Bicycle Physics

On gear shifting, force exchange and power

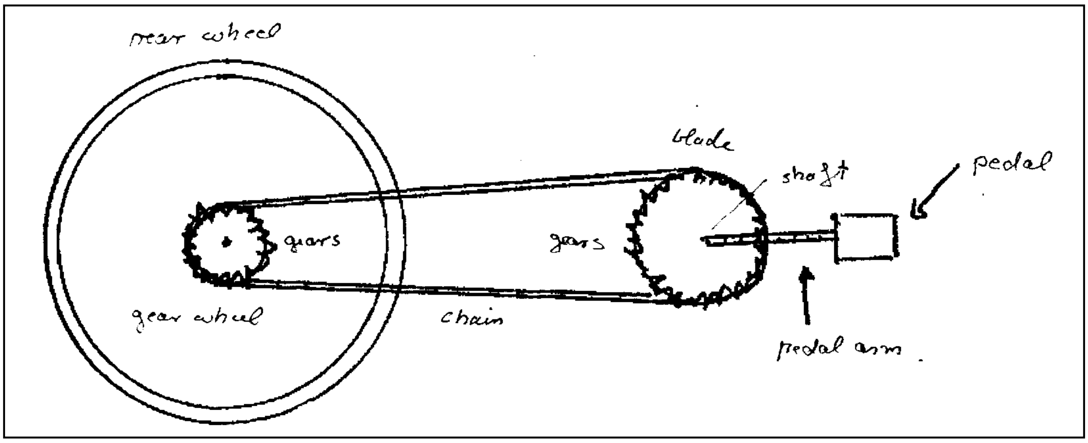
This is an article from my home page: www.olewitthansen.dk

Ole Witt-Hansen 2009

Contents

1. Gear exchange on a bicycle

The shaft on which the pedals are mounted is called the crank. The gears that are mounted on the crank are called blades.
On a bike without a gear or a bike with internal gear there is only one blade. The sprockets which are mounted on the rear shaft are called gearwheels. The sprockets on the blade and the sprockets on the rear wheel are connected by the bike chain.



The radius in the blade is denoted r_{blade}, the radius on gear wheel is r_{gear} and the radius on the rear wheel is r_{wheel}. The number of teeth on the blades is n_{blade}, and the number of teeth on the gearwheel is n_{gear}. The distance between the teeth on the sprockets is the same and is denoted *d.*
Since the perimeter of a circle with radius *r* is $2\pi r$, we have:

(1.1) $$n_{blade}d = 2\pi \cdot r_{blade} \quad and \quad n_{gear}d = 2\pi \cdot r_{gear}$$

When the pedals, and thus the blades, have moved one round, then the chain has moved a distance $s_{blade} = 2\pi\, r_{blade} = n_{blade}d$, and since the blade and the gear wheel are connected with the chain, the gear wheel has moved the same distance *s*.

If the gear wheel has turned N_{gear} rounds then $s_{gear} = N_{gear}\, 2\pi\, r_{\text{gear}} = N_{gear}\, n_{\text{gear}}d$.
The number of rounds that the gear wheel has moved, when the blade has moved one round is then found by setting: $s = s_{blade} = s_{gear}$.

(1.2) $$s = 2\pi\, r_{\text{blade}} = N_{gear}\, 2\pi\, r_{\text{gear}} \quad \Rightarrow \quad N_{gear} = \frac{2\pi \cdot r_{blade}}{2\pi \cdot r_{gear}} = \frac{n_{blade}}{n_{gear}}$$

This ratio of exchange is thus the number of rounds that the gear wheel turns, when the pedals (the blade) turns one round.

To determine the distance that the bike moves when the pedals turn one round, we must multiply N_{gear} by the perimeter of the bicycle wheel, which is $2\pi\, r_{wheel}$.

The distance s_{wheel}, that the wheel and thereby the bike has moved, when the pedals have turned one round is therefore.

(1.3) $$s_{bike} = s_{wheel} = 2\pi\, r_{wheel} \frac{n_{blade}}{n_{gear}}$$

2. Power transfer and work, when riding a bike

We shall then analyze the transmission of power from the pedals to the rear wheel, but first we must clarify some physical conditions.

1. The force that drives the bike forward is the frictional force that the underlay acts on the rear wheel.

2. The reaction force to that force is (according to Newton's 3. law) equal to the force with which the wheel acts on the underlay.

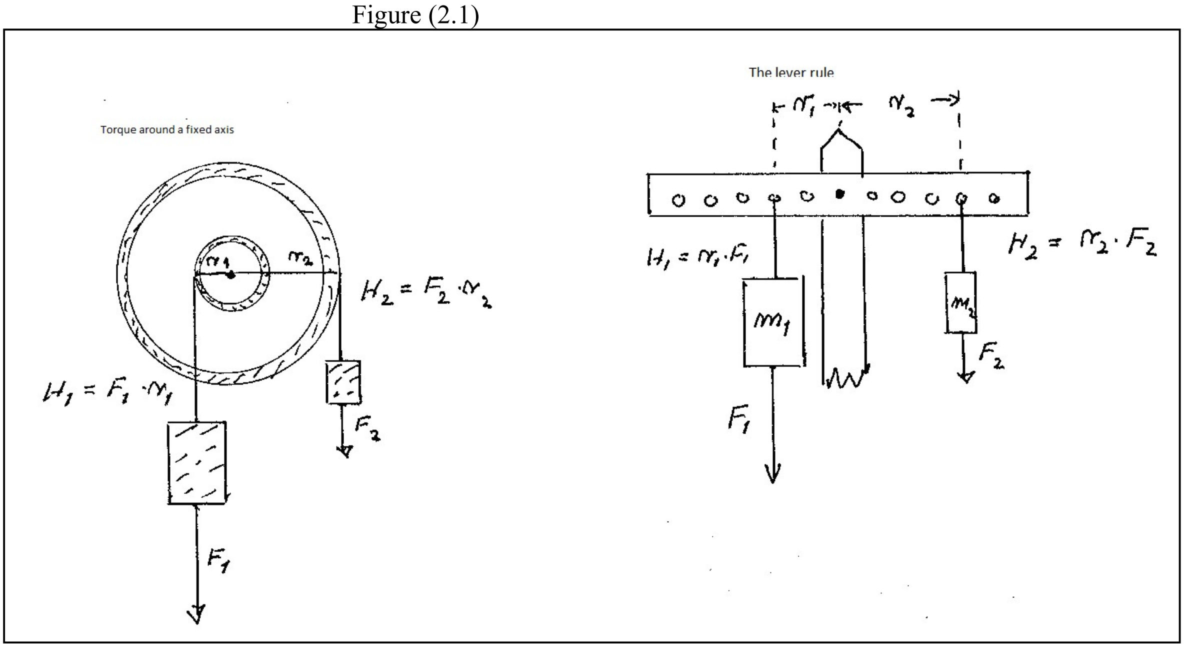
3. The force that drives the bike forward is eventually delivered by the cyclist via the power transmission from the pedals to the chain and to the sprocket gears.

Without physical insight one might think that the force F_p, delivered by the cyclists is the same as the force F_c that drives the bike forward, but that is not at all so.

To make a physical analysis, it is necessary to introduce the concept of *torque* (moment of force).

For a rotation around a fixed axis the concept of *torque* is analogous to the concept of force regarding a linear motion. We shall denote the torque by H, (but it is also written as τ in many textbooks on physics).

Figure (2.1)



The moment of force (torque) is loosely defined as: *Force · lever*. We shall denote the moment of force by *H.* When the force *F* and the lever *r* are orthogonal, it is given by:

(2.1) $$H = F \cdot r$$

In the figure above is shown a disc, which can turn around a frictionless shaft, and tangentially affected by two forces F_1 and F_2 at the distances r_1 and r_2 from the centre (having levers r_1 and r_2).

It shows up, that the disc will be in balance, not when the forces F_1 and F_2 are equal, but when the two moments of force $H_1 = F_1 \cdot r_1$ and $H_2 = F_2 \cdot r_2$ are equal to each other.

This fundamental law is called the *lever rule*.

The lever rule is often demonstrated in class with a device shown in the figure (2.1) to the right, where a symmetric lever is balanced on an axis.

If weights m_1 and m_2 with gravity $F_1 = m_1 g$ and $F_2 = m_2 g$ are placed in different distances r_1 and r_2 from the axis, then the condition of balance is again given by the lever rule.

(2.2) $$F_1 \cdot r_1 = F_2 \cdot r_2$$

For a linear motion it is, as you know, the force given by Newton's 2. law $F = ma$, which determines the acceleration of a body.

If s is the position of the body, then the velocity v and the acceleration are given by:

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt}$$

For a rotation around an axis, the angle of rotation φ is analogous to the position s, the angular velocity $\omega = \frac{d\varphi}{dt}$ is analogous to the velocity and the angular acceleration $\alpha = \frac{d\omega}{dt}$ is analogous to the acceleration for a linear motion.

For the rotation around a fixed axis it is the *moment of force* and not the force that determines the angular acceleration.
We shall, however, not go deeper into the theory of rotation, since it is a rather comprehensive subject, but a thorough review of the subject can be found in:

www.olewitthansen.dk: The theory of rotation.

If the lever of the pedal has the size r_{pedal} , and a force F_{pedal} is acting on the pedal, then the blade gets a moment of force $H_{pedal} = r_{pedal} \cdot F_{pedal}$.

The force with which the blade is affected can then be calculated from:

(2.3) $$H_{pedal} = H_{blade} \quad \Leftrightarrow \quad r_{pedal} \cdot F_{pedal} = r_{blade} \cdot F_{blade} \quad \Rightarrow \quad F_{blade} = \frac{r_{pedal}}{r_{blade}} F_{pedal}$$

If the sprocket chain is free from friction (as we shall assume), then this force will be the same as the force acting on the gear wheel and it induces the force that drives the rear wheel forward.

(2.4) $F_{gear} = F_{blade}$ => $H_{gear} = r_{gear}F_{gear} = r_{gear}F_{blade} = r_{gear}\dfrac{r_{pedal}}{r_{blade}}F_{pedal}$

It is this moment of force that drives the rear wheel forward

To find the force F_{wheel} that the wheel acts on the underlay, we only need to express that moment of force on the wheel is the same as the moment of force on the gear:

(2.5) $$H_{wheel} = H_{gear} \quad \Rightarrow \quad r_{wheel}F_{wheel} = r_{gear}F_{gear}$$

To get an expression for F_{wheel}, we just need to insert the obtained expression for $H_{gear} = r_{gear}F_{gear}$ in (2.4) and divide with r_{wheel}. Then we obtain the expression:

(2.5) $$F_{wheel} = \frac{r_{gear}}{r_{blade}}\frac{r_{pedal}}{r_{wheel}}F_{pedal} \quad \Leftrightarrow \quad F_{wheel} = \frac{n_{gear}}{n_{blade}}\frac{r_{pedal}}{r_{wheel}}F_{pedal} \quad \Leftrightarrow \quad F_{pedal} = \frac{n_{blade}}{n_{gear}}\frac{r_{wheel}}{r_{pedal}}F_{wheel}$$

The second expression comes about, because the radius according to (1.1) is proportional to the number of teeth in the sprocket, and the distance between the teeth are the same on the blade-sprocket and the gear-sprocket.

The radii in the pedal lever and in the rear wheel are independent of which gear you are using, so the force on the rear wheel is (not surprisingly) proportional to the force acting on the pedal having a constant of proportionality, which is ratio of exchange in the transmission of the force.

3. The work done when biking

First we wish to calculate the work done, when the rear wheel is turned one round, and the force between the tire and the underlay is held constant.
When the wheel has turned one round, and the backward force is F_{wheel} , then the biker has performed a work, calculated from: $W=F\cdot s$. (Work = Force· Displacement)

(3.1) $$2\pi r_{wheel}F_{wheel} = 2\pi r_{wheel}\frac{n_{gear}}{n_{blade}}\frac{r_{pedal}}{r_{hjul}}F_{pedal} = 2\pi\frac{n_{gear}}{n_{blade}}r_{pedal}F_{pedal}$$

Now $\dfrac{n_{gear}}{n_{blade}}$ is the number of rounds the blade has turned, when the gear wheel (and thus the rear wheel) has done one round and therefore $\dfrac{n_{gear}}{n_{blade}}2\pi r_{pedal} = s_{pedal}$ is the distance that the pedals has completed. If the left side of the equation is inserted in (3.1), we find a formula for the pedal work.

(3.2) $$Pedal\ Work = 2\pi r_{wheel}F_{wheel} = 2\pi\frac{n_{gear}}{n_{blade}}r_{pedal}F_{pedal} = F_{pedal}s_{pedal}$$

The last expression seems to be entirely trivial (and it is) but it is a important (but not surprising) theoretical result that the work done by turning the rear wheel one round is independent of the gearing chosen.

Physiologically there is, however, a great difference, since static work (slow movement, using a high gearing) is far more strenuous than dynamic work (rapid movement, using a low gearing)

4. Numerical examples for a racing bike

The formulas derived above can be illustrated, by using data from a racing bike, and I have used my own bike as an example. This bike has 3 sprockets on the blade and 7 sprockets one the gear wheel.

On the blade, the number of teeth is: n_1 =30, n_2 =42, n_3 =52.

On the gear wheel, the number of teeth is: n_1 =24, n_2 =22, n_3 =20, n_4 =18, n_5 =16, n_6 =14, n_7 =13.

The diameter on the largest of the sprockets on the blade is $d_{blade} = 0.21\ m$.

All other diameters (if necessary) can be found by taking the ration between the numbers of teeth, since the diameter is directly proportional to the number of teeth.

The Pedal lever is $r_{pedal} = 0.19\ m$.
Radius in the rear wheel is: $r_{wheel} = 0.34$ m.

I have learned that a professional bike rider contribute with a power of about 200 W, but for an ordinary biker it is rather 100 W.
Having a speed of v =18 $km/$h =5.0 m/s the necessary force to move the cyclist can be found from the equation: $Fv=P$
For a common cyclist, we shall examine the force on the pedals for various gearing, at a speed of 18 km/h = 5.0 m/s.
First the power delivered by the rear wheel. From:

$$P = F_{wheel}\, v \qquad \text{it follows} \qquad F_{wheel} = \frac{P}{v} = \frac{100\,W}{5.0\,m/s} = 20\,N$$

And from (2.5):

$$F_{pedal} = \frac{n_{blade}}{n_{gear}} \frac{r_{wheel}}{r_{pedal}} F_{wheel}$$

If we choose a medium gearing: n_{blade} = 42 and n_{gear} = 20, we find by inserting the numerical values

$$F_{pedal} = \frac{42}{20}\frac{0{,}34}{0{,}19} 20\,N = 75\,N \ \cong 7.5kp$$

Which corresponds to the weight of 7.5 kg.

If we next look at the highest gearing (n_3 =52. n_7 =13) and the lowest gearing (n_1 =30, n_1 =24)

Highest gearing: $F_{pedal} = \frac{52}{13}\frac{0{,}34}{0{,}19} 20\,N = 143N$ Lowest gearing: $F_{pedal} = \frac{30}{24}\frac{0{,}34}{0{,}19} 20\,N = 45\,N$

The number of rounds that the pedals must turn can be found from (3.2), which gives the distance that the wheel turns, when the pedals turn one round.

$$s_{wheel} = 2\pi\, r_{wheel} \frac{n_{blade}}{n_{gear}}$$

When the pedals turn N_{pedal} then the distance is:

$$s_{wheel} = 2\pi\, r_{wheel} \frac{n_{blade}}{n_{gear}} N_{pedal}$$

which can be solved for N_{pedal}:

$$N_{pedal} = \frac{n_{gear}}{n_{blade}} \frac{s_{wheel}}{2\pi\, r_{wheel}}$$

When the wheel turns N_{wheel} rounds then $s_{wheel} = 2\pi r_{wheel} N_{wheel}$ and we find (what is actually quite obvious):

$$N_{pedal} = \frac{n_{gear}}{n_{blade}} N_{wheel}$$

At the speed of 5.0 *m/s*: $N_{wheel} = \dfrac{v}{2\pi r_{wheel}}$ rounds so,

$$N_{wheel} = \frac{5.0\ m/s}{2\pi r_{wheel}} = 2.35 \text{ rps}$$

For the three cases mentioned above we find: $N_{pedal} = 1.1$ rps, $N_{pedal} = 0.58$ rps and $N_{pedal} = 1.9$ rps

5. It is hard to go uphill on a bike

Every biker knows that even moderate rises upwards requires an excessive amount of power. Likewise the speed of the bike normally decreases dramatically

So if the mass of the bike and the biker is *m*, and if you ride upwards on a hill which rises an angle $\alpha = 5.0^0$, (which is a substantial rise, corresponding to 8.75%), then the component of gravity, acting against the motion is $F_1 = mg\sin\alpha$, and with a total mass 80 kg it amounts to 68.5 *N*.

Moving with a speed 12.0 *km/h* = 3.33 *m/s* it requires a power: $P = F\cdot v = 68.5\cdot 3.33$ W = 228 *W*,

Together with the standard power 100 *W* delivered by an ordinary cyclist, it gives: $P_{biker} = 328$ *W*.

The force with which the rear wheel acts on the underlay can be found from:

$$F_{wheel}\, v = 328\ W, \text{ which gives: } F_{wheel} = 98.5\ N$$

From the equation (2.5)

$$F_{pedal} = \frac{n_{blade}}{n_{gear}} \frac{r_{wheel}}{r_{pedal}} F_{wheel}$$

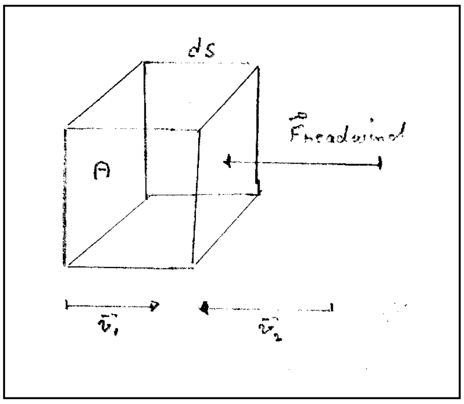
one may then determine the force acting on the pedals in the 3 cases mentioned above: (The numbers in parenthesis, are the masses that give the same forces of gravity)

Medium: ($n_{blade} = 42$, $n_{gear} = 20$): $F_{pedal} = 411$ *N* (41 *kg*).
Highest: ($n_{blade} = 52$, $n_{gear} = 20$) $F_{pedal} = 705$ *N* (70.5 *kg*).
Lowest: ($n_{blade} = 30$, $n_{gear} = 20$) $F_{pedal} = 220$ N (22 *kg*).

If we lower the speed to the half, which is probably more realistic, it is nevertheless hard to go uphill on a bike.

6. The nuisance of Headwind

It is a general experience that a strong headwind makes it substantial harder to ride a bike.
We shall try to make an estimate of the effect of the headwind, but it remains an estimate since the turbulent behaviour of the wind makes an analytic calculation impossible.
We shall represent the bicycle rider by a rectangular square having a area A.



The force from the wind on the cyclist is denoted F.
The cyclist drive with a speed v_1 and the headwind has the speed v_2. Thus the resulting speed of the wind against the cyclist is $v = v_1 + v_2$. Assuming that the wind decreases from v_1 to $-v_2$ we have $\Delta v = -(v_2 + v_1)$.
The air that hits the cyclist in front of the cyclist in time dt has a mass $dm = \rho A ds$.
From mechanics we know that: $Fds = d(\frac{1}{2}mv^2)$ so,
$Fds = \frac{1}{2}\rho A ds v^2 \quad \Rightarrow \quad F = \frac{1}{2}\rho A v^2$, where ρ is the density of the air.
And the force per unit area $f = \dfrac{F}{A}$ is: $f = \dfrac{F}{A} = \frac{1}{2}\rho v^2$

The assumption assume that the wind that hits the cyclist has a change of velocity: $\Delta v = -(v_2 + v_1)$ can not be uphold. The factor in which it is reduced can only be determined by experiment.
We shall in the following use a factor of one half.

For an average cyclist the cyclists area is estimated to $A = 0.4$ m^2, we find for a cyclist riding with a speed of 18 km/h=5.0 m/s the force needed is 20 N = 2.0 kp. (weight of 2.0 kg). 20 N is therefore a number, we can compare to.

If there is no wind, the wind resistance comes only from the cyclist motion with the speed $v_1 = 5\ m/s$.

$$F_{headwind} = \tfrac{1}{2} f_{headwind} \cdot A = \tfrac{1}{2} \cdot 0.4 \cdot \tfrac{1}{2}\rho v^2 = 0.4 \cdot \tfrac{1}{2} \cdot 1.29 \cdot 5^2 = \tfrac{1}{2} \cdot 6.45N \cong 0.32kp$$

Next we look at a situation where $v_1 = 5\ m/s\ and\ v_2 = 5\ m/s$, and the cyclist area is estimated to 0.4 m^2, we find:

$$F_{headwind} = f_{headwind} \cdot A = 0.4 \cdot \tfrac{1}{2}\rho v^2 = 0.4 \cdot \tfrac{1}{2} \cdot 1.29 \cdot 10^2 = 25.8N \cong 26kp$$

But the force of the headwind grows with the square of the relative velocity, so we look at the situation where, $v_1 = 5\ m/s\ and\ v_2 = 10\ m/s$, and the cyclist area is estimated to 0.4 m^2, we find:

$$F_{headwind} = \tfrac{1}{2} \cdot f_{headwind} \cdot A = \tfrac{1}{2} \cdot 0.4 \cdot \tfrac{1}{2}\rho v^2 = 0.4 \cdot \tfrac{1}{2} \cdot 1.29 \cdot 15^2 = \tfrac{1}{2} \cdot 58.5N = 29{,}5kp$$

If: $v_1 = 5\,m/s \quad and \quad v_2 = 15\,m/s$

$$F_{headwind} = \tfrac{1}{2}\cdot f_{headwind}\cdot A = \tfrac{1}{2}\cdot 0.4\cdot\tfrac{1}{2}\rho v^2 = 0.4\cdot\tfrac{1}{2}1.29\cdot 20^2 = \tfrac{1}{2}\cdot 103N \cong 50.2kp$$

We shall now estimate the supplementary force which must be exerted on the pedals in the three examples above.
The force with which the rear wheel acts on the underlay can be found from: $Fv = P$
The force exerted on the underlay from the wheel is the force required to move the bicycle.
From the equation (2.5). We shall now look at how it influences the force on the pedals on a standard cycle with 3 gears.

$$F_{pedal} = \frac{n_{blade}}{n_{gear}}\frac{r_{wheel}}{r_{pedal}}F_{wheel}$$

One may then determine the force acting on the pedals in the 3 cases mentioned above: (The numbers in parenthesis, are the masses that give the same forces of gravity)

$$F_{pedal} = \frac{n_{blade}}{n_{gear}}\frac{r_{wheel}}{r_{pedal}}\tfrac{1}{2}\cdot F_{headwind}$$

<u>At v =5.0 m/s</u>:

$F_{headwind} = 6.45N$,
$r_{pedal} = 0.19$,
$r_{wheel} = 0.34m$,
$n_{blade} = 42$,
$n_{gear} = 20$:

$F_{pedal} = \dfrac{42}{20}\dfrac{0.34}{0.19}\tfrac{1}{2}\cdot 6.45N = \tfrac{1}{2}\cdot 24.3N = 1.6kp$ (Supplementary force due to the headwind)

At v =10.0 m/s:

$F_{headwind} = 25.8N$
$r_{pedal} = 0.19$
$r_{wheel} = 0.34m$, $n_{blade} = 42$
$n_{gear} = 20$

$F_{pedal} = \dfrac{42}{20}\dfrac{0.34}{0.19}25.8N = 96{,}8N = 9.7kp$ (Supplementary force due to the headwind)

At $v = 15m/s \quad and\ 20m/s$, we can make use of the fact that force of the headwind grows with the square of the wind. So we get:

At v =15.0 m/s: $F_{pedal} = \frac{42}{20}\frac{0.34}{0.19}\frac{1}{2}\cdot 58.5N = \frac{1}{2}\cdot 218N = 11kp$

At v =20.0 m/s: $F_{pedal} = \frac{42}{20}\frac{0.34}{0.19}\frac{1}{2}\cdot 103N = \frac{1}{2}\cdot 387N = 19.5kp$

From these estimates it is clear that a strong headwind requires a substantial force to uphold the motion of the bike.

6. Simple experiments performed with a racing bike

Experiment (4.1). Examination of the relation (1.3): The distance the bike moves by one turn of the pedals

(1.3) $$s_{bike} = s_{wheel} = 2\pi\, r_{wheel}\frac{n_{blade}}{n_{gear}}$$

Material: A measuring tape.
We start by counting the number of teeth on the blade and on the sprocket gear. We measure the diameter on the blade and the rear wheel. It is not necessary to measure the diameter on the other gear wheels, since the ratio between the diameters is the same as the ratio between the numbers of teeth on the gear wheels

The bike is kept upright. The position of the rear wheel is marked, and with a tight chain the pedals are moved one round as the bike moves forward. This is repeated with various choices of gearing, to verify relation (1.3).

Experiment (4.2). Examination of the relation (2.5)

(2.5) $$F_{wheel} = \frac{n_{gear}}{n_{blade}}\frac{r_{pedal}}{r_{wheel}}F_{pedal}$$

Material: 5 *kg* or 10 *kg* weights. Newton meter 20 – 50 *N.*

The length of the pedal lever is measured. The Newton meter is fixed at the rear of the bike. The rear wheel must stand solid on the underlay. The bike must be supported, so that it does not overthrow, but there must be no external forces along the bike. The pedal must be strictly horizontal and on the pedal is placed one of the weights with mass, and (almost) without touching the bike the Newton meter at the rear is read.

The experiment may be repeated with different masses and different choices of gear.

The force with which the mass acts on the pedal is: $F_{pedal} = mg$, and F_{wheel} is read on the Newton meter.

The aim of this experiment is to verify the relation (2.5) above