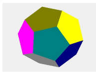
Circular waves and the Bessel function



Ole Witt-Hansen 2018

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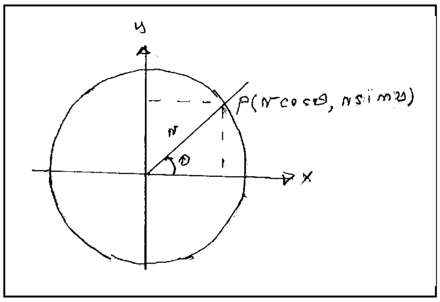
1. The wave equation

In this article, we shall look into the physics of *plane* circular waves based on the wave equation:

(1.1) $$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \quad \Leftrightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

where u is a scalar field.

2. Polar coordinates in the plane and space



The polar coordinates in the plane are usually chosen as (r,θ) as shown in the figure: We have:

(2.1) $$x = r\cos\theta \quad and \quad y = r\sin\theta$$

The infinitesimal distance vector is seen geometrically to be: $d\vec{s} = (rd\theta, dr)$

Since the direction of $rd\theta$ *and* dr are orthogonal the square of the distance element is.

(2.2) $$ds^2 = r^2 d\theta^2 + dr^2$$

And the area element dA is:

(2.3) $$dA = rd\theta dr$$

These results obtained geometrically, may also be formally proven, by making a coordinate transformation.

$$dx = \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial r}dr = -r\sin\theta d\theta + \cos\theta dr$$

$$dy = \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial r}dr = r\cos\theta d\theta + \sin\theta dr$$

$$ds^2 = dx^2 + dy^2 = (-r\sin\theta d\theta + \cos\theta dr)^2 + (r\cos\theta d\theta + \sin\theta dr)^2 \quad \Rightarrow$$

(2.4) $$ds^2 = r^2 d\theta^2 + dr^2$$

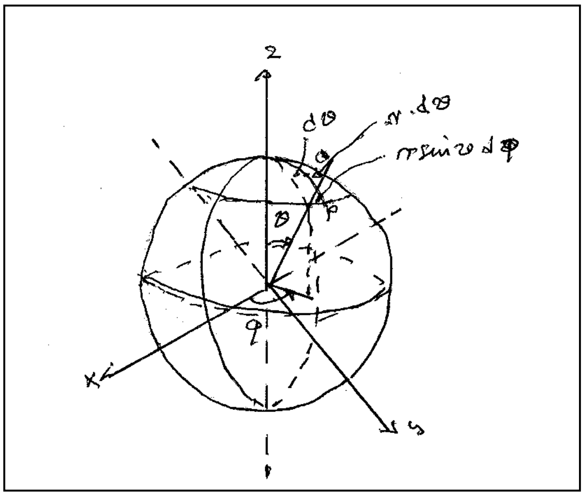
Since we shall consider a rotational symmetric wave, we shall begin by rewriting the Laplace operator in polar coordinates.

For general *orthogonal* curvilinear coordinates in the plane (p_1, p_2) or in space (p_1, p_2, p_3), the distance element is $d\vec{s} = (g_1 dp_1, g_2 dp_2)$ in the plane, and in space is $d\vec{s} = (g_1 dp_1, g_2 dp_2, g_3 dp_3)$ - Here the g*'s* are functions of the coordinates (p_1, p_2) or (p_1, p_2, p_3).

If the curvilinear coordinates are orthogonal we have:

(2.5) $ds^2 = g_1{}^2 dp_1{}^2 + g_2{}^2 dp_2{}^2$ and $ds^2 = g_1{}^2 dp_1{}^2 + g_2{}^2 dp_2{}^2 + g_3{}^2 dp_3{}^2$

In *space* a point $P(x, y, z)$ has the polar coordinates (r, θ, φ). See figure below. θ is the polar angle and φ is the azimuth angle.
The coordinates are, when expressed by the two angles.



(5.1) $x = r\sin\theta\cos\varphi,\ \ y = r\sin\theta\sin\varphi, z = r\cos\theta$

To come from one point $P\ (r,\theta,\varphi)$ to another $Q(r+dr,\ \theta + d\theta,\ \varphi + d\varphi)$, you first go "horizontally" $r\sin\theta\ d\varphi$ then "vertically" the distance $rd\theta$, and finally the radial distance dr.
The infinitesimal distance vector therefore becomes

(5.2) $d\vec{s} = (r\sin\theta d\phi,\ rd\theta,\ dr)$

Since the three "steps" are perpendicular to each other, the square of the distance is:

(5.3) $ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 + dr^2$

Likewise the infinitesimal volume element is the product of the three orthogonal steps.

(2.6) $dV = r^2\sin\theta d\theta d\varphi dr$

The gradient in the *plane* is defined as:

(2.7) $\vec{\nabla} u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$

The significance of the gradient is: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \vec{\nabla} u \cdot (dx, dy)$

That is, the infinitesimal increase in u resulting from the displacement (dx, dy).

The infinitesimal displacement along the curvilinear coordinates is in the plane $d\vec{s} = (g_1 dp_1, g_2 dp_2)$ and in space: $d\vec{s} = (g_1 dp_1, g_2 dp_2, g_3 dp_3)$, dx is replaced by $g_1 dp_1$ and so on, and therefore the gradient of a scalar field U in plane curvilinear becomes.

$$\vec{\nabla} U = (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}) \quad \rightarrow \quad \vec{\nabla} U = (\frac{1}{g_1}\frac{\partial U}{\partial p_1}, \frac{1}{g_2}\frac{\partial U}{\partial p_2})$$

And in space:

(2.8) $$\vec{\nabla} U = (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}) \quad \rightarrow \quad \vec{\nabla} U = (\frac{1}{g_1}\frac{\partial U}{\partial p_1}, \frac{1}{g_2}\frac{\partial U}{\partial p_2}, \frac{1}{g_3}\frac{\partial U}{\partial p_3})$$

Since $d\vec{s} = (rd\theta, dr)$ in polar coordinates in the plane we have:

(2.9) $$\vec{\nabla} U = (\frac{1}{r}\frac{\partial U}{\partial \theta}, \frac{\partial U}{\partial r})$$

And in polar coordinates in space: $d\vec{s} = (r\sin\theta d\varphi, rd\theta, dr)$

(2.10) $$\vec{\nabla} U = (\frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}, \frac{1}{r}\frac{\partial U}{\partial d\theta}, \frac{\partial U}{\partial r})$$

Concerning the divergence, things become somewhat more complicated: This comes about because the lines inclosing the "rectangular" square or box in curvilinear coordinates the area of the front end and the back end are not necessarily equal.

In space the areas of a box with sides (1-2) (1-3) and (2-3) are $\Delta A_{23} = g_2 g_3 \Delta p_2 \Delta p_3$, $\Delta A_{13} = g_1 g_3 \Delta p_1 \Delta p_3$, and $\Delta A_{12} = g_1 g_2 \Delta p_1 \Delta p_2$, so in the calculation of the "change of flux", along one of the coordinates in a Cartesian coordinate system: $\frac{\partial u_x}{\partial x}\Delta x \Delta y \Delta z$ must be replaced by $\frac{1}{g_1}\frac{\partial}{\partial p_1}(g_2 g_3 u_1) g_1 \Delta p_1 \Delta p_2 \Delta p_3 = \frac{\partial}{\partial p_1}(g_2 g_3 u_1)\Delta p_1 \Delta p_2 \Delta p_3$, because $g_2 g_3$ may depend on p_1.

The divergence of a vector field, however, is defined as the flux out of the volume dV: $\int \vec{\nabla}\cdot \vec{u} dV$

The flux in curvilinear coordinates calculated in this manner from a "rectangular" box is, however:

(2.11) $$\begin{aligned}&\int (\frac{\partial}{\partial p_1}(g_2 g_3 u_1) + \frac{\partial}{\partial p_2}(g_1 g_3 u_2) + \frac{\partial}{\partial p_3}(g_1 g_2 u_3)) dp_1 dp_2 dp_3 = \\ &\int \frac{1}{g_1 g_2 g_3}(\frac{\partial}{\partial p_1}(g_2 g_3 u_1) + \frac{\partial}{\partial p_2}(g_1 g_3 u_2) + \frac{\partial}{\partial p_3}(g_1 g_2 u_3)) g_1 dp_1 g_2 dp_2 g_3 dp_3\end{aligned}$$

Since $dV = g_1 dp_1 g_2 dp_2 g_3 dp_3$ is the correct expression for the volume element, the divergence becomes.

(2.12) $$\vec{\nabla}\cdot\vec{u} = \frac{1}{g_1 g_2 g_3}\left(\frac{\partial}{\partial p_1}(g_2 g_3 u_1) + \frac{\partial}{\partial p_2}(g_1 g_3 u_2) + \frac{\partial}{\partial p_3}(g_1 g_2 u_3)\right)$$

In the *plane* we get similarly:

(2.13) $$\vec{\nabla}\cdot\vec{u} = \frac{1}{g_1 g_2}\left(\frac{\partial}{\partial p_1}(g_2 u_1) + \frac{\partial}{\partial p_2}(g_1 u_2)\right)$$

In polar coordinates in the plane $d\vec{s} = (rd\theta, dr)$, we find:

(2.14) $$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\left(\frac{\partial u_1}{\partial\theta} + \frac{\partial}{\partial r}(ru_2)\right) = \frac{1}{r}\frac{\partial u_1}{\partial\theta} + \frac{1}{r}\frac{\partial(ru_2)}{\partial r}$$

And for the Laplace operator in the plane we find:

$$\nabla^2 U = \vec{\nabla}\cdot\vec{\nabla}U = \vec{\nabla}\cdot(\frac{1}{r}\frac{\partial U}{\partial\theta}, \frac{\partial U}{\partial r}) = \frac{1}{r}\frac{\partial}{\partial\theta}(\frac{1}{r}\frac{\partial U}{\partial\theta}) + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial U}{\partial r})$$

(2.15) $$\nabla^2 U = \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial U}{\partial r})$$

The wave equation $\nabla^2 U = \frac{1}{v^2}\frac{\partial^2 U}{dt^2}$, where v is the speed of propagation is in polar coordinates:

(2.16) $$\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial U}{\partial r}) = \frac{1}{v^2}\frac{\partial^2 U}{dt^2}$$

3. Solving the wave equation for plane waves.

Assuming that the time dependence is independent of r and θ, we write: $U(r,\theta,t) = u(r,\theta)\psi(t)$. Inserting in (2.16) and subsequently dividing by $u\psi$, we have:

(3.1) $$\psi\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \psi\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = u\frac{1}{v^2}\frac{\partial^2\psi}{dt^2} \quad\Leftrightarrow$$
$$\frac{1}{u}(\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})) = \frac{1}{v^2}\frac{1}{\psi}\frac{\partial^2\psi}{dt^2}$$

In this equation the left side depends only on r and θ, whereas the right side depends only on t. But that means that they are both equal to the same constant, which we for physical reasons put to $-k^2$, where k is the wave number: $k = \frac{\omega}{v}$. So we have the two equations:

(3.2) $$\frac{1}{v^2}\frac{1}{\psi}\frac{\partial^2\psi}{dt^2} = -k^2 \quad\Leftrightarrow\quad \frac{\partial^2\psi}{dt^2} = -k^2 v^2\psi \quad\Leftrightarrow$$
$$\psi = Ae^{ikvt} \quad\Leftrightarrow\quad \psi = Ae^{i\omega t}$$

(3.3) $$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = -k^2 u \quad\Leftrightarrow\quad \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + k^2 u = 0$$

If we further assume that $u = u(r, \theta)$ does not depend explicitly on θ, such that: $\frac{\partial u}{\partial\theta} = 0$, we have the equation:

(3.4) $$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + k^2 u = 0 \quad \Leftrightarrow \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + k^2 u = 0$$

If we put $\rho = kr$ we can eliminate the constant *k*.

(3.5) $$k^2 \frac{\partial^2 u}{\partial \rho^2} + k^2 \frac{1}{r}\frac{\partial u}{\partial \rho} + k^2 u = 0 \quad \Leftrightarrow$$
$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{r}\frac{\partial u}{\partial \rho} + u = 0$$

The solution to this differential equation cannot be expressed with already known functions, but the solution is called the Bessel function of zero order and is denoted $J_0(\rho)$ or $J_0(x)$.
We can see that if $J_0(\rho)$ is a solution to (3.5) , then $J_0(kr)$ is a solution to (3.4).

Shifting the independent variable from ρ to *x*, and the dependent variable from *u* to *f* , we can making a power series expansion of *f*:

(3.6) $$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad .$$

Inserting this expression in the differential equation: $\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x} + f = 0$, collecting terms with the same power of *x*, we find in the following steps:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + ...$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + ...$$

$$\frac{f'(x)}{x} = \frac{a_1}{x} + 2a_2 + 3a_3 x + 4a_4 x^2 + 5a_5 x^3 + 6a_6 x^4 + ...$$
$$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + ...$$

We shall assume that $J_0(0) = 1 \quad and \quad J_0'(0) = 0$

Inserting in the differential equation:
$$2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + ... \frac{a_1}{x} + 2a_2 + 3a_3 x + 4a_4 x^2 + 5a_5 x^3 + 6a_6 x^4 + ...$$
$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + ... = 0$$

We shall multiply this equation with *x*.

$$2a_2 x + 6a_3 x^2 + 12a_4 x^3 + 20a_5 x^4 + 30a_6 x^5 + ... a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + ...$$
$$a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + a_5 x^6 + a_6 x^7 + ... = 0$$

The condition $J_0(0)=1$ gives $a_0 = 1$, and from above we see that $a_1 = 0$. We then collect the terms with x.

$$(2a_2+2a_2+a_0)x=0 \quad \Rightarrow \quad a_2=-\tfrac{1}{4}$$

$$(6a_3+3a_3+a_1)x^2=0 \quad \Rightarrow 9a_3=0 \quad \Leftrightarrow \quad a_3=0$$

$$(12a_4+4a_4+a_2)x^3=0 \quad \Rightarrow \quad a_4=\tfrac{1}{64}$$

Continuing we get the first 4 terms in the series expansion of the Bessel function of zero order.

(3.7) $$J_0(x)=1-(\tfrac{x}{2})^2+\tfrac{1}{2!^2}(\tfrac{x}{2})^4-\tfrac{1}{3!^2}(\tfrac{x}{2})^6+\ldots$$

On the other hand, if $u(r,\theta)$ is not independent of θ, we have the differential equation:

(3.8) $$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+k^2u=0$$

We replace $u(r,\theta)$ with $u_n(r)e^{in\theta}$, finding:

$$\frac{1}{r^2}u_n(-n^2)e^{in\theta}+e^{in\theta}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+e^{in\theta}k^2u=0 \qquad \Leftrightarrow$$

(3.9) $$\frac{\partial^2 u_n}{\partial r^2}+\frac{1}{r}\frac{\partial u_n}{\partial r}+(k^2-\frac{n^2}{r^2})u_n=0$$

Or when putting $x=kr$:

(3.10) $$\frac{\partial^2 u_n}{\partial x^2}+\frac{1}{x}\frac{\partial u_n}{\partial x}+(1-\frac{n^2}{x^2})u_n=0$$

Let $J_n(x)$ denote the solution of this differential equation. $J_n(x)$ is then denoted the Bessel function of order n. The series expansions for $J_n(x)$ turn out to be:

(3.11) $$J_n(x)=\tfrac{1}{n!}(\tfrac{x}{2})^n-\tfrac{1}{1!(n+1)!}(\tfrac{x}{2})^{n+2}+\tfrac{1}{2!(n+2)!}(\tfrac{x}{2})^{n+4}-\ldots$$

We shall only apply the zero order Bessel function in the following, but below we state some theorems for the Bessel functions, (without proof).

(3.12) $$J_1(x)=-\frac{d}{dx}J_0(x)$$ (Can be verified, by applying (3.11)

(3.13) $$xJ_0(x) = \frac{d}{dx}(xJ_1(x))$$ (Can be verified, by applying (3.11)

This relation integrated to an arbitrary upper limit, yields the following formula.

$$\int_0^{x_0} xJ_0(x)dx = x_0 J_1(x_0)$$

Below are shown the graphs of the first three Bessel functions

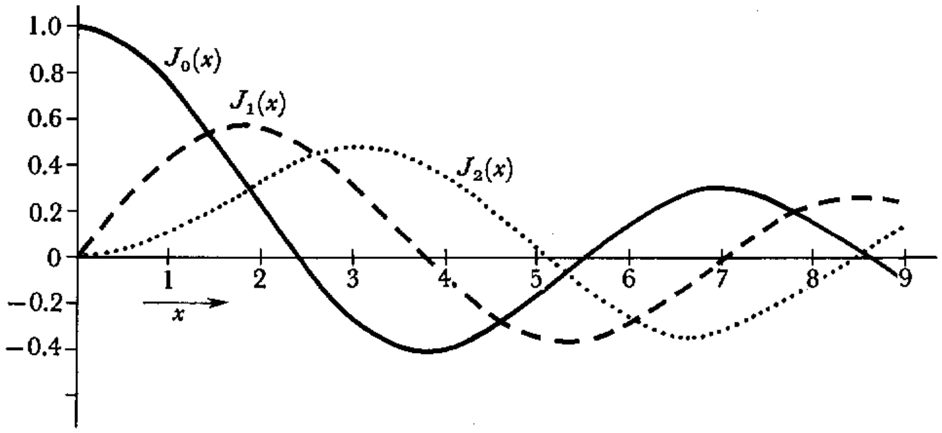


Figure 7–1 The Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$

4. The drum head

Assuming that we have a circular drum head having radius *R*, and the drum skin has the elastic velocity of propagation *v*, and assuming that the solution has no explicit dependence on θ, then the solution, (as demonstrated above), will obey the following differential equations.

$$\frac{1}{v^2}\frac{1}{\psi}\frac{\partial^2\psi}{dt^2} = -k^2 \quad \Leftrightarrow \quad \frac{\partial^2\psi}{dt^2} = -k^2v^2\psi \quad \Leftrightarrow$$

$$\psi = Ae^{ikvt} \quad \Leftrightarrow \quad \psi = Ae^{i\omega t}$$

And

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + k^2 u = 0$$

The last equation having the solution $J_0(kr)$.

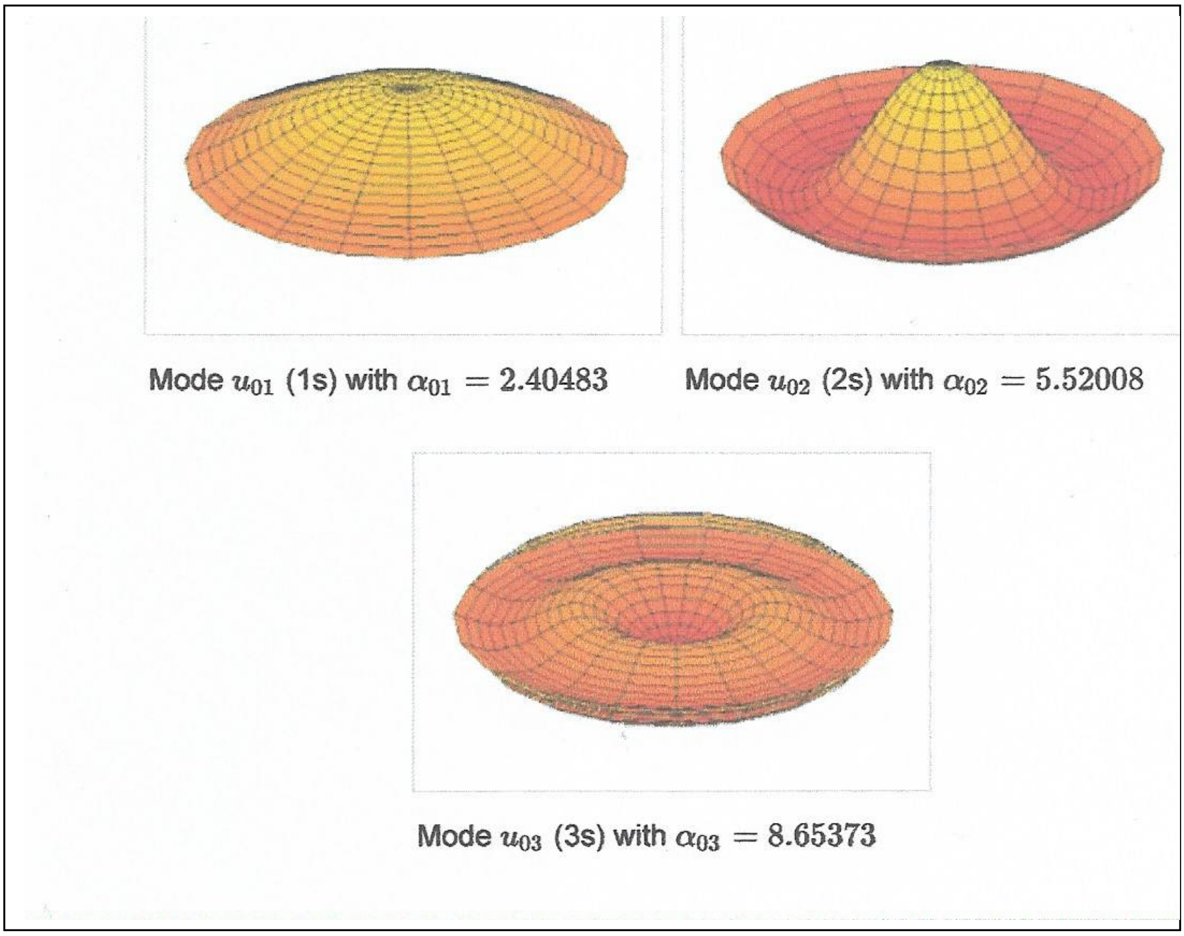
The wave number is $k = \frac{\omega}{v}$, so that the wave number is proportional to ω.

Any solution must obey the boundary condition $J_0(kR) = 0$, since the oscillations must vanish, at the edge.

The first three zero points of $J_0(r)$ occur at: $r = 2.40$, $r = 5.52$ and $r = 8.65$.

If $R = 0.10\ m$, then it gives: $kR = 2.40$, $kR = 4.52$, and $kR = 8.65$, resulting in the values for the wave number: $k = 24.0\ m^{-1}$, $k = 45.2\ m^{-1}$ and $k = 86.5\ m^{-1}$.

The first three modes are shown below in a 3-dim mapping. (Borrowed from Wikipedia: “Vibrations of a circular membrane”)



Mode u_{01} (1s) with $\alpha_{01} = 2.40483$ Mode u_{02} (2s) with $\alpha_{02} = 5.52008$

Mode u_{03} (3s) with $\alpha_{03} = 8.65373$

Assuming that the modes correspond to standing waves having: $R = \frac{1}{4}\lambda$, $R = \frac{3}{4}\lambda$ and $R = \frac{5}{4}\lambda$, the corresponding wavelengths are 0.40 *m*, 0.133 *m* and 0.80 *m*.

If the speed of propagation in the drum skin is 100 *m/s*, we find for the frequencies:
$\nu = \frac{v}{\lambda}$ = 250 *Hz*, 750 *Hz* and 1250 *Hz*, which does not seem unreasonable, but we have really no way of estimating the speed of propagation, without measuring the frequency.