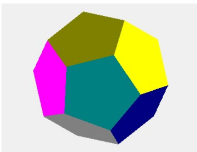
Hydrostatics
Does the boat capsize

This is an article from my home-page: www.olewitthansen.dk



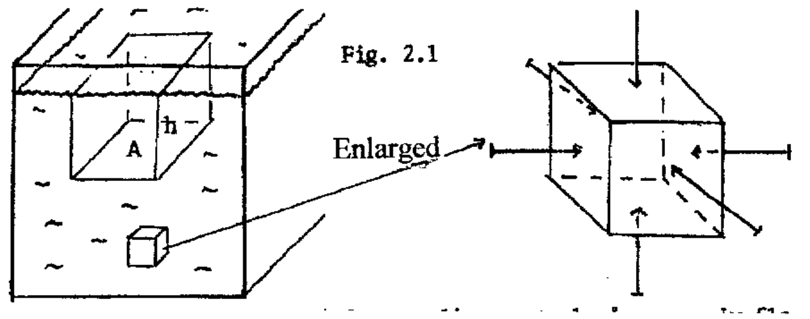
Ole Witt-Hansen (2021)

Contents

1. Introduction

Before we enter the problem of the equilibrium of a boat, we shall give an elementary treatment of some of the main result of hydrostatics. If you are familiar with the laws of pressure in gasses and Archimedes law, you may skip this first sections until section 5. Section 2 and 4 are taken from the first volume of
Ole Witt-Hansen: Elementary physics. http://olewitthansen.dk/Physics/Pressure_EF1.pdf

2. Pressure in liquids



Pressure in liquids and gasses are defined in the same manner as pressure on a solid surface, as the force normal to the surface per unit area.

In a certain depth of a liquid, the pressure is the same in all directions. Because, if we consider a small cube of liquid at rest, as depicted in the figure, so small that we may ignore its gravity. Then the forces on opposite sides will be equal, since otherwise the cube would move, and the pressure on adjacent sides must also be the same, since otherwise it would be deformed.

Since the pressure is the same in all directions, we simply speak of the pressure in a certain depth. We shall then seek a formula for the pressure p_h in the depth h of a liquid.
We put the density to ρ, and the pressure at the surface of the liquid to p_0 .

We then consider a rectangular volume of liquid, where the upper side coincides with the surface of the liquid. The area of that side and of the bottom side is A, and the height (depth) of the rectangular volume is h.
The volume of the box is then $V = A{\cdot}h$. We then apply the defining equation for pressure:

$$p = \frac{F_N}{A} \quad \Leftrightarrow \quad F_N = pA\,.$$

The force on the upper side is the atmospheric pressure times the area: $F_0 = p_0A$.
The mass of the liquid in the box is: $m_v = \rho{\cdot}V = \rho{\cdot}A{\cdot}h$
The gravity of the liquid in the rectangular volume is thus: $F_T = m_vg = \rho{\cdot}A{\cdot}h{\cdot}g$.
The normal force on the bottom side of the rectangular volume must be the force normal to that side, which is the force on the upper side plus the gravity of the liquid in the rectangular volume.

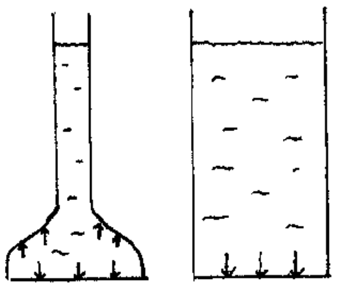
$$F_N = F_T + F_0 = \rho{\cdot}A{\cdot}h{\cdot}g + p_0{\cdot}A$$

Since $p = \frac{F_N}{A}$, we can find the pressure in the depth h by inserting F_N and dividing with the area A.

(2.2)	$p_h = p_0 + \rho g h$	(Pressure in the depth h of a liquid)

It is worth noticing that the pressure depends only on the depth, but not on the design of the container, nor on how much liquid, there is in the container.

2.3 Example



The two vessels, shown to the left, have the same ground surface, but different volumes. If they are filled with liquid to the same height, then according to the formula (2.2), the pressure at the two bottoms should be the same. Since the two bottom surfaces have the same area there should also be the same force on the bottom surface!
But how can this be true if the gravity of the liquid in the other vessel is much larger?
It can of course be investigated by placing the two vessels on a weight and the argument above could point towards that the weight would show the same, (which it of course not does!).
So what is wrong with the reasoning? The solution is of course that the weight does not measure the force on the bottom side but the resulting force of gravity.
But for the vessel to the left, the forces from pressure also affects the vessel upwards, forces that must be subtracted from the forces acting from the pressure at the bottom. But it is still correct that the two vessels have the same pressure at the bottom.

3. Units for pressure. Conversions for units

The SI-unit for pressure is, (as already mentioned), *Pascal* (*Pa)* equal to (N/m^2), but especially when gasses are concerned, there are several other units, which have their origins in how the atmospheric air pressure was measured earlier.

3.1 Definition: By the pressure 1 atmosphere, we understand the pressure of a 760 *mm* high quick silver column. To make the conversion to the SI-unit, we apply the formula for the pressure in a liquid with density ρ in the depth h.

$$P(760\ mm\ Hg) = \rho_{Hg} g h = 13.6\ 10^3\ kg/m^3 \cdot 9.82\ m/s^2 \cdot 0.760\ m = 1.013\ 10^5\ Pa$$

(3.2) $$1\ atm = 760\ mm\ Hg = 1.013\ 10^5\ Pa$$

$$1\ mm\ Hg = \frac{1}{760}\ atm = 133.3\ Pa$$

(3.3) (Definition) $1\ Bar = 1\ b = 10^5\ Pa$. $1\ mb$ (1 *milibar)* $= 10^2\ Pa$

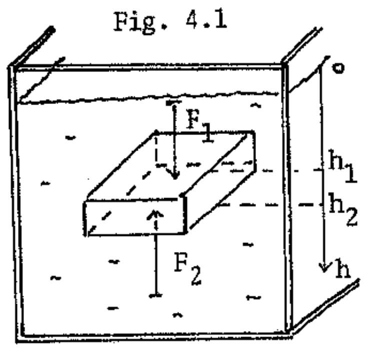
1 *at* is the pressure exerted by 1 *kg* on 1 cm^2.

(3.4) $$1\ at = 1\ kp/cm^2 = 9.80665\ N/(10^{-4}\ m^2) = 9.80665\ 10^4\ Pa$$

We can see that 1 *atm*, 1 *Bar* and 1 *at* are almost equal to each other, which do not make it easier. Earlier the air pressure was mostly given in *mb*, and even earlier in *atm*.
Nowadays the air pressure is measured in *hPa* (*hecto-Pascal)*, which is almost the same numerical number as *mb*, a unit the fishermen have used for decades, so they did not really have to make a conversion, listening to the weather forecast. The unit 1 *at* has mostly been used in engineering.

For the pressure in car tires is often used the unit *psi* (pounds per square inch) (also in Europe outside the UK). 1 *psi* =6.895 10^3 *Pa*. Also in daily language pressure is stated as kg/m^2. However, since this it not a physical unit for pressure, then presumably is meant kp/m^2.

4. Archimedes law



The figure shows a rectangular box immersed in a liquid having density ρ.
The pressure on the top side and on the bottom side of the box can be determined from (2.2) $p_h = p_0 + \rho gh$.
The top side is located in the depth h_1, and the bottom side in the depth h_2. Thus we find the pressure on the two sides:

$$p_1 = p_0 + \rho g h_1 \quad \text{and} \quad p_2 = p_0 + \rho g h_2$$

The forces that act on the two sides may be found by multiplying with their common area. A:

$$F_1 = p_1 A \quad \text{and} \quad F_2 = p_2 A.$$

The difference between the forces on the top and the bottom, is called the *buoyancy* and it is denoted F_{up}. We shall then calculate the magnitude of F_{up}.

(4.2) $$F_{up} = F_2 - F_1 = (p_0 + \rho g h_2)A - (p_0 + \rho g h_1)A = \rho g(h_2 - h_1)A$$

The volume V of the box is the height times the ground surface. $V = (h_2 - h_1)A$.
It can therefore hold mass of liquid $m_v = \rho V$. Then we can find an expression for the buoyancy:

(4.3) $$F_{up} = \rho g(h_2 - h_1)A = \rho g V = m_v g \quad \Leftrightarrow \quad F_{up} = m_v g \quad \Leftrightarrow$$

This is Archimedes law:

A body that is immersed in a liquid is affected by a buoyancy which is equal to the gravity of the displaced amount of water..

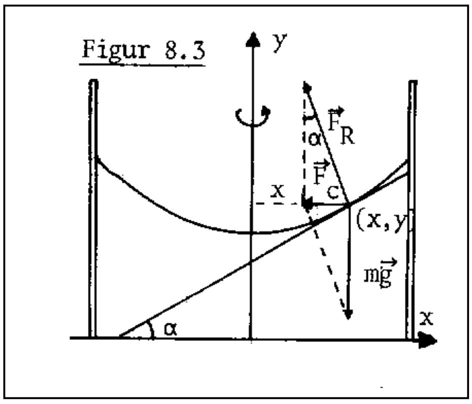
We have performed a rather detailed explanation of Archimedes law, but only for a rectangular box. However Archimedes law is valid for a body of any shape.
If you (mathematically) confine a volume of liquid exactly the same as the shape of the immersed body, then the liquid is affected by gravity and the pressure from the surrounding liquid. Since it is at rest in the liquid the pressure forces from the liquid must exactly cancel the gravity of the liquid in the volume, which is equal to $m_v g$.

4.4 Exercises

1. a) Find the force that the atmosphere exerts on a 40 x 40 cm^2 seat of a chair.
   b) Is it more or less than the gravity of an elephant?
   c) Why does the chair not crash?

2. What is pressure in bottom of the Pilipino graves, (depth 10.5 *km*)? State the result in *atm*.

8.3 Example. Rotating cylinder with liquid.



The figure shows a cylinder with a liquid, which rotates with constant angular velocity ω around its axis of symmetry. The figure shows a vertical cut through the axis of the cylinder. We want to determine the equation $y = f(x)$ for the intersecting curve in the cut with the surface. We consider a small particle of liquid at the surface. This particle is affected by gravity: $\vec{F}_T = m\vec{g}$ and the reaction forces $\vec{F}_R$ from the liquid particles around it. We assume that the liquid particles have no mutual movement, and $\vec{F}_R$ must therefore be directed perpendicular to the surface, since the liquid particle is at rest relative to the liquid. The sum of the forces $m\vec{g}$ and $\vec{F}_R$ must consequently be equal to the centripetal force on the particle.

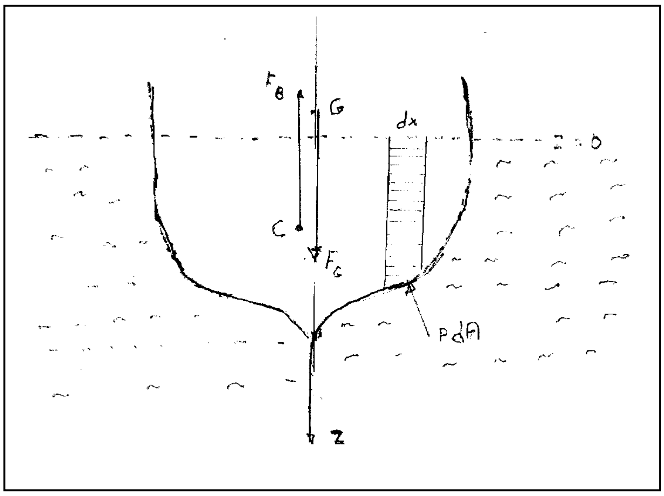
In the figure is marked the angle α so that $f'(x)= \alpha$ is the slope of the tangent. On the other hand as seen from the triangle: $\tan \alpha = F_c/mg$, from which follows.

(8.3.1) $$\frac{dy}{dx} = \frac{F_c}{mg} = \frac{m\omega^2 x}{mg} = \frac{\omega^2}{g}x \qquad \Rightarrow \qquad y = \frac{\omega^2}{g}x^2 + y_0$$

The intersecting curve is a parabola, and surface of the liquid is called a rotational paraboloid.
We should notice that this curve is not what you see, when you stir in a glass of liquid, (even if they look alike), since the viscosity in the liquid plays a significant role. Here the velocity of the liquid particles will have a decreasing velocity until it becomes zero at the edge of the glass.

5. Equilibrium and Buoyancy for a boat

We now turn to the problem of equilibrium of a boat, where gravity is the only *external* force.
The figure below represents a cross section normal to the longitudinal axis, which is chosen as the *y*-axis pointing out of the paper. The *x*-axis is chosen at the water-level, and the *z*-axis is the depth.
We consider the hull being a cylindrical surface, parallel to the *y*-axis. *G* is the centre of mass of the boat, and *C* is the centre of mass of the displaced water, when the boat is upright.
We define the buoyancy F_B on the boat as the resulting force, coming from the pressure, acting on the part of the boat that is below the water. We will then show that the resultant of the pressure from the water passes through *C,* the centre of mass of the displaced water.



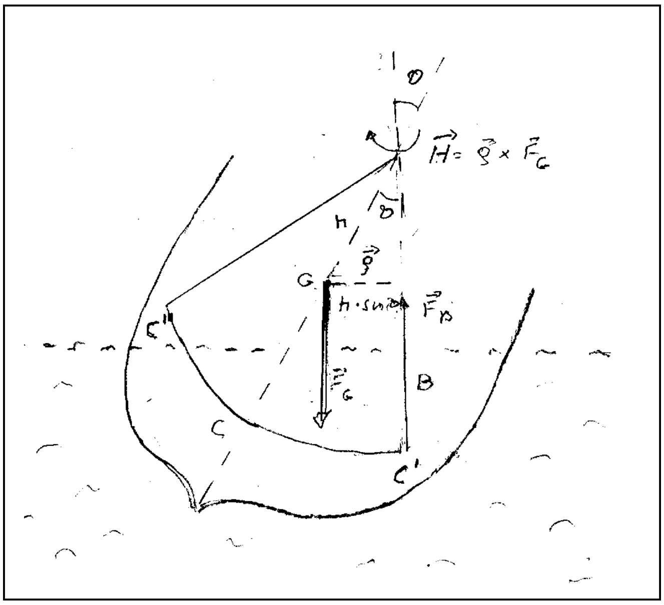
Let $dA = dsdy$ be the surface element on the side of the boat, where ds is the line element of the cross section. The pressure $p(z)$ in the depth z, is $p(z) = \rho gz$, where ρ is the density of the liquid (water) and g is the acceleration of gravity. The pressure $p(z)$ acts along the normal to the surface element dA, and the pressure force is therefore $\rho gzdA$, and its vertical component is $\rho gzdA\cos(n, z)$, where n is the inward normal. Since $dA \cos(n, z)$ is numerically equal to the projection of dA on the horizontal, then $\rho gzdA\cos(n, z)$ the vertical component of the force element can be written as $\rho gzdxdy$. The buoyancy can be evaluated as:

$$F_B = \rho g \int_{surface} zdxdy = \rho g V_{boat}$$

This equation expresses of course the law of Archimedes.
The two vectors gravity $\vec{F}_G$, and the buoyancy $\vec{F}_B$ have opposite direction, but otherwise they are equal to each other $\vec{F}_B = -\vec{F}_G$, since the boat is at rest.
When the boat is at equilibrium (at rest), the centre of gravity G and the centre of mass of the displaced water C lie on the same vertical line. The condition $F_B = \rho gV = F_G = mg$, determines the height of the waterline. We also note that the pressure acting on the two horizontal opposite area element, must balance each other, since the pressure is the same in the same depth, and the boat is at rest.

6. When the boat rolls

We shall now turn to the dynamics of the boat, if the equilibrium position is disturbed.



We look therefore at a situation, where the boat heels, and oscillates from side to side along the z -axis.
This can be done within the framework of hydrostatics, since the oscillations are so slow that it can be considered as a quasi static situation.
We thus consider a rolling motion along a horizontal axis, the y-axis. The symmetry plane of the boat now subtends an angle θ with the vertical. During the roll, the shape of the displacements and the position of C will vary between C' and C'', where C as before is the centre of mass of the

displaced water. C' is the position at the end of the roll and C'' is the corresponding opposite point of the roll. During the roll the centre of displacement describes a curve, having the midpoint C.
In the figure above C' passes through the centre of buoyancy $\vec{F}_B$.
The force from gravity $\vec{F}_G$ always passes through the centre of mass G.
The two vectors $\vec{F}_G$ and $\vec{F}_B$ exerts, however, a moment of force $\vec{H} = \vec{\rho} \times \vec{F}_G$ and we can see, that $H = mgh\sin\theta$ and is directed along the positive y-axis.
In the figure the moment will then always tend to bring the boat back to equilibrium, excluding a capsize of the boat.
However, if the buoyancy is to the left of the centre of mass, the moment will change direction an turn over the boat. If the shape of the boat is more or less like the one depicted in the figure this is hardly possible. Thus a boat built by sensible constructors will never capsize from rolling in sea waves.
So the (theoretical) answer to the question: Can the boat capsize, submitted to sea waves. The answer is no, for traditional constructed boats.

For moderate heeling, we may approximate $H = mgh\sin\theta$ by $H = mgh\theta$
If I denotes the moment of inertia of the boat along the y-axis, we can use the moment of force theorem $I\ddot{\theta} = H$, and in this case:

$$I_{boat}\ddot{\theta} = -m_{boat}g\theta$$

But this is just the differential equation for a physical pendulum:

$$I\ddot{\theta} = -k\theta \quad \Leftrightarrow \quad \ddot{\theta} = -\omega^2\theta \qquad \text{where} \quad \omega = \sqrt{\frac{k}{I}}$$

In the case of a boat $k = m_{boat}g\theta$, so we get:

$$\omega = \sqrt{\frac{m_{boat}g}{I_{boat}}} \qquad \text{or} \qquad T = 2\pi\sqrt{\frac{I_{boat}}{m_{boat}g}}$$

Notice that the Period T is independent of the amplitude in the roll, as is the case of harmonic oscillations.

Inserting numerical values are not really meaningful, but if we make some very crude assumptions, we may get some order of magnitude. We consider the boat as half a circular cylinder with radius r =10 m, and the length l = 100 m.
The moment of inertia of a circular disc is $\frac{1}{2}mr^2$ and the moment of inertia of half a cylinder with length l is therefore $\frac{1}{4}mr^2l$. The masses cancel in the expression for T, so we find:

$$T = 2\pi\sqrt{\frac{\frac{1}{4}10^2 \cdot 100}{g}} \approx 50s$$

To my experience, this is not so far from reality.

Reference: Arnold Sommerfeld: Mechanics of deformable bodies