Emptying a barrel from a tab at the bottom

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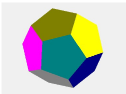

Ole Witt-Hansen 2018

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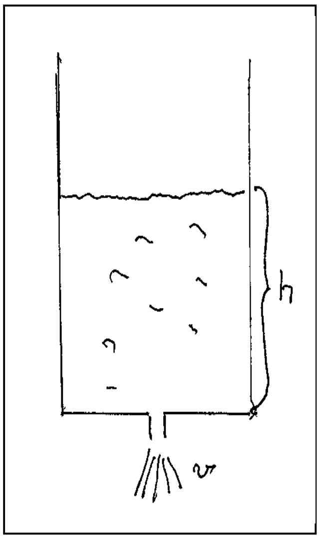
1. Introduction to the problem

As a former teacher of mathematics and physics in the 9-12 year of the Danish high school, one often encountered professional challenges, when the students should write their 14 days project on a subject, where the subject was mainly chosen by the students themselves.
It happened frequently that the students presented subjects that the teacher had no knowledge of whatsoever, (and neither had the student of course).
If there did not exists a book on the subject suitable for the professional level, they had to resort to the Internet, which, however, is most superficial and seldom gives a tangible theoretical explanation. University textbooks are usually beyond their theoretical skills.

When the student was about to give up, more or less, and since the teacher in the end was responsible for the formulation of the project, the only way to settle this issue was often, that you had to write some notes yourself that covered the content of the project, and which the students could more or less copy. (Not a ideal professional situation, but that was how it was, and still is)

One year a student had chosen the subject of emptying a vessel by a tap placed at the bottom.
Since I could find no reference to this problem in the books on my shelves, I had to deal with the problem by hand.
Since I had earlier written an article on Bernouilli's law, I knew, however, very well where to begin.

2. Emptying a barrel by a tap placed at the bottom



We consider a container filled with a non viscous liquid (water) up to a height h over the tap at the bottom. The liquid has density ρ, p denotes the external pressure, and v is velocity of the liquid, at dept y. We shall then apply Bernouilli's law at the positions (1) and (2)

(2.1) $$p_1 + ½\rho\, v_1^2 + \rho\, g\, y_1 = p_2 + ½\, \rho\, v_2^2 + \rho\, g\, y_2$$

The content of Bernoulli's law is then that $p + ½\rho\, v^2 + \rho g y$ = constant (along a streamline), but we shall discard the term with the external pressure, since it is considered to be the same from top to bottom.

Replacing the dept y with h, we then have:

(2.2) $$½\rho v_1^{\,2} + \rho g h_1 = ½\rho v_2^{\,2} + \rho g h_2$$

At the surface $h = 0$, $v = 0$, and in the dept h the velocity is v.

Then Bernoulli's law gives the same result as a free gravitational fall.

(2.3) $$½\, \rho\, v^2 + \rho\, g(-h) = 0 \quad \Leftrightarrow \quad v = \sqrt{2gh}$$

Where v is the velocity in which the liquid leaves the tap.

If the mass of liquid in the container is $m = m(t)$ and if the pipe to the tap has cross section D, then the equation of continuity requires that for the amount of liquid dm that leaves the tap in the time interval dt applies:

(2.4) $$\frac{dm}{dt} = -\rho D v.$$ (The minus sign applies, because m decreases)

If the cylindrical container has the cross section A, then at the same time: $m = \rho A h$, so that

(2.5) $$\frac{dm}{dt} = \rho A \frac{dh}{dt}$$

Setting the two expressions for $\frac{dm}{dt}$ equal to each other, we find:

(2.6) $$\rho A \frac{dh}{dt} = -\rho D v$$

And inserting the expression for $v = \sqrt{2gh}$, then we get a differential equation for the dept h.

(2.7) $$\rho A \frac{dh}{dt} = -\rho D \sqrt{2gh} \quad \Leftrightarrow \quad \frac{dh}{dt} = -\sqrt{2g}\frac{D}{A}\sqrt{h}$$

The equation may be solved by separation of the variables:

$$\frac{dh}{\sqrt{h}} = -\sqrt{2g}\frac{D}{A}dt \quad \Leftrightarrow \quad \int_{h_0}^{h} \frac{dh}{\sqrt{h}} = -\sqrt{2g}\frac{D}{A}\int_0^t dt \quad \Leftrightarrow$$

$$2\sqrt{h} - 2\sqrt{h_0} = -\sqrt{2g}\frac{D}{A}t \quad \Leftrightarrow \quad \sqrt{h} = \sqrt{h_0} - \sqrt{2g}\frac{D}{2A}t \quad \Leftrightarrow$$

(2.8) $$h = \left(\sqrt{h_0} - \frac{D\sqrt{2g}}{2A}t\right)^2$$

The container is empty when $h = 0$, and this happens when:

(2.9) $$t = \frac{A}{D}\sqrt{\frac{2h_0}{g}}$$

For a container having a cross section A = 50 x 50 cm^2, h_0 =1.0 m and D=2.0 cm^2, this results in a time for emptying which is: t = 564 s.

3. Emptying a barrel into another barrel, where the let out tap is

We may elaborate a little on this result, if we consider two containers with different cross sections and different taps, and where the liquid from the primary container is led into in a second container, and then let out by a tap in the bottom of that container.
This leads to two coupled differential equations of first order.

Since it is the same amount of water, which leaves the upper container that enters the second container, then for the corresponding two volumes of liquid applies: $dV_1 = dV_2 \Leftrightarrow A_1 dh_1 = A_2 dh_2$.

The differential equation for the first container is the same as before, but for the second container, we must add a positive contribution from the first container.

(3.1) $$\frac{dh_1}{dt} = -\sqrt{2g}\frac{D_1}{A_1}\sqrt{h_1} \quad and \quad \frac{dh_2}{dt} = \frac{A_1}{A_2}\frac{dh_1}{dt} - \sqrt{2g}\frac{D_2}{A_2}\sqrt{h_2}$$

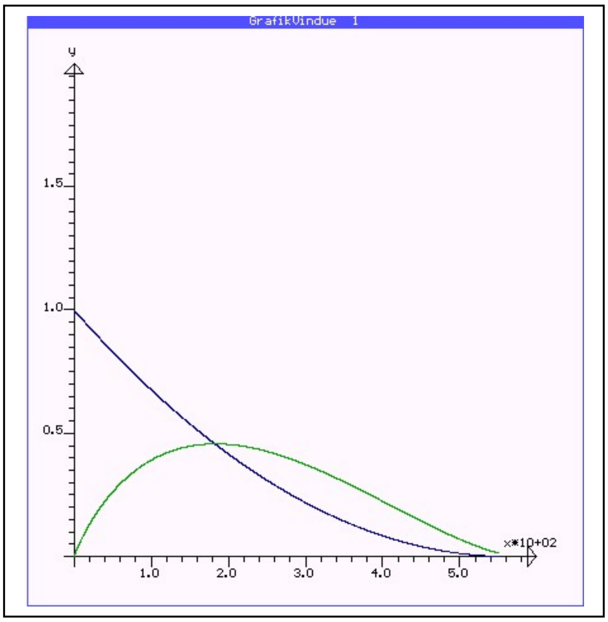
Which gives:

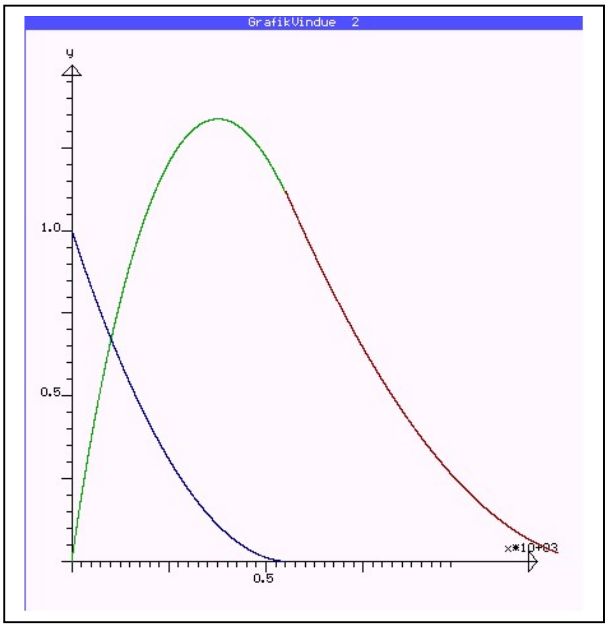
(3.2) $$\frac{dh_1}{dt} = -\sqrt{2g}\frac{D_1}{A_1}\sqrt{h_1} \quad and \quad \frac{dh_2}{dt} = \sqrt{2g}\frac{D_1}{A_2}\sqrt{h_1} - \sqrt{2g}\frac{D_2}{A_2}\sqrt{h_2}$$

It is not possible to solve these two coupled differential equations, (at least not by standard methods), and we have to resort to numerical methods, where two results are shown below.

The first curve refers to the primary container as is obvious from the graphs.

If the figure to the left the cross sections of the tabs are (almost) equal, whereas in the second container, the cross section of the tab of the second container is smaller than for the first one
One should notice the almost parabolic form of the curves, which is the result of (2.8)





4. Emptying a barrel, which is not a cylinder with circular cross section

As demonstrated in (2.7) above emptying a barrel leads to the differential equation:

$$(4.1)\qquad \rho A\frac{dh}{dt} = -\rho D\sqrt{2gh} \quad\Leftrightarrow\quad \frac{dh}{dt} = -\sqrt{2g}\,\frac{D}{A}\sqrt{h}$$

If the container does not have a circular cross section, then $A = A(h)$. This we may insert in the differential equation.

$$(4.2)\qquad \frac{dh}{dt} = -\sqrt{2g}\,\frac{D}{A(h)}\sqrt{h} \;\Leftrightarrow\; \frac{A(h)}{\sqrt{h}}dh = -D\sqrt{2g}\;dt$$

A differential equation, which possibly may be integrated-

4.1 A cone shaped barrel

We shall think of a cone having height H with the radius in the bottom R.
The tap is in the bottom, and r is the radius in the cross section at the water level $H - h$.
We must then have:

$$\frac{r}{R} = \frac{H-h}{H} \;\Rightarrow\; r = \frac{H-h}{H}R \quad \text{so that} \quad A(h) = \pi r^2 = \pi\frac{R^2}{H^2}(H-h)^2$$

Inserted in (4.2) above, we get:

$$\frac{\pi\frac{R^2}{H^2}(H-h)^2}{\sqrt{h}}dh = -D\sqrt{2g}\;dt \qquad\Leftrightarrow\qquad \frac{(H-h)^2}{\sqrt{h}}dh = -D\frac{H^2}{\pi R^2}\sqrt{2g}\;dt$$

$$\frac{(H-h)^2}{\sqrt{h}}dh = -\lambda dt \quad \text{where} \quad \lambda = \frac{H^2}{\pi R^2}D\sqrt{2g}$$

$$\int_H^h \frac{(H-h)^2}{\sqrt{h}}dh = -\lambda\int_0^t dt \quad\Leftrightarrow\quad \left[2H\sqrt{h} + \tfrac{4}{5}h^{\frac{5}{2}} - \tfrac{4}{3}Hh^{\frac{3}{2}}\right]_H^h = -\lambda t$$

$$2H\sqrt{h} + \tfrac{4}{5}h^{\frac{5}{2}} - \tfrac{4}{3}Hh^{\frac{3}{2}} - (2H\sqrt{H} + \tfrac{4}{5}H^{\frac{5}{2}} - \tfrac{4}{3}H^{\frac{5}{2}}) = -\lambda t$$

Since this is a fifth order equation in square root h, it cannot be solved analytically.

4.2 The container is an upside down cone

Then we shall consider a upside down cone having the height H and radius in the bottom R. The tap is placed at the bottom, and r is the radius in the cross section, at the water level $H - h$. We must have:

$$\frac{r}{R} = \frac{h}{H} \;\Rightarrow\; r = \frac{h}{H}R \quad \text{so that} \quad A(h) = \pi r^2 = \pi\frac{R^2}{H^2}h^2$$

When inserted in (4.2): $\frac{A(h)}{\sqrt{h}}dh = -\sqrt{2g}\ dt$, we get

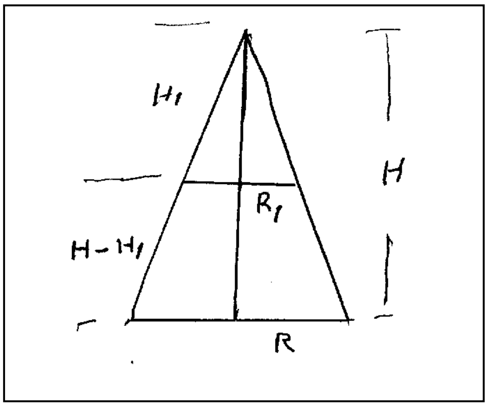
$$\frac{\pi\frac{R^2}{H^2}h^2}{\sqrt{h}}dh = -D\sqrt{2g}\ dt \quad \Leftrightarrow \quad \frac{h^2}{\sqrt{h}}dh = -\frac{H^2}{\pi R^2}D\sqrt{2g}\ dt$$

$$h\sqrt{h}dh = -\lambda dt \quad where \quad \lambda = \frac{H^2}{\pi R^2}D\sqrt{2g}$$

$$\int_H^h h\sqrt{h}dh = -\lambda\int_0^t dt \quad \Leftrightarrow \quad \left[\tfrac{2}{5}h^{\frac{5}{2}}\right]_H^h = -\lambda t$$

$$\tfrac{2}{5}h^{\frac{5}{2}} - \tfrac{2}{5}H^{\frac{5}{2}} = -\lambda t \quad where \quad \lambda = \frac{H^2}{\pi R^2}D\sqrt{2g} \qquad \Rightarrow \qquad h = \left(H^{\frac{5}{2}} - \tfrac{5}{2}\lambda t\right)^{\frac{2}{5}}$$

4.3 When the container is a frustum



In principle there is only little differenrence between a upside down cone and an upside down frustum.
Since, if you know the height of the frustum $H_2 = H - H_1$, where H_1 is the height of the cut off , and R and R_1 are radii in the end faces, then you can determine the height H of the entire cone, and then insert this in the formula for the upside down cone.

$$\frac{H}{H_1} = \frac{R}{R_1} \quad \Rightarrow \quad H = H_1\frac{R}{R_1} \quad \Rightarrow$$

$$H_2 = H - H_1 = H_1(\frac{R}{R_1} - 1)$$

$$H_1 = \frac{H_2}{(\frac{R}{R_1} - 1)} \Rightarrow H = H_1\frac{R}{R_1}$$

If this H is used in the expression $A(h) = \pi r^2 = \pi\frac{R^2}{H^2}h^2$

And the same formula as before is applied with the new H, we get a very similar differential equation.
When integrating, however, H must be replaced with H_2, the height of the frustum.

$$h\sqrt{h}dh = -\lambda dt \quad where \quad \lambda = \frac{H^2}{\pi R^2}D\sqrt{2g}$$

$$\int_{H_2}^h h\sqrt{h}dh = -\lambda\int_0^t dt \quad \Leftrightarrow \quad \left[\tfrac{2}{5}h^{\frac{5}{2}}\right]_{H_2}^h = -\lambda t$$

$$\tfrac{2}{5}h^{\frac{5}{2}} - \tfrac{2}{5}H_2^{\frac{5}{2}} = -\lambda t \quad \textit{where} \quad \lambda = \frac{H^2}{\pi R^2} D\sqrt{2g} \qquad \Rightarrow \qquad h = \left(H_2^{\frac{5}{2}} - \tfrac{5}{2}\lambda t\right)^{\frac{2}{5}}$$