the image and f is the focal width, we have:

(7.2) $$\frac{y'}{y} = \frac{s'}{s} \quad \wedge \quad \frac{y'}{y} = \frac{f - s'}{f} \quad \Rightarrow$$
$$\frac{s'}{s} = \frac{f - s'}{f} \quad \Leftrightarrow \quad s'f = sf - ss'$$

Dividing by $ss'f$, we get the image equation.

(7.3) $$\frac{1}{s} - \frac{1}{s'} = -\frac{1}{f}$$

The minus sign on the focal length, is caused by the fact that the focus point is on the same side as the object, so does the image, so the image is virtual.
The magnification can be found by solving the last of the equations (7.2) with respect to s' and insert the result in the first of the equations (7.2).

(7.4) $$s' = \frac{sf}{f + s} \quad \wedge \quad \frac{y'}{y} = \frac{s'}{s} \quad \Rightarrow \quad \frac{y'}{y} = \frac{f}{f + s}$$

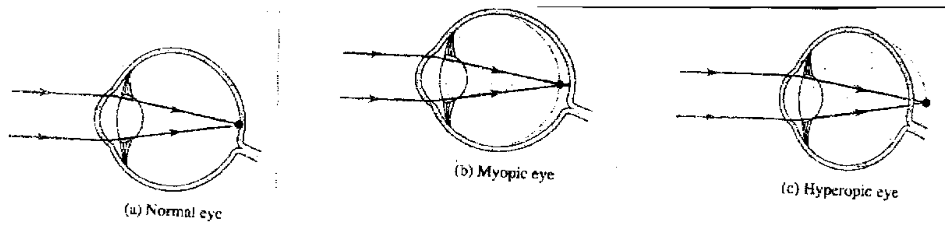
For a diverging lens the image is always less than the object.

Ch. 2. Optical instruments

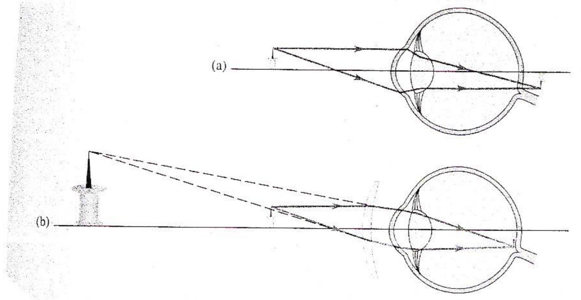
8. The principle in using glasses.

If you are farsighted you should wear condensor lenses, while nearsighted persons should wear diverging lenses. In the figures below is illustrated a normal eye, where the distance from the lens in the eye to the retinal is equal to the focal length of the lens.
Also is shown a farsighted eye where the distance from the lens in the eye to the retinal is (a little) larger than to the focal length of the lens, and a nearsighted eye, where the distance from the lens in the eye to the retinal is (a little) less to the focal length of the lens. In both cases, the image of the object, will be blurred in the retinal.

In the figures below is demonstrated how you may remedy these problems by using glasses and placing a condensor lens or diverging lens before the eye put into a pair of spectacles.
Above the age of 50, most people have to use reading glasses, because you get farsighted with age. The reason is that the lens is not made of solid material, but contains liquid.
The eye has the ability automatically to accommodate, that is, to make small magnification or diminishing of the focal length of the lens in the eye, while observing an object quite near or far away.
This ability is diminished with age, so that you no longer can press the lens together in an attempt to read "small letters".
Nearsighted people avoid in a certain degree this old age sight, but not entirely, since old age sight is a stiffness of the lens, while nearsightedness comes from a diminished distance from the lens to the retinal. The two disabilities can not compensate each other entirely.



Refractive errors for a myopic (nearsighted) and a hyperopic (farsighted) eye viewing a very distant object. The dashed blue curve indicates the correct position of the retina.



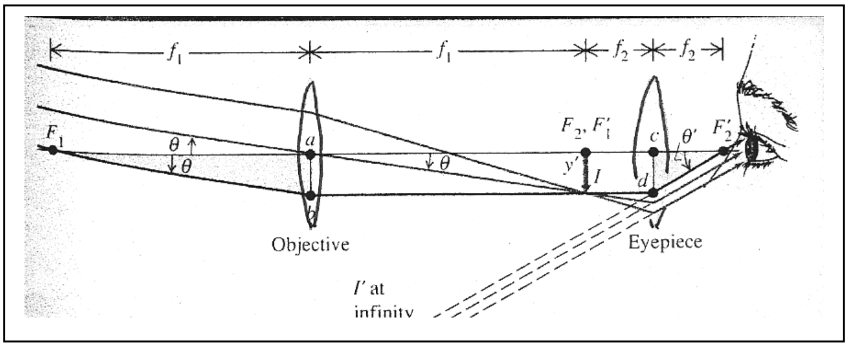
(a) An uncorrected hyperopic (farsighted) eye. (b) A positive lens gives the extra convergence needed for a hyperopic eye to focus the image on the retina. The virtual image formed by the converging lens acts as an object at or past the near point.

9. The astronomical binocular (The telescope)

The astronomical binocular is built up by two condensor lenses, placed in a tube so that the distance between the lenses is the sum of their focal length. See the figure below.
The lens facing the object is called the *lens*, and the other lens facing the eye is called the eyepiece. The image of a distant object will lie in the focus point of the lens, where it can be observed through the eyepiece, serving as a magnifying glass. Most commonly the lens has a much larger focal length than the eyepiece.
The image is however upside down, which makes the astronomical binocular unsuitable for any other purpose than astronomy(at sea, for hunting or for military purposes).
The magnification is defined as the ratio between the angles under which the object and the image are seen. As can be read from the figure:

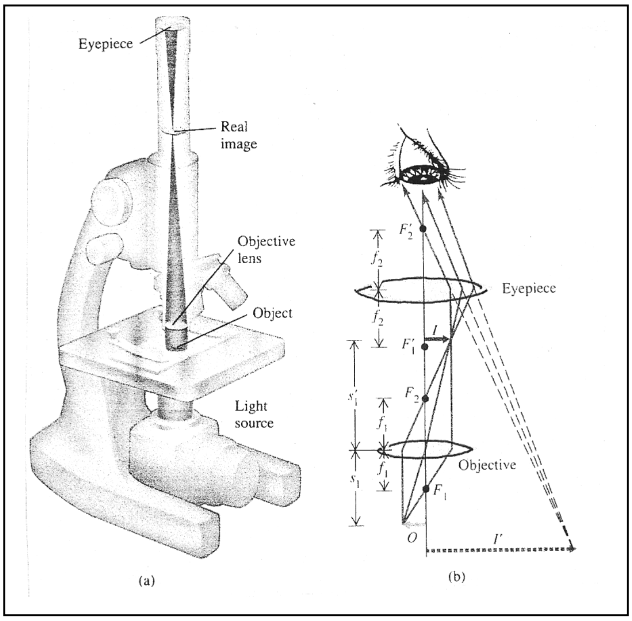
$$\tan\theta \approx \theta = \frac{y'}{f_1} \quad and \quad \tan\theta' \approx \theta' = \frac{y'}{f_2} \quad \Rightarrow \quad \frac{\theta'}{\theta} = \frac{f_1}{f_2} \tag{1.1}$$

So the magnification is simply the ratio between the focal lengths of the lenses and the eyepiece.



10. The microscope

The microscope has also two condenser lenses, but their roles are switched, such that the lens has the least focal length. The two lenses are placed such that their distance is larger than the sum of the focal widths f_1 and f_2. Consult the figure below.



In the microscope the object is placed near but outside the focal length of the lens.
The magnification from the lens is according to (6.4) $\frac{y'}{y} = \frac{O}{I} = \frac{f_1}{s_1 - f_1}$.
The reversed image is mapped inside but near the focal length of the eyepiece. The image is here observed through the eyepiece, functioning as a magnifying glass.
The formula for the magnification is the same as before: $\frac{y'}{y} = \frac{I'}{I} = \frac{f_2}{s_2 - f_2}$. From the two ratios we then have: (Consult the figure for *I'* and *O*)

(8.1) $$\frac{I'}{O} = \frac{f_2}{f_1}\frac{s_1 - f_1}{s_2 - f_2}$$

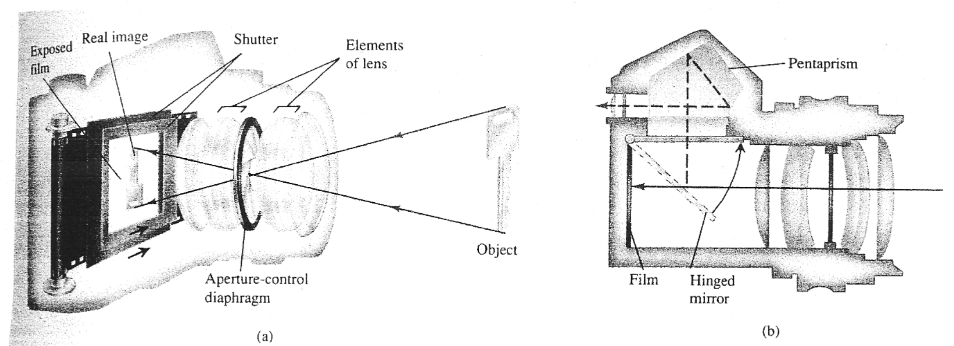
The ratio is negative since the image is virtual. If we choose $f_2 >> f_1$ or/and $s_2 \approx f_2$, we get a big magnification. There are however many other practical details to consider, when designing a microscope.

11. Lens error

The manufacturing of lenses, have during the last hundred years been an advanced and refined craft and technology.
Unfortunately it is not sufficient to make the lenses with the required accuracy, this is cause by a phenomenon called *aberration.* For lenses this is the so called chromatic aberration, which is caused by the fact that the index of refraction for visible light depends slightly on the wavelength, so that the focal length is slightly different for red, yellow and blue light.
For cheap binoculars it is easy to observe colour rings at the edge of the image.
A lens where the aberration is eliminated is called an *achromat*, (but the don't come cheap).
It turns out however, that different sorts of glass e.g. crown glass and flint glass have a different refraction and also a different *dispersion* (colour resolution).
One way to make an achromat, is to construct the condensing lens from two different sorts of glass, one condensing lens with a large refraction, but a small dispersion, and a diverging lens with small refraction but a larger dispersion.

12. The Camera

Without going into details, the figures below show a (lens) camera as it was manufactured in the 1980'ties.

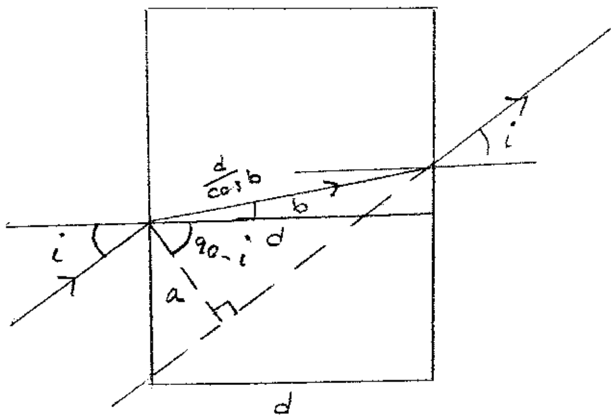


36–1 Camera with aperture control. (a) The lens forms a real image of the object in the plane of the film. (b) The Zeiss Tessar lens, a typical compound lens, and the pentaprism for viewing through the lens. The hinged mirror swings up out of the way as the shutter

Ch 3. Experimental optics

1. The light path through a block of glass

Materials: A rectangular block of glass, a soft underlay for the pins, paper and small pins.
The purpose of the experiment: is to illustrate the light path through a block of glass and determine the refractive index air/glass.



Trial description: The glass block is placed on the paper on the soft underlay, and the position of the block is marked with a pencil. To pins are placed on one side of the block, so their line of sight makes an angle less than 60^0 with edge of the glass block. These two pins are then observed from the other side of the block, such that they are aligned. On the same side is then placed two other pins, such that all four pins are visually aligned. The position of the block and the line of sight between the two sets of pins are marked with a pencil. The broken line is the path of light through the block.

The width of the block is *d*. The angle of the entrance of the ray is *i*. The angle of refraction is *b*, and *a* is the distance between the parallel rays on entrance and leaving the block. (See the figure).
From the transition from the material (1) to material (2) the law of refraction applies: $\frac{\sin i}{\sin b} = n_{12}$,

Where n_{12} is the index of refraction: We have $n_{21} = \frac{1}{n_{12}}$ so the path of the light is symmetric.

From the figure we see that:

$$a = \frac{d}{\cos b}(\cos(90 - i + b).$$

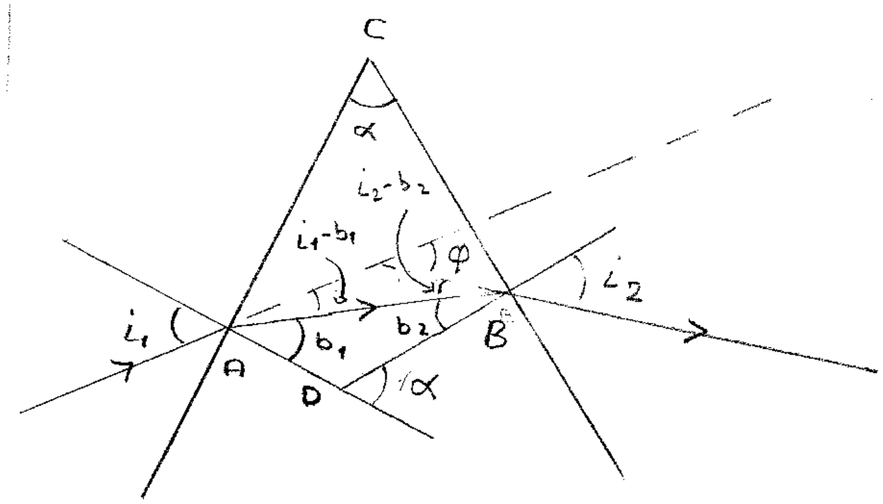
If we prolong the path of the ray through the block, the entrance angle and the refraction angle can be read from the lines drawn on the paper, using a protractor, and the index of refraction can then be calculated from the law of refraction.
Further the displacement *a* is measured with a ruler, and the agreement with the above can be verified. The experiment should be made more than once.

2. Prism in its main position.

Material: One or two glass (or massive plastic) prisms. (Crownglass/flintglass). A soft underlay, paper, and needle pins.

The purpose of the experiment is to illustrate the light path trough a prism and to determine the index of refraction air/glass.

Below is shown a prism with a non symmetric passage of a light ray.



We consider the square *ACBD*. We shall apply that the supplement angle (180^0 minus the angle) to the angle in triangle is (evidently) equal to the sum of the other two angles in the triangle.

If we apply this to ΔABD, and since A=B=90^0 it appears from the figure. (and from the geometry).

$$D = 180 - \alpha \;\; and \;\; 180 - D = b_1 + b_2 \Rightarrow \alpha = b_1 + b_2$$

The deviation φ of the beam is seen to be:

(2.1) $$\varphi = i_1 - b_1 + i_2 - b_2 = i_1 + i_2 - (b_1 + b_2) = i_1 + i_2 - \alpha$$

One may experimentally show, (and theoretically prove, as demonstrated below) that the deviation of the beam is least, when the passage of the beam through the prism is symmetric, that is, when:

$$i_1 = i_2 = i \quad \text{and} \quad b_1 = b_2 = b.$$

This is called the main position of the prism. In that case:

(3.2) $$\varphi = \varphi_{\min} = 2 \cdot i - \alpha, \quad \text{so} \quad i = \frac{\varphi_{\min} + \alpha}{2} \quad \text{and} \quad b = \frac{\alpha}{2}.$$

Applying the law of refraction to these angles, we get:

(3.3) $$n = \frac{\sin i}{\sin b} = \frac{\sin(\frac{\varphi_{\min} + \alpha}{2})}{\sin(\frac{\alpha}{2})}$$

2.1 A theoretical argument for the main position

According to the derivation above, we have in all cases

$$\varphi = i_1 + i_2 - (b_1 + b_2) \quad and \quad \alpha = (b_1 + b_2)\ ,$$

And according to the law of refraction:

$$\frac{\sin i_1}{\sin b_1} = n \quad and \quad \frac{\sin i_2}{\sin b_2} = n \quad \Rightarrow$$
$$\sin i_1 = n \sin b_1 \quad and \quad \sin i_2 = n \sin b_2 = n \sin(\alpha - b_1)$$

From the last equation, we may however consider i_2 as an implicit function of b_1, and from the first equation we may consider b_1 as an implicit function of i_1. Formally written:

$$i_2 = i_2(b_1) \quad og \quad b_1 = b_1(i_1) \quad => \quad i_2 = i_2(b_1(i_1))$$

The functional dependence of the various variables has been stated above.
Hereafter we may implicitly express φ as a function of i_1 and by differentiating implicitly we may find the minimum of φ.

$$\varphi = i_1 + i_2 - (b_1 + b_2) \quad and \quad \alpha = (b_1 + b_2) \quad \Rightarrow$$
$$\varphi = i_1 + i_2 - \alpha = i_1 + i_2(b_1(i_1)) - \alpha$$

Doing implicit differentiation of the composite function with respect to i_1 , we get:

$$\frac{d\phi}{di_1} = 1 + \frac{di_2}{db_1}\frac{db_1}{di_1}$$

By implicit differentiation of the equations:

$$\sin i_1 = n \sin b_1 \quad and \quad \sin i_2 = n \sin b_2 = n \sin(\alpha - b_1)$$

We find:

$$\cos i_2 \frac{di_2}{db_1} = -n\cos(\alpha - b_1) \quad and \quad n\cos b_1 \frac{db_1}{di_1} = \cos i_1$$

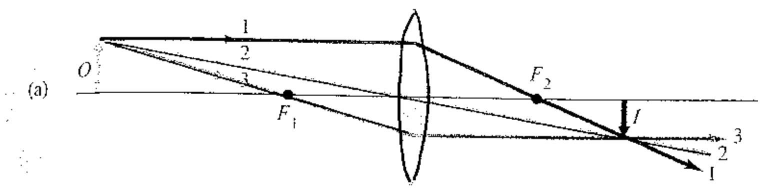
Solving these equations with respect to $\frac{di_2}{db_1}$ and $\frac{db_1}{di_1}$ inserting in the expression for φ, we have

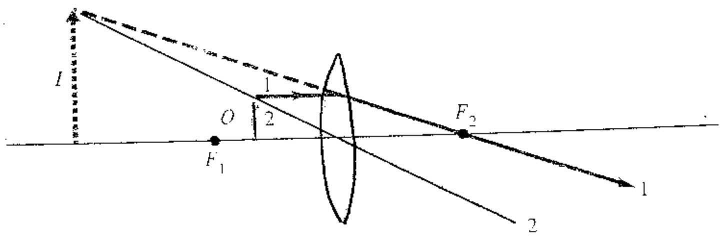
$$\frac{d\phi}{di_1} = 1 + \frac{di_2}{db_1}\frac{db_1}{di_1}$$
$$= 1 - \frac{\cos i_1}{\cos i_2}\frac{\cos(\alpha - b_1)}{\cos b_1}$$
$$= 1 - \frac{\cos i_1}{\cos i_2}\frac{\cos b_2}{\cos b_1}$$

We can now see from the last equation above that $\frac{d\phi}{di_1} = 0$ has the solution $i_1 = i_2 \wedge b_1 = b_2$, since in that case both fraction becomes equal to one, and the solution correspond to the path of the beam, when the prism is in its main position.

3. The images created by condensing lenses

Material: Condensing lenses, an optical bench with accessories, candle lights and a ruler.
The purpose of the experiment is to illustrate the images created from objects, when the light passes through a condensing lens, and to verify the image equation.





Conducting the experiment: Using an optical bench, you should place a condensing lens, a candle light and a screen on metal riders.
The candle light should be placed outside the focal width of the lens. The screen is adjusted until the image of the candle light flame is as sharp as possible. With a ruler the distances from the lens to the image and to the object are measured.
The focal length of the lens is either read from the lens or measured by sending a parallel light beam through the lens and moving the screen until the image of the beam is least (a point). The distance from the lens to the screen is the focal length.

If *f* designates the focal width of the lens, and as usual *s* designates the distance to the object and *s'* designates the distance to the image, we have from the object-image equation.

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$$

There should be conducted at least 5 experiments using lenses with different focal lengths and with varying distances from the lens to the object.

Estimate in each experiment the height (larger/smaller) of the size between the object and the image. Theoretically we should have:

$$\frac{y'}{y} = \frac{s'}{s} = \frac{f}{s-f} \ ,$$

where y and y' designates the height of the object and the image respectively.

4. The astronomical binocular

The purpose of the experiment is to construct an astronomical binocular.
Material: Optical bench, condensing lenses having various focal widths.
Conducting the experiment:
Two condensing lenses having different focal length should be placed on the optical bench with a distance which is the sum of the two focal lengths.

The lens having the largest focal width should be placed furthest away from the observing eye and the ocular nearest the observing eye.

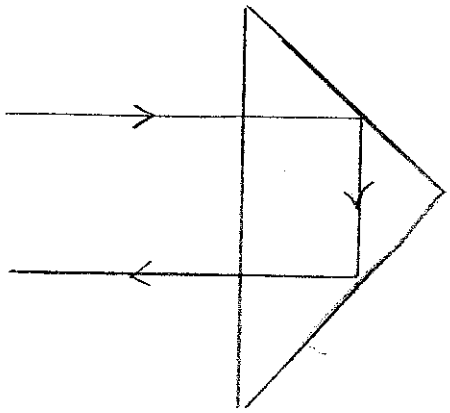
Looking through the ocular is observed a faint object. The ocular is displaced until the object is seen sharply.

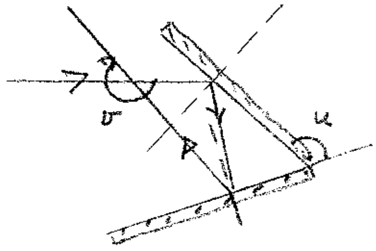
The magnification of the object seen through the binocular is estimated, and is compared with the ratio between the two angles $\alpha' \ and \ \alpha$ in which the image and the object is seen.
According to theory we should have: $\frac{\alpha'}{\alpha} = \frac{f_1}{f_2}$

5. Small experiments with mirrors, concave mirrors and prisms

5.1 Total reflection in a prism

Material: Prism having a refractive angle of 90^0, and a Laser.
Place the prism on the optical bench and let the laser beam come in perpendicular to the bottom of the prism.
Observe that there is total reflection on the refracting surfaces (Why?) and that the beam is reflected exactly 180^0





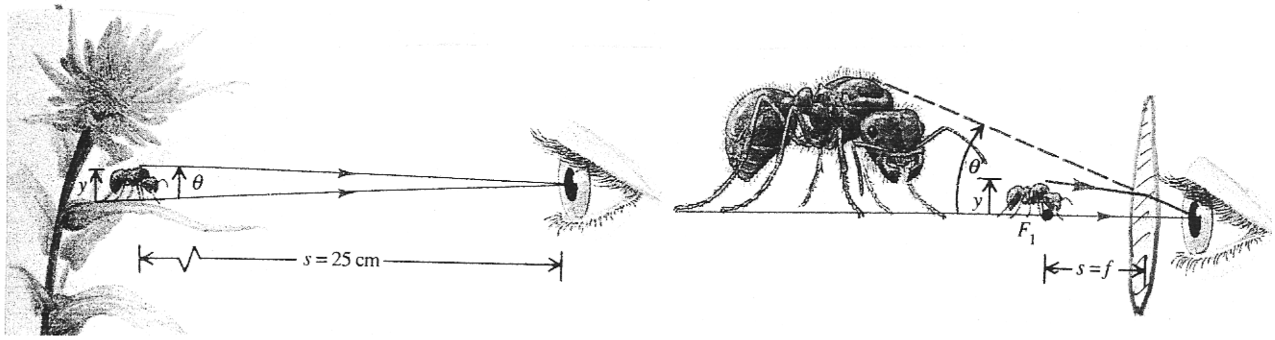
5.2 Double reflection in two mirrors

Material: A double mirror and a laser.

Send a beam from the laser towards the mirrors perpendicular to the planes of the two mirrors. You may then verify the relations, which were derived on page 2. Especially when the angle between the mirrors is 90^0.

5.3 Experiment with a magnifying glass

Choose a condensing lens having a known focal length.
Observe an object in various distances from the condensing lens. Verify that the observations are qualitatively in correspondence with the formulas presented on page 9-10.



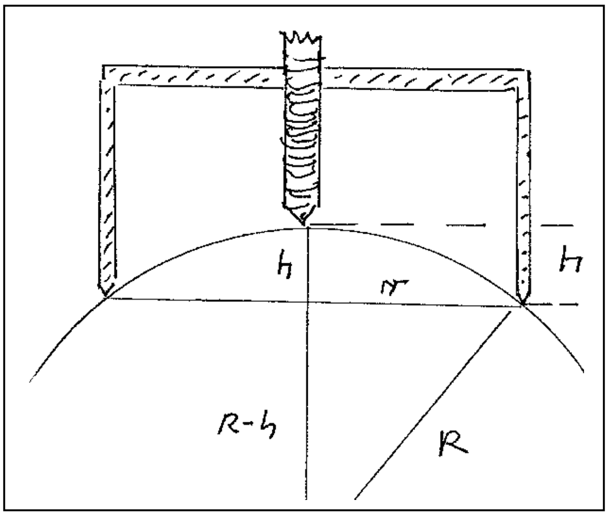
5.4 Experiments with concave mirrors

Materiel: Concave mirror and possibly an optical bench and candle lights.
Place a light source displaced a little from the symmetry axis of the concave mirror.
Try if it possible to see the image on a screen, and verify if possible the relation presented earlier.

5.5 Determination of the radii in the spherical surfaces on a lens

Using a micrometer screw it is possible, and with great precision to determine the radii in the spheres in which the lenses are built. The legs of the micrometer screw are positioned at an angle of

120^0 in a circle with radius *r*. The radius is written on the instrument. The screw which is placed in the centre of the circle is supplied with a circular scale, which can be read with an accuracy of 1/1000 *mm* (1 μm)



The apparatus is first placed on an absolute plane surface, where the zero point is controlled. Then the screw is loosened, before it is placed on the lens, so that the peak of the screw is on the axis of the lens. With care the screw is turned down, until there is contact with the lens.
The piece that the screw has been lifted in relation to the legs on the micrometer screw is *h*. The radius in the spherical surfaces of the lens is *R*. From the figure above is then seen:

$$(R-h)^2 + r^2 = R^2 \iff 2Rh = h^2 + r^2 \iff$$
$$R = \frac{h^2 + r^2}{2h}$$

Since *h* and *r* are known, the radius *R* can be determined. If this is done on both sides of the lens, then, we ca determine the two radii R_1 and R_2 , and the focal width can be determined from the lens makers formula.

$$\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$$

The refractive index for yellow light in crown glass is ca. 1.51 and for flint glass ca. 1.61
‘