Chaotic motion
A dynamic analysis

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Chaotic motion

Chaotic motion appears frequently in a dynamic system, where the equations of motion are non linear second order differential equations.
This phenomena has already been studied by Poincaré (in his study of the three body problem) in the end of the seventeen century, but the interest in "chaos theory" and "fractal structures" (also outside the scientific society) virtually exploded, as a consequence of the articles of Mandelbroot and Feigenbaum in the mid 80'ties.
What they discovered (but as most mathematicians already knew) was that a even very small alteration in the initial conditions of non linear equation of motion may in sight lead to a radically different solution of a dynamic system.
The statement that: "If a butterfly barks with its wings in Brazil, it may eventually lead to a hurricane in the Caribbean", was repeated almost endlessly in the next 15 years.
This statement should by no means be taken literally, but it is more apt to illustrate the core of a chaotic development.
The notion of chaotic development was introduced (and maybe earlier) by Steven Spielberg in his movie Jurassic Park from 1993, and it has played a role (but less successful) in several other movies. Spielberg demonstrated in his movie that minor incidents, sloppiness, oversights and "harmless" sabotage may lead to a chaotic development.
Spielberg's movie, however, had nothing to do whatever with the mathematics of chaos theory.

In the years 1985 – 2000, chaos theory, fractals (and dinosaurs) was something that occupied not only the scientific world, but everyone from public school to professors at the universities. Especially the creation of the wondrous fractal Mandelbroot set, which every student in science having a graphic computer could create.
The chaos theory became a virtual hype, and was introduced in almost all branches of sciences and social live, like the stock market, physics, economics, psychology, war strategy the interaction between spices (also humans) as a widespread notion.

I was teaching physics, mathematics and computer science in that period, and it was almost mandatory that I also gave lectures in chaos theory. Although I also made the Mandelbrot set as computer graphics, I was never really convinced that chaos theory had any impact apart from entertainment.
The interest in chaos theory declined dramatically after the year 2000, because lack of specific theoretical results in the fields, where it was applied.

The purpose of this article, however, is not to go into the mathematics of chaos theory, but to give a simple example of how an chaotic motion can be analyzed using classical physics.
The non linear equations of motion cannot be solved analytically, but since the beginning of the 1990'ties it has been possible to give a graphic representation to the numerical solution of the equations of motion that lead to chaotic, that is, non periodic motion.
Another (and more complex) example is presented in:
www.olewitthansen.dk /Physics/Coupled_pendulums_and_coupled_harmonic_oscillations.pdf

This example is a gadget, shown in the front page, which I acquired some 25 years ago.

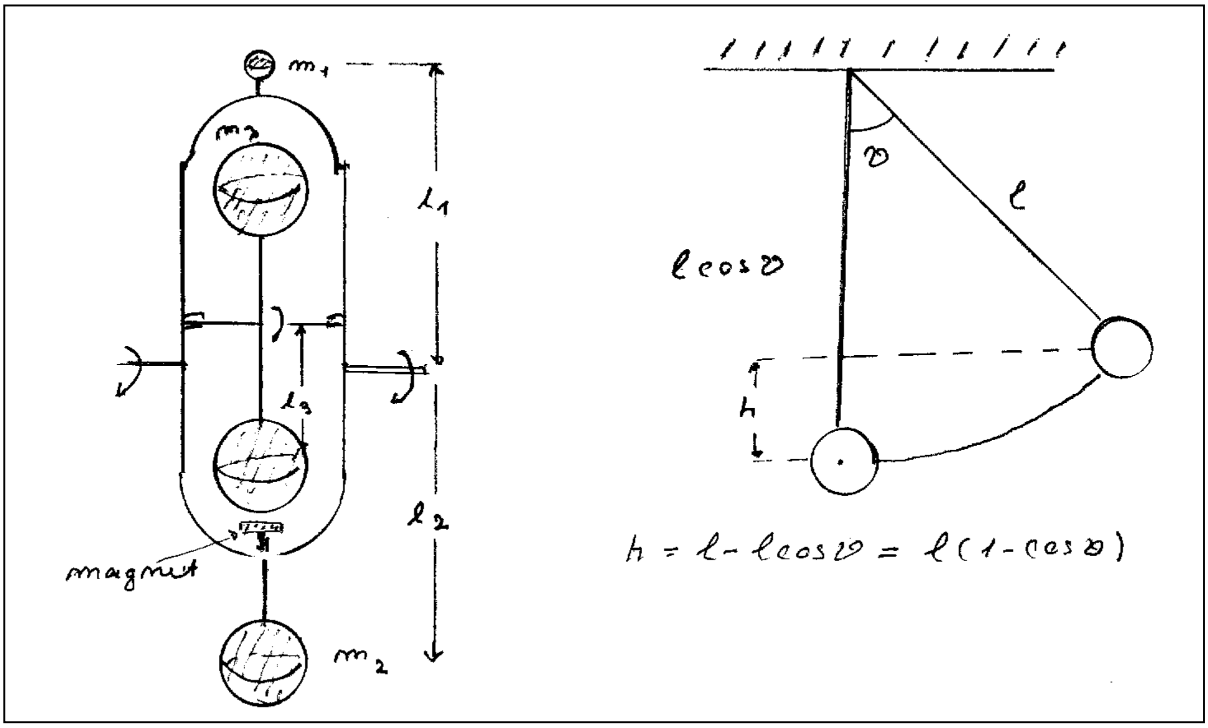
It consists of a frame which a heavy bullet at the bottom, and a minor bullet at the top, acting as a physical pendulum. In the middle of the frame a rod having two light bullets can move freely around a horizontal axis.
Above the heavy bullet is placed a magnet with the north pole pointing upward, and likewise a north pole is placed in the rear end of the two rotating bullets. When one of the bullets are in the bottom position, it is repelled from the magnet.
When the apparatus is set into motion the resulting motion appears chaotic.

It is not possible to establish a correct analytic expression for the magnetic repulsion between the magnet and the bullet, but this is not necessary, since any repulsion will lead to a chaotic motion of the rotating bullets. The only important thing is that the force decreases fast (quadratic?) with the distance between the two magnets.

The angle of rotation for the frame (physical pendulum) is denoted θ and angle of rotation for the rotating bullets is denoted φ.
We have chosen that the magnetic force to be proportional to $\cos(\theta-\varphi)$.
The force then has a max when $\theta-\varphi=0$, that is when a bullet passes the magnet, and the force is zero when $\theta-\varphi=\frac{1}{2}\pi$. A max (with opposite sign) is however also reached when $\theta-\varphi=\pi$ as the other bullet is above the magnet, but since it comes from the other side, it will again be a repulsion: We therefore put:

(1.1) $$F_{mag}=\mu\cos(\theta-\varphi)$$

If the force is conservative: $F_{mag}=-\frac{\partial U_{mag}}{\partial\theta}$ then $U_{mag}=-\mu\sin(\theta-\varphi)+c$.
This expression for the potential energy will be applied below.
Below is shown a schematic drawing of the apparatus together with a pendulum.



The notation used in the analysis can be read from the figure.

Our aim is not really to make a correct description of the actual motion (which also would be impossible to verify), but rather to demonstrate how the Lagrange formalism may be used to establish the equation of motion.
The development of the Lagrange formalism is treated in some detail in the article:
www.olewitthansen.dk /Mathematics/Calculus_of_Variations.pdf

To make use of the Lagrange equations, we must first establish an expression for the kinetic energy *T* and the potential energy *U* of the system. The Lagrange function is then:

(1.2) $$L = T - U$$

The equations of motion is expressed in generalize coordinates q_i , where a dot above a variable as usual means differentiation with respect to time. The Lagrange equations of motion are then:

(1.3) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

The kinetic energy is the kinetic energy of the three bullets, and the potential energy has two contributions one from gravity acting on the frame, which consists of two masses m_1 and m_2, and one from the magnetic repulsion.

(1.4)
$$\begin{aligned}
&E_{kin}(1) = \tfrac{1}{2}m_1 l_1^{\,2}\dot{\theta}^2 \qquad E_{kin}(2) = \tfrac{1}{2}m_2 l_2^{\,2}\dot{\theta}^2 \qquad E_{kin}(3) = 2\tfrac{1}{2}m_3 l_3^{\,2}\dot{\varphi}^2 \quad \Rightarrow \\
&T = E_{kin}(1) + E_{kin}(2) + E_{kin}(3) = \tfrac{1}{2}(m_1 l_1^{\,2} + m_2 l_2^{\,2})\dot{\theta}^2 + m_3 l_3^{\,2}\dot{\varphi}^2 \\
&U = m_2 g l_2(1-\cos\theta) + m_1 g l_1(1-\cos(\theta+\pi)) - \mu\sin(\theta-\varphi) \\
&U = (m_2 g l_2 - m_1 g l_1)(1-\cos\theta) - \mu\sin(\theta-\varphi) + 2m_1 g l_1 \\
&U = (m_2 g l_2 - m_1 g l_1)(1-\cos\theta) - \mu\sin(\theta-\varphi)
\end{aligned}$$

The last omission of the term $2m_1 g l_1$ is caused by the fact that the potential energy is only determined apart from a constant, whish vanish when differentiating. We write hereafter, with an abbreviation of the constant factors.

(1.5)
$$\begin{aligned}
&T = \tfrac{1}{2}(m_1 l_1^{\,2} + m_2 l_2^{\,2})\dot{\theta}^2 + m_3 l_3^{\,2}\dot{\varphi}^2 = k_1\dot{\theta}^2 + k_3\dot{\varphi}^2 \\
&U = k_4(1-\cos\theta) - \mu\sin(\theta-\varphi)
\end{aligned}$$

Then we establish the Lagrange equations of motion.

(1.6)
$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \Leftrightarrow \quad 2k_1\ddot{\theta} + k_4\sin\theta - (-(-\mu\cos(\theta-\varphi))) = 0 \quad \Leftrightarrow \\
&2k_1\ddot{\theta} + k_4\sin\theta - \mu\cos(\theta-\varphi) = 0
\end{aligned}$$

$$(1.7)\qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \quad \Leftrightarrow$$
$$2k_3\ddot{\varphi} + \mu\cos(\theta - \varphi) = 0$$

The problem is then reduced to solving two relatively simple coupled differential equations.

$$(1.8)\qquad \ddot{\theta} = -\frac{k_4}{2k_1}\sin\theta + \frac{\mu}{2k_1}\cos(\theta - \varphi)$$
$$\ddot{\varphi} = -\frac{\mu}{2k_3}\cos(\theta - \varphi)$$

The equations will be solved numerically and displayed graphically with a Turbo Pascal 7 DOS-program, written for that purpose in 1996. The program was originally run on Windows 95, and it could also run on Windows 98.
It can still run on Windows XP, but without the facility of making a screen dump.
After Windows XP, it can no longer run on the Windows platform. (And that's why I have kept both a Windows 98 machine and a Windows XP machine)
There may exist programs which (among other things) may be able to solve up to 6 coupled second order differential equations, but I have never encountered any with the same spectrum of facilities, as my own.

If we measure the masses i gram and the lengths in cm we have:

$$m_1 = 20\,g,\ m_2 = 50\,g,\ m_3 = 10\,g,\ l_1 = 6\,cm,\ l_2 = 7\,cm,\ l_3 = 5\,cm,\ g = 100\,cm/s^2 \quad \mu = 10$$

and

$$k_1 = 1585,\ k_3 = 250,\ \ k_4 = 23000\ ,$$

Dividing all constant by 1000 we have:

$$(1.9)\qquad \ddot{\theta} = -7.25\sin\theta + \frac{\mu}{3.17}\cos(\theta - \varphi)$$

$$(1.10)\qquad \ddot{\varphi} = -\frac{\mu}{0.5}\cos(\theta - \varphi)$$

The results will naturally depend on the value of μ. Experimenting a bit we find that $\mu = 10$ gives results, that you might expect.
On the graph to the left the blue graph represents the vertical position of the frame taken from the bottom position. The deviations from a harmonic oscillation is significant.

The green graph is the vertical position of the one of the rotating bullets. Even if the bullet moves up and down, the deviations from a uniform circular motion is significant. It is clearly chaotic.

On the graph to the left is depicted the angular velocities of the bullet in the frame, and one of the rotating bullets. The motions are clearly chaotic.

The pictures are of low quality! Yes! They are photos taken from a cell phone from a 10’’ screen.

The graph to the right depicts the same motions, but represent the angular velocities of the bottom bullet and one of the rotating bullets.