Coupled pendulums
and
Coupled harmonic oscillations

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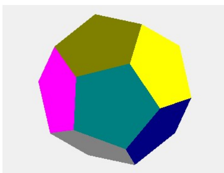

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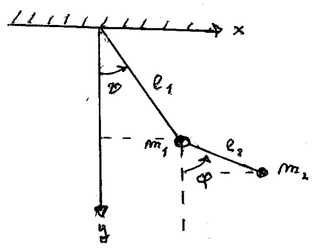
1. The Euler-Lagrange formalism. Generalized coordinates.



If a problem is described by generalized coordinates, usually denoted by q_i, then the Euler-Lagrange equations determining the extrema of a functional F, are

(1.1) $$\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_i} - \frac{\partial F}{\partial q_i} = 0$$

As usual, $\dot{q}_i$ means differentiating with respect to time t.

If the system is a mechanical system, then the Euler-Lagrange equations are the equations of motion in the generalized coordinates. For such a system, the Lagrange function is defined by the expression:

(1.2) $$L = T - U$$

In this context the kinetic energy is usually denoted by T, and the potential energy by U. Expressed with L, the Euler-Lagrange equations become.

(1.3) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Which are the equations of motion to be solved.

In the figure above a system of two coupled pendulums are displayed. The generalized coordinates are θ, which is the angular displacement from equilibrium for the upper pendulum, while φ is the angular displacement from equilibrium for the lower pendulum.
The masses for the two pendulums are m_1 and m_2, and the two pendulum lengths are l_1 and l_2.

First we determine the potential energy of the system:.

(1.4) $$\begin{aligned} E_{pot}(1) &= m_1 g h_1 = m_1 g l_1 (1-\cos\theta) \\ E_{pot}(2) &= m_2 g h_2 = m_2 g l_1 (1-\cos\theta) + m_2 g l_2 (1-\cos\varphi) \end{aligned}$$

The kinetic energy is a bit more circumstantial, and therefore we initially determine the position of the two weights as a function of the angles θ and φ.

(1.5) $$\begin{aligned} (x_1, y_1) &= (l_1 \sin\theta, l_1 \cos\theta) \\ (x_2, y_2) &= (l_1 \sin\theta, l_1 \cos\theta) + (l_2 \sin\varphi, l_2 \cos\varphi) \end{aligned}$$

From which we get from differentiating:

(1.6) $$\begin{aligned} (\dot{x}_1, \dot{y}_1) &= (l_1 \cos\theta\,\dot{\theta}, -l_1 \sin\theta\,\dot{\theta}) \\ (\dot{x}_2, \dot{y}_2) &= (l_1 \cos\theta\,\dot{\theta}, -l_1 \sin\theta\,\dot{\theta}) + (l_2 \cos\varphi\,\dot{\varphi}, -l_2 \sin\varphi\,\dot{\varphi}) \end{aligned}$$

(1.7) $$E_{kin}(1) = \tfrac{1}{2}m_1(\dot{x}_1{}^2 + \dot{y}_1{}^2) = \tfrac{1}{2}m_1 l_1{}^2\dot{\theta}^2(\cos^2\theta + \sin\theta^2) = \tfrac{1}{2}m_1 l_1{}^2\dot{\theta}^2$$

To obtain the result for (1), we had actually not needed to write down the coordinates, since the generalize coordinate to (1) is $r_\theta = l_1\theta$, so that $\dot{r}_\theta = l_1\dot{\theta}$, but with respect to (2) it is a bit more complicated. We determine the kinetic energy from the generalized velocities:

(1.8) $$E_{kin}(2) = \tfrac{1}{2}m_2(\dot{x}_2{}^2 + \dot{y}_2{}^2) = \tfrac{1}{2}m_2((l_1\cos\theta\,\dot{\theta} + l_2\cos\varphi\,\dot{\varphi})^2 + (-l_1\sin\theta\,\dot{\theta} - l_2\sin\varphi\,\dot{\varphi})^2)$$

The sum of the two squares in the first term gives: $\tfrac{1}{2}m_2 l_1{}^2\dot{\theta}^2$, as before, and the sum of the two squares in the second term gives $\tfrac{1}{2}m_2 l_2{}^2\dot{\varphi}^2$, (because $\cos^2 + \sin^2 = 1$), and these would be the only terms, if there was no coupling, and the Lagrange equation would separate into two equations one for each variable. So far so good, but we are left with the two double products.

$$m_2 l_1 l_2\dot{\theta}\dot{\varphi}\cos\theta\cos\varphi \quad and \quad m_2 l_1 l_2\dot{\theta}\dot{\varphi}\sin\theta\sin\varphi$$

$$E_{kin}(2) = \tfrac{1}{2}m_1 l_1{}^2\dot{\theta}^2 + \tfrac{1}{2}m_2 l_2{}^2\dot{\varphi}^2 + m_2 l_1 l_2\dot{\theta}\dot{\varphi}\cos\theta\cos\varphi + m_2 l_1 l_2\dot{\theta}\dot{\varphi}\sin\theta\sin\varphi$$

$$E_{kin}(2) = \tfrac{1}{2}m_1 l_1{}^2\dot{\theta}^2 + \tfrac{1}{2}m_2 l_2{}^2\dot{\varphi}^2 + m_2 l_1 l_2\dot{\theta}\dot{\varphi}(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$$

(1.9) $$E_{kin}(2) = \tfrac{1}{2}m_1 l_1{}^2\dot{\theta}^2 + \tfrac{1}{2}m_2 l_2{}^2\dot{\varphi}^2 + m_2 l_1 l_2\dot{\theta}\dot{\varphi}\cos(\theta - \varphi)$$

Especially the last term removes every aspiration of solving the equations analytically.

2. The Lagrange equations for the coupled pendulums.

Below we have established the Lagrange function.

(2.1) $$\begin{aligned} L = T - U = \tfrac{1}{2}m_1 l_1{}^2\dot{\theta}^2 + \tfrac{1}{2}m_2 l_1{}^2\dot{\theta}^2 + \tfrac{1}{2}m_2 l_2{}^2\dot{\varphi}^2 + m_2 l_1 l_2\dot{\theta}\dot{\varphi}\cos(\theta - \varphi) - \\ m_1 g l_1(1 - \cos\theta) - m_2 g l_1(1 - \cos\theta) - m_2 g l_2(1 - \cos\varphi) \end{aligned}$$

We have chosen to do the calculations in some detail, otherwise it is quite easy to get lost – and even then.

$$\frac{\partial L}{\partial\theta} = -m_1 g l_1\sin\theta - m_2 g l_1\sin\theta - m_2 l_1 l_2\dot{\theta}\dot{\varphi}\sin(\theta - \varphi) = -(m_1 + m_2)g l_1\sin\theta - m_2 l_1 l_2\dot{\theta}\dot{\varphi}\sin(\theta - \varphi)$$

$$\frac{\partial L}{\partial\varphi} = -m_2 g l_2\sin\varphi - m_2 l_1 l_2\dot{\theta}\dot{\varphi}\sin(\theta - \varphi)$$

$$\frac{\partial L}{\partial\dot{\theta}} = m_1 l_1{}^2\dot{\theta} + m_2 l_1{}^2\dot{\theta} + m_2 l_1 l_2\dot{\varphi}\cos(\theta - \varphi)$$

$$\frac{\partial L}{\partial \dot{\varphi}} = m_2 l_2{}^2 \dot{\varphi} + m_2 l_1 l_2 \dot{\theta} \cos(\theta - \varphi)$$

We proceed to establish the equations of motion step by step, first for the variable θ.

(2.2)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(m_1 l_1{}^2 \dot{\theta} + m_2 l_1{}^2 \dot{\theta} + m_2 l_1 l_2 \dot{\varphi} \cos(\theta - \varphi)) = m_1 l_1{}^2 \ddot{\theta} + m_2 l_1{}^2 \ddot{\theta} + m_2 l_1 l_2 \ddot{\varphi} \cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\varphi} \sin(\theta - \varphi)(\dot{\theta} - \dot{\varphi})$$

(2.3)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \Leftrightarrow \quad l_1{}^2 (m_1 + m_2)\ddot{\theta} + m_2 l_1 l_2 \ddot{\varphi} \cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\varphi} \sin(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) + (m_1 + m_2) g l_1 \sin\theta + m_2 l_1 l_2 \dot{\theta}\dot{\varphi} \sin(\theta - \varphi) = 0$$

$$l_1{}^2 (m_1 + m_2)\ddot{\theta} + m_2 l_1 l_2 \ddot{\varphi} \cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\varphi} \sin(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) + (m_1 + m_2) g l_1 \sin\theta + m_2 l_1 l_2 \dot{\theta}\dot{\varphi} \sin(\theta - \varphi) = 0$$

Then we proceed to establish the equations of motion step by step for the variable φ

(2.4)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{d}{dt}(m_2 l_2{}^2 \dot{\varphi} + m_2 l_1 l_2 \dot{\theta} \cos(\theta - \varphi)) = m_2 l_2{}^2 \ddot{\varphi} + m_2 l_1 l_2 \ddot{\theta} \cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\theta} \sin(\theta - \varphi)(\dot{\theta} - \dot{\varphi})$$

(2.5)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \quad \Leftrightarrow \quad m_2 l_2{}^2 \ddot{\varphi} + m_2 l_1 l_2 \ddot{\theta} \cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\theta} \sin(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) - (-m_2 g l_2 \sin\varphi - m_2 l_1 l_2 \dot{\theta}\dot{\varphi} \sin(\theta - \varphi) = 0$$

$$m_2 l_2{}^2 \ddot{\varphi} + m_2 l_1 l_2 \ddot{\theta} \cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\theta} \sin(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) + m_2 g l_2 \sin\varphi + m_2 l_1 l_2 \dot{\theta}\dot{\varphi} \sin(\theta - \varphi) = 0$$

As it is always the case, when we have oscillations with moderate deviations of the angle, we can approximate $\sin x \approx x \quad and \quad \cos x \approx 1$.

The equation of motion for the variable ***θ***.

(2.6)
$$l_1{}^2 (m_1 + m_2)\ddot{\theta} + m_2 l_1 l_2 \ddot{\varphi} - m_2 l_1 l_2 \dot{\varphi}(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) + (m_1 + m_2) g l_1 \theta + m_2 l_1 l_2 \dot{\theta}\dot{\varphi}(\theta - \varphi) = 0$$

The equation of motion for the variable φ.

(2.7)
$$m_2 l_2{}^2 \ddot{\varphi} + m_2 l_1 l_2 \ddot{\theta} - m_2 l_1 l_2 \dot{\theta}(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) + m_2 g l_2 \varphi + m_2 l_1 l_2 \dot{\theta}\dot{\varphi}(\theta - \varphi) = 0$$

3. Eliminating either *θ''* or *φ''* from the two equations to obtain two second order differential equations as: *θ''* = *F*(*θ, θ', φ, φ'*) *and φ''* = *G*(*θ, θ', φ, φ'*)

The equations (2.6) and (2.7) are two coupled second order differential equations, which determine the motion of the two pendulums. However since both θ'' and φ'', appear in both equations, we have to solve the equations for θ'' and φ''. This is a somewhat algebraic exercise, so it will be illustrated with the same number of reductions that I have done myself.

First we eliminate $\ddot{\varphi}$ from the two equations by multiplying (2.6) with l_2 and multiplying (2.7) with $-l_1$

(3.1) $l_1^2 l_2(m_1+m_2)\ddot{\theta}+m_2 l_1 l_2^2\ddot{\varphi}-m_2 l_1 l_2^2\dot{\varphi}(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+(m_1+m_2)g l_1 l_2\theta+m_2 l_1 l_2^2\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$

(3.2) $-m_2 l_1 l_2^2\ddot{\varphi}-m_2 l_1^2 l_2\ddot{\theta}+m_2 l_1^2 l_2\dot{\theta}(\theta-\varphi)(\dot{\theta}-\dot{\varphi})-m_2 g l_1 l_2\varphi-m_2 l_1^2 l_2\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$

Then we add the two equations:

(3.1)+(3.2): $$(l_1^2 l_2(m_1+m_2)-m_2 l_1^2 l_2)\ddot{\theta}+(m_2 l_1^2 l_2\dot{\theta}-m_2 l_1 l_2^2\dot{\varphi})(\theta-\varphi)(\dot{\theta}-\dot{\varphi})-$$
$$m_2 g l_1 l_2\varphi+(m_1+m_2)g l_1 l_2\theta+(m_2 l_1 l_2(l_2-l_1)\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$$

Next we eliminate $\ddot{\theta}$ by multiplying (2.6) with $-m_2 l_1 l_2$ and multiplying (2.7) with $l_1^2(m_1+m_2)$.

(3.3) $$-l_1^2(m_1+m_2)m_2 l_1 l_2\ddot{\theta}-(m_2 l_1 l_2)^2\ddot{\varphi}+(m_2 l_1 l_2)^2\dot{\varphi}(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+$$
$$m_2 l_1 l_2(m_1+m_2)g l_1\theta+(m_2 l_1 l_2)^2\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$$

(3.4) $$l_1^2(m_1+m_2)m_2 l_2^2\ddot{\varphi}+l_1^2(m_1+m_2)m_2 l_1 l_2\ddot{\theta}-l_1^2(m_1+m_2)m_2 l_1 l_2\dot{\theta}(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+$$
$$l_1^2(m_1+m_2)m_2 g l_2\varphi+l_1^2(m_1+m_2)m_2 l_1 l_2\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$$

(3.3)+ (3.4): $$(l_1^2(m_1+m_2)m_2 l_2^2-(m_2 l_1 l_2)^2)\ddot{\varphi}+((m_2 l_1 l_2)^2\dot{\varphi}-l_1^2(m_1+m_2)m_2 l_1 l_2\dot{\theta})(\theta-\varphi)(\dot{\theta}-\dot{\varphi})-$$
$$m_2 l_1 l_2(m_1+m_2)g l_1\theta+l_1^2(m_1+m_2)m_2 g l_2\varphi+((m_2 l_1 l_2)^2+l_1^2(m_1+m_2)m_2 l_1 l_2)\dot{\theta}\dot{\varphi}(\theta-\varphi)$$

For convenience we introduce some shorthand notations for the parameters in the two equations where we have eliminated $\ddot{\theta}$ *and* $\ddot{\varphi}$. First we repeat the two equations below:

The equations of motion for the variable ***θ***:

$$(l_1^2 l_2(m_1+m_2)-m_2 l_1^2 l_2)\ddot{\theta}+(m_2 l_1^2 l_2\dot{\theta}-m_2 l_1 l_2^2\dot{\varphi})(\theta-\varphi)(\dot{\theta}-\dot{\varphi})-m_2 g l_1 l_2\varphi+(m_1+m_2)g l_1 l_2\theta+$$
$$(m_2 l_1 l_2(l_2-l_1)\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$$

We divide with *l_1 l_2*:

$$l_1 m_1\ddot{\theta}+(m_2 l_1\dot{\theta}-m_2 l_2\dot{\varphi})(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+m_2 g\varphi+(m_1+m_2)g\theta-(m_2(l_2-l_1)\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$$

Introducing *a_1,… a_6* for the constants in the equation.

$$a_1\ddot{\theta}+(a_2\dot{\theta}-a_3\dot{\varphi})(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+a_4\varphi+a_5\theta-a_6\dot{\theta}\dot{\varphi}(\theta-\varphi)=0$$

(3.5) $\ddot{\theta} = -(a_2/a_1\dot{\theta} - a_3/a_1\dot{\varphi})(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) - a_4/a_1\varphi - a_5/a_1\theta + a_6/a_1\dot{\theta}\dot{\varphi}(\theta-\varphi) = 0$

And introducing the numerical data: $l_1 = 0{,}50\,m, l_2 = 0{,}20\,m, m_1 = 0{,}10\,kg, m_2 = 0{,}030\,kg$

$a_1 = l_1 m_1 = 0{,}05,\quad a_2 = m_2 l_1 = 0{,}015,\quad a_3 = m_2 l_2 = 0{,}006,\quad a_4 = m_2 g = 0{,}295,$
$a_5 = (m_1+m_2)g = 1{,}28,\quad a_6 = m_2(l_2 - l_1) = -0.009$

$a_1/a_1 = 1,\quad a_2/a_1 = 0{,}300,\quad a_3/a_1 = 0{,}120,\quad a_4/a_1 = 5.9,\quad a_5/a_1 = 25.6,\quad a_6/a_1 = -0.18$

The resulting reduced differential equation for *θ*.

(3.6) $\ddot{\theta} = -(0{,}3\dot{\theta} - 0{,}120\dot{\varphi})(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) - 5{,}9\varphi - 25{,}6\theta - 0{,}18\dot{\theta}\dot{\varphi}(\theta-\varphi) = 0$

The equation of motion for the variable *φ*.

$$(l_1^2(m_1+m_2)m_2 l_2^2 - (m_2 l_1 l_2)^2)\ddot{\varphi} + ((m_2 l_1 l_2)^2\dot{\varphi} - l_1^2(m_1+m_2)m_2 l_1 l_2\dot{\theta})(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) -$$
$$m_2 l_1 l_2(m_1+m_2)g l_1\theta + l_1^2(m_1+m_2)m_2 g l_2\varphi + ((m_2 l_1 l_2)^2 + l_1^2(m_1+m_2)m_2 l_1 l_2)\dot{\theta}\dot{\varphi}(\theta-\varphi)$$

We divide by $l_1\, l_2$**:**

$$l_1 l_2 m_1\ddot{\varphi} + (m_2^2 l_1 l_2\dot{\varphi} - l_1^2(m_1+m_2)m_2\dot{\theta})(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) -$$
$$m_2(m_1+m_2)g l_1\theta + l_1(m_1+m_2)m_2 g\varphi + (m_2^2 l_1 l_2 + l_1^2(m_1+m_2)m_2)\dot{\theta}\dot{\varphi}(\theta-\varphi) = 0$$

Introducing b_1**,…** b_6 **for the constants in the equation.**

$$b_1\ddot{\varphi} + (b_2\dot{\varphi} - b_3\dot{\theta})(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) - b_4\theta + b_5\varphi + b_6\dot{\theta}\dot{\varphi}(\theta-\varphi) = 0$$

$$\ddot{\varphi} = -(b_2/b_1\dot{\varphi} - b_3/b_1\dot{\theta})(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) - b_4/b_1\theta - b_5/b_1\varphi + b_6/b_1\dot{\theta}\dot{\varphi}(\theta-\varphi)$$

$l_1 = 0.50\,m, l_2 = 0.20\,m, m_1 = 0.10\,kg, m_2 = 0.030\,kg$

$b_1 = l_1 l_2 m_1 = 0.1,\quad b_2 = m_2^2 l_1 l_2 = 0.00009,\quad b_3 = l_1^2(m_1+m_2)m_2 = 0.000975,$
$b_4 = m_2(m_1+m_2)g_1 l = 0.0192,\quad b_5 = l_1(m_1+m_2)m_2 g = 0.0192,\quad b_6 = m_2^2 l_1 l_2 + l_1^2(m_1+m_2)m_2 = 0.00107$

$b_1/b_1 = 1,\quad b_2/b_1 = 0009,\quad b_3/b_1 = 0{,}0975,\quad b_4 = 1{,}92_1,\quad b_5/b_1 = 1{,}92,\quad b_6/b_1 = 0{,}107$

The resulting reduced differential equation for *φ*.

$$(3.7) \qquad \ddot{\varphi} = -(0.009\dot{\varphi} - 0.975\dot{\theta})(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) - 1.92\theta - 1.92\varphi + 0.107\dot{\theta}\dot{\varphi}(\theta - \varphi)$$

We thus have two coupled differential equations of second order:

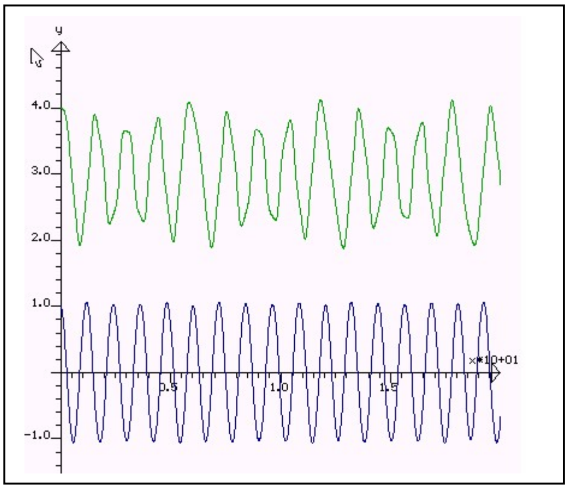
$$(3.8) \qquad \begin{aligned} \ddot{\theta} &= -(0.3\dot{\theta} - 0.12\dot{\varphi})(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) - 5.9\varphi - 25.6\theta - 0.18\dot{\theta}\dot{\varphi}(\theta - \varphi) = 0 \\ \ddot{\varphi} &= -(0.009\dot{\varphi} - 0.975\dot{\theta})(\theta - \varphi)(\dot{\theta} - \dot{\varphi}) - 1.92\theta - 1.92\varphi + 0.107\dot{\theta}\dot{\varphi}(\theta - \varphi) \end{aligned}$$

It is enough to cast a qualified glance at the two equations to exclude any analytic solution, however a numeric graphic solution is within reach, with the right computer program.

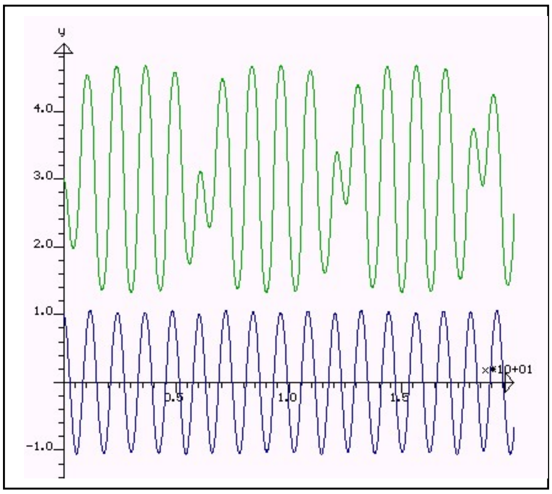
Although such programs may exist, I prefer to use my own, which among many other things is able to solve up to 6 coupled second order differential equations. It was written in Turbo Pascal in 1994 -1995, before the windows interface, so the windows –like interface is handmade.
The program can however not run Windows above Windows XP, and to make screen dumps one has to use Windows 98.
I have tried other programs, but I don't think they give the same possibilities.

In the numeric solution, the two pendulums are first assumed to have the same deflection, namely *pi*/6 at $t = 0$.
I have displayed three graphs for θ *and* φ in the printout from the program.
All the graphs are made with the numeric values from above.
The first graph below shows the oscillations of the largest pendulum at the bottom, and the oscillations of the smaller pendulum above.
The next graph is done with the same parameters, but the upper graph is the difference in deflection between the two pendulums.

Graph (3.9)

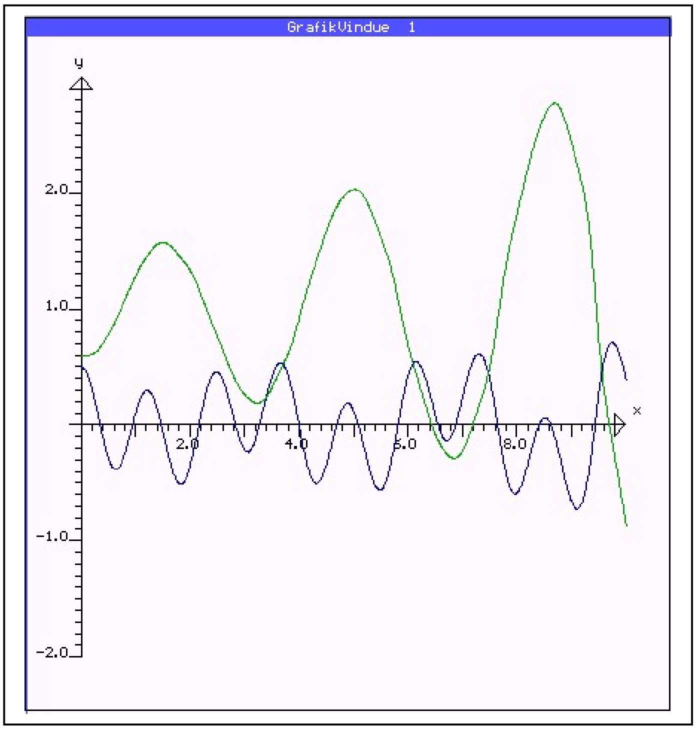


Graph (3.10)



If we however initially choose a moderate different deflection angle, the oscillations become chaotic, as shown in the third graph below.
However the graph is probably not realistic, since the deflections become so large, that the approximations $\sin x \approx x$ *and* $\cos x \approx 1$ can not be used any longer.

Graph (3.11)



3. Eliminating either *θ''* or *φ''* from the two equations, without the assumptions of small oscillations

We start out from the equations with no approximations. First we establish the equation of motion for the variable θ

(2.2) $$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{d}{dt}(m_1 l_1{}^2\dot{\theta} + m_2 l_1{}^2\dot{\theta} + m_2 l_1 l_2\dot{\varphi}\cos(\theta-\varphi)) =$$
$$m_1 l_1{}^2\ddot{\theta} + m_2 l_1{}^2\ddot{\theta} + m_2 l_1 l_2\ddot{\varphi}\cos(\theta-\varphi) - m_2 l_1 l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})$$

(2.3) $$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \iff l_1{}^2(m_1+m_2)\ddot{\theta} + m_2 l_1 l_2\ddot{\varphi}\cos(\theta-\varphi) - m_2 l_1 l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) +$$
$$(m_1+m_2)g l_1\sin\theta + m_2 l_1 l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi) = 0$$

$$l_1{}^2(m_1+m_2)\ddot{\theta} + m_2 l_1 l_2\ddot{\varphi}\cos(\theta-\varphi) - m_2 l_1 l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) + (m_1+m_2)g l_1\sin\theta + m_2 l_1 l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi) = 0$$

Then we proceed to establish the equations of motion step by step for the variable φ

(2.4)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = \frac{d}{dt}(m_2 l_2{}^2 \dot\varphi + m_2 l_1 l_2 \dot\theta \cos(\theta-\varphi)) =$$
$$m_2 l_2{}^2 \ddot\varphi + m_2 l_1 l_2 \ddot\theta \cos(\theta-\varphi) - m_2 l_1 l_2 \dot\theta \sin(\theta-\varphi)(\dot\theta-\dot\varphi)$$

(2.5)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = 0 \quad \Leftrightarrow \quad m_2 l_2{}^2 \ddot\varphi + m_2 l_1 l_2 \ddot\theta \cos(\theta-\varphi) - m_2 l_1 l_2 \dot\theta \sin(\theta-\varphi)(\dot\theta-\dot\varphi) -$$
$$(-m_2 g l_2 \sin\varphi - m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi) = 0$$

$$m_2 l_2{}^2 \ddot\varphi + m_2 l_1 l_2 \ddot\theta \cos(\theta-\varphi) - m_2 l_1 l_2 \dot\theta \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + m_2 g l_2 \sin\varphi + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi) = 0$$

And our aim is as before to solve the two equations for $\ddot\theta$ *and* $\ddot\varphi$.
We write again the two equations below each other:

$$m_2 l_2{}^2 \ddot\varphi + m_2 l_1 l_2 \ddot\theta \cos(\theta-\varphi) - m_2 l_1 l_2 \dot\theta \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + m_2 g l_2 \sin\varphi + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi) = 0$$
$$l_1{}^2 (m_1+m_2)\ddot\theta + m_2 l_1 l_2 \ddot\varphi \cos(\theta-\varphi) - m_2 l_1 l_2 \dot\varphi \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + (m_1+m_2) g l_1 \sin\theta + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi) = 0$$

We do this by multiplying the first equation by $l_1 \cos(\theta-\varphi)$ and the second equation by $-l_2$ and add the two equations to eliminate $\ddot\varphi$

I:
$$m_2 l_2{}^2 l_1 \cos(\theta-\varphi)\ddot\varphi + m_2 l_1 l_2 \ddot\theta l_1 \cos(\theta-\varphi)\cos(\theta-\varphi) +$$
$$l_1 \cos(\theta-\varphi)(-m_2 l_1 l_2 \dot\theta \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + m_2 g l_2 \sin\varphi + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi)) = 0$$

II:
$$-l_2 l_1{}^2 (m_1+m_2)\ddot\theta - m_2 l_1 l_2{}^2 \ddot\varphi \cos(\theta-\varphi) -$$
$$l_2(-m_2 l_1 l_2 \dot\varphi \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + (m_1+m_2) g l_1 \sin\theta + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi)) = 0$$

Adding the two equations gives:

III:
$$m_2 l_1 l_2 \ddot\theta l_1 \cos(\theta-\varphi)\cos(\theta-\varphi) - l_2 l_1{}^2 (m_1+m_2)\ddot\theta +$$
$$l_1 \cos(\theta-\varphi)(-m_2 l_1 l_2 \dot\theta \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + m_2 g l_2 \sin\varphi + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi)) -$$
$$l_2(-m_2 l_1 l_2 \dot\varphi \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + (m_1+m_2) g l_1 \sin\theta + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi)) = 0$$

We then isolate $\ddot\theta$ from III:

$$l_2 l_1{}^2 (m_2 \cos^2(\theta-\varphi) - (m_1+m_2))\ddot\theta +$$
$$l_1 \cos(\theta-\varphi)(-m_2 l_1 l_2 \dot\theta \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + m_2 g l_2 \sin\varphi + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi)) -$$
$$l_2(-m_2 l_1 l_2 \dot\varphi \sin(\theta-\varphi)(\dot\theta-\dot\varphi) + (m_1+m_2) g l_1 \sin\theta + m_2 l_1 l_2 \dot\theta\dot\varphi \sin(\theta-\varphi)) = 0$$

$$\text{III:}\quad \begin{aligned}\ddot{\theta} = -\frac{l_1\cos(\theta-\varphi)(-m_2l_1l_2\dot{\theta}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+m_2gl_2\sin\varphi+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))}{l_2l_1^2(m_2\cos^2(\theta-\varphi)-(m_1+m_2))}+\\ \frac{l_2l_1\cos(\theta-\varphi)(-m_2l_1l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+(m_1+m_2)gl_1\sin\theta+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))}{l_2l_1^2(m_2\cos^2(\theta-\varphi)-(m_1+m_2)}\end{aligned}$$

We shall then isolate $\ddot{\varphi}$. We write again the two Lagrange equations:

$$\text{I:}\quad m_2l_2^2\ddot{\varphi}+m_2l_1l_2\ddot{\theta}\cos(\theta-\varphi)-m_2l_1l_2\dot{\theta}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+m_2gl_2\sin\varphi+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi)=0$$

$$\text{II:}\quad \begin{aligned}&l_1^2(m_1+m_2)\ddot{\theta}+m_2l_1l_2\ddot{\varphi}\cos(\theta-\varphi)-m_2l_1l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+\\&(m_1+m_2)gl_1\sin\theta+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi)=0\end{aligned}$$

We multiply the first equation by $-l_1(m_1+m_2)$ and the second equation by $m_2l_2\cos(\theta-\varphi)$

$$\text{I:}\quad \begin{aligned}&-l_1(m_1+m_2)m_2l_2^2\ddot{\varphi}-l_1(m_1+m_2)m_2l_1l_2\ddot{\theta}\cos(\theta-\varphi)-\\&l_1(m_1+m_2)(-m_2l_1l_2\dot{\theta}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+m_2gl_2\sin\varphi+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))=0\end{aligned}$$

$$\text{II:}\quad \begin{aligned}&m_2l_2\cos(\theta-\varphi)l_1^2(m_1+m_2)\ddot{\theta}-m_2l_1l_2\cos(\theta-\varphi)\ddot{\varphi}\cos(\theta-\varphi)+\\&m_2l_2\cos(\theta-\varphi)(-m_2l_1l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+(m_1+m_2)gl_1\sin\theta+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))=0\end{aligned}$$

$$\text{I:}\quad \begin{aligned}&-m_2l_1l_2^2(m_1+m_2)\ddot{\varphi}-m_2l_1^2l_2(m_1+m_2)\cos(\theta-\varphi)\ddot{\theta}-\\&l_1(m_1+m_2)(m_2l_1l_2\dot{\theta}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+m_2gl_2\sin\varphi+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))=0\end{aligned}$$

Adding the two equations:

$$\text{IV:}\quad \begin{aligned}&-m_2l_1l_2\cos(\theta-\varphi)\ddot{\varphi}\cos(\theta-\varphi)-l_1(m_1+m_2)m_2l_2^2\ddot{\varphi}-\\&l_1(m_1+m_2)(-m_2l_1l_2\dot{\theta}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+m_2gl_2\sin\varphi+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))+\\&m_2l_2\cos(\theta-\varphi)(-m_2l_1l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+(m_1+m_2)gl_1\sin\theta+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))=0\end{aligned}$$

And reducing a bit

$$\text{IV:}\quad \begin{aligned}&-m_2l_1l_2(\cos^2(\theta-\varphi)+l_2(m_1+m_2))\ddot{\varphi}-\\&l_1(m_1+m_2)(-m_2l_1l_2\dot{\theta}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+m_2gl_2\sin\varphi+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))+\\&m_2l_2\cos(\theta-\varphi)(-m_2l_1l_2\dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi})+(m_1+m_2)gl_1\sin\theta+m_2l_1l_2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi))=0\end{aligned}$$

We find

IV:
$$\ddot{\varphi} = -\frac{l_1(m_1+m_2)(-m_2 l_1 l_2 \dot{\theta}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) + m_2 g l_2 \sin\varphi + m_2 l_1 l_2 \dot{\theta}\dot{\varphi}\sin(\theta-\varphi))}{m_2 l_1 l_2(\cos^2(\theta-\varphi) + l_2(m_1+m_2))} - \frac{m_2 l_2 \cos(\theta-\varphi)(-m_2 l_1 l_2 \dot{\varphi}\sin(\theta-\varphi)(\dot{\theta}-\dot{\varphi}) + (m_1+m_2) g l_1 \sin\theta + m_2 l_1 l_2 \dot{\theta}\dot{\varphi}\sin(\theta-\varphi))}{m_2 l_1 l_2(\cos^2(\theta-\varphi) + l_2(m_1+m_2))}$$

These two coupled second order differential equations III and IV look certainly frightening, but solving numerically should be possible.
Unfortunately the computer program I use (written in Borland Pascal Turbo 7.0) can only have 255 characters in a string. And that is exceeded for the general differential equations.

4. Horizontally coupled harmonic oscillations

Below is a sketch, showing an example of horizontal coupled harmonic oscillations.
If the masses m_1 and m_2 were only fixed to the wall with independent springs, having the *strengths* (spring constants) k_1 and k_3, then both masses would perform harmonic oscillations with the periods:

(4.1) $$T_1 = 2\pi\sqrt{\frac{m_1}{k_1}} \qquad \text{and} \qquad T_2 = 2\pi\sqrt{\frac{m_2}{k_3}}$$

This is because of Hooke's law: $F_{spring} = -kx$, where k [N/m] is the spring constant (strength).
This result in the equation of motion.

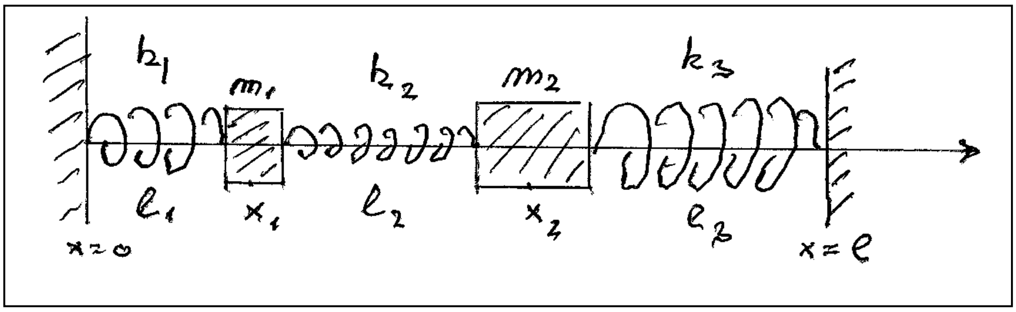
(4.2) $$F_{res} = F_{spring} \quad \Leftrightarrow \quad m\frac{d^2x}{dt^2} = -kx \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This equation has the solution: $x = A\cos(\omega t + \varphi_0)$, if and only if $\omega^2 = \frac{k}{m}$. As one may easily see by inserting the expression for x in (4.2). Since $\omega = \frac{2\pi}{T}$, where T is the period, we find:

$$T = 2\pi\sqrt{\frac{m}{k}} \ , \qquad \text{as stated above.}$$

The potential energy of a spring, when it is prolonged or compressed an amount x from its initial length, can be found by using Hooke's for the force.

(4.3) $$E_{pot} = \int_0^x F_{spring}dx = \int_0^x kxdx = \tfrac{1}{2}kx^2$$



The case of two independent harmonic oscillators both with a constant period, will no longer hold, when the who masses are bound together with a third spring with strength k_2.

The system may be analyzed in various ways, here we shall initially use the Lagrange formalism, as we did for the coupled pendulums, (where it was necessary). The Lagrange function is:

(4.3) $$L = T - U$$
Where:

$T = E_{kin}$ (The kinetic energy) and $U = E_{pot}$ (The potential energy).

A bullet above a variable means as usual differentiating with respect to time.
For example: $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$.

If the system is described with the generalized coordinates q_i, then the Lagrange equations of motion are:

(4.4) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For a system of two masse, bound together with springs obeying Hooke's law, the (generalized) coordinates are x_1 and x_2, which are the positions of the two masses.
So all that we have to do is to express the kinetic and potential energy of the system in these coordinates and plug it into (4.4).
The rest lengths of the three springs are set to be l_1 , l_2 , l_3 . and we put $l = l_1 + l_2 + l_3$.

(4.5) $$E_{kin} = \tfrac{1}{2}m_1\dot{x}_1^{\,2} + \tfrac{1}{2}m_2\dot{x}_2^{\,2}$$

The potential energy of a spring, stretched by Δx is according to (4.2) $E_{pot} = \tfrac{1}{2}k\Delta x^2$.
So we find for the potential energy of the three springs:

(4.6) $$E_{pot}(1) = \tfrac{1}{2}k_1(l_1 - x_1)^2 \quad , \quad E_{pot}(2) = \tfrac{1}{2}k_2(l_2 - (x_2 - x_1))^2 \quad , \quad E_{pot}(3) = \tfrac{1}{2}k_3(l_3 - (l - x_2))^2$$

And consequently:

(4.7) $$L = T - U = \tfrac{1}{2}m_1\dot{x}_1^{\,2} + \tfrac{1}{2}m_2\dot{x}_2^{\,2} - \tfrac{1}{2}k_1(l_1 - x_1)^2 - \tfrac{1}{2}k_2(l_2 - (x_2 - x_1))^2 - \tfrac{1}{2}k_3(l_3 - (l - x_2))^2$$

(4.8) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \quad \Leftrightarrow \quad m_1\ddot{x}_1 - k_1(l_1 - x_1) + k_2(l_2 - (x_2 - x_1)) = 0$$

(4.9) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0 \quad \Leftrightarrow \quad m_2\ddot{x}_2 - k_2(l_2 - (x_2 - x_1)) + k_3(l_3 - (l - x_2)) = 0$$

However this system may also be analyzed directly by writing down Newton's 2. law for the two masses of the system.

$$F_1 = m_1\ddot{x}_1 = k_1(l_1 - x_1) - k_2(l_2 - (x_2 - x_1))$$

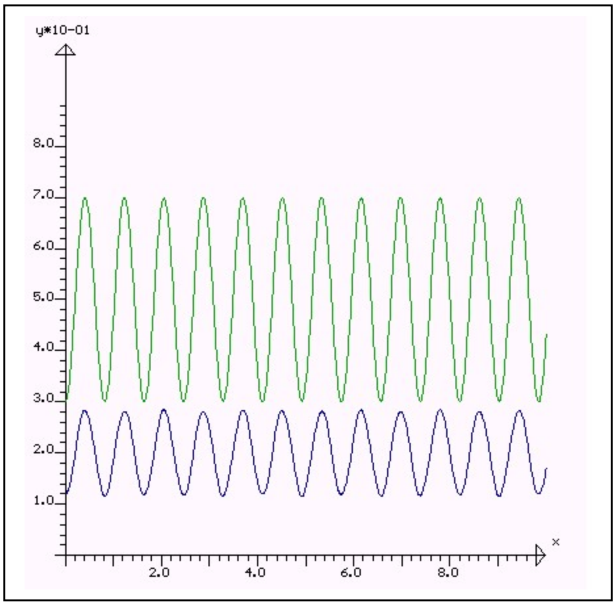
$$F_2 = m_2\ddot{x}_2 = k_2(l_2 - (x_2 - x_1)) - k_3(l_3 - (l - x_2))$$

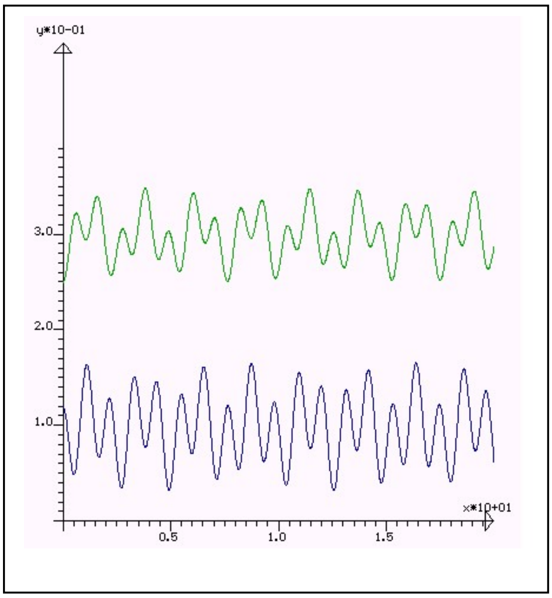
It is hardly possible to establish any analytic solutions to these coupled differential equations, so we settle for a graphic computer solution, using some suitable constants for the parameters of the system. First we have to reduce the equations a bit.

$$m_1\ddot{x}_1 = -(k_1 + k)x_1 + k_2x_2 + k_1l_1 - k_2l_2 \quad \Leftrightarrow$$
$$\ddot{x}_1 = -\frac{(k_1 + k)}{m_1}x_1 + \frac{k_2}{m_1}x_2 + \frac{k_1l_1 - k_2l_2}{m_1}$$

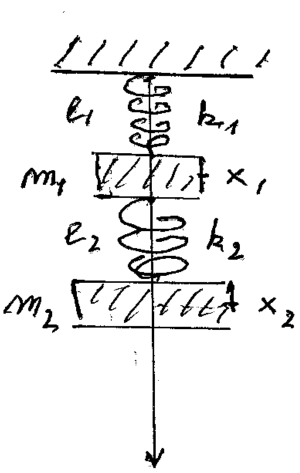
$$m_2\ddot{x}_2 = -(k_2 + k_3)x_2 + k_2x_1 + k_2l_2 - k_3(l_3 - l) \quad \Leftrightarrow$$
$$\ddot{x}_2 = -\frac{(k_2 + k_3)}{m_2}x_2 + \frac{k_2}{m_2}x_1 + \frac{k_2l_2 - k_3(l_3 - l)}{m_2}$$

Below is shown some computer generated solution, the first one with a rather weak coupling, and the second with a stronger coupling. The latter having a strong deviation from a pure harmonic oscillation.





5. Vertically coupled harmonic oscillations



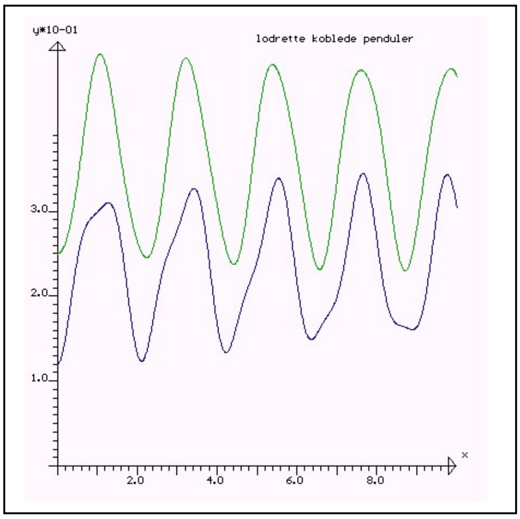
This system is simpler, compared to the previous one, because there are only two springs instead of three. We therefore establish the equation of motion directly from the acting forces, which are the forces from the springs and gravity.

$$F_1 = m_1\ddot{x}_1 = k_1(l_1 - x_1) - k_2(l_2 - (x_2 - x_1)) + m_1g$$

$$F_2 = m_2\ddot{x}_2 = k_2(l_2 - (x_2 - x_1)) + m_2g$$

The equations are reduced to:

$$
(5.1)\qquad \begin{aligned} m_1\ddot{x}_1 &= -(k_1+k_2)x_1 + k_2x_2 + k_1l_1 - k_2l_2 + m_1g \\ m_2\ddot{x}_2 &= k_2x_1 - k_2x_2 + k_2l_2 + m_2g \end{aligned}
$$



The figure to the left shows a computer generated solution to the equations (5.1)
The drawback with a numerical solution is of course that the parameters of the system can be chosen in an infinite number of ways. We may never reach the grand overview, when making an analysis of an analytic solution.

The deviation from the pure harmonic solution are however less significant, than that of the horizontally coupled oscillations.