Bernoulli’s law

With applications to hydrodynamics and aerodynamics

This is an article from my home page: www.olewitthansen.dk

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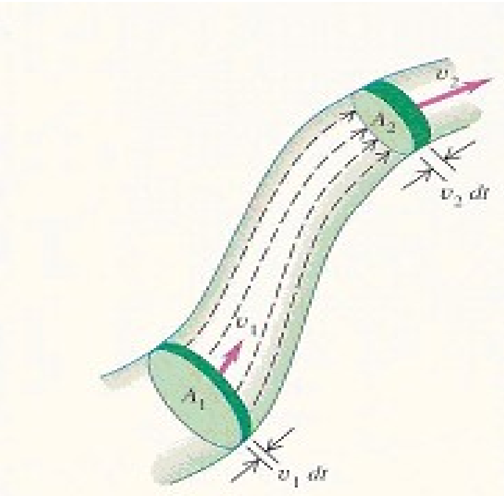
1. Introduction

In the Danish 9 to 12 year high school the teaching of physics has since the year of 2005 changed drastically from the traditional theoretical and experimentally method to dealing with group projects, not covered in the their textbooks, including other (not necessarily scientific) subjects as history and biology. This pedagogically and social ideological change in the education has mildly speaking been a challenge for the university educated teachers, and it has predictably ruined the traditional academic teaching of physics and mathematics, since proofs and theoretical explanations has been abolished in favour of computers.
In 2008, I was the physics teacher together with a historian teacher in a cross professional subject of The Vietnamese war. But how could I bring some theoretical physics into this subject. The (completely absurd) way out was that since The Vietnamese war was the first almost completely airborne war, my contribution was to demonstrate why an airplane can fly!
To demonstrate this, one has to introduce Bernouilli’s law for laminar flow of liquids.

2. Flow of liquids

A stationary flow means that the velocity of any liquid particle in a given position is unchanged in magnitude and direction.
A streamline is a curve that follows a liquid particle. In a laminar streaming the streamlines cannot be closed curves.
In the following we shall rather consequently use differentials prefixed by a *d*, rather than finite increments prefixed by a “delta” Δ to denote small changes. E.g. dv instead of Δv, ds instead of Δs.
In this way the ratio between two small increments become a differential quotient.

E.g. $v = \frac{ds}{dt}$ rather than: $v = \frac{\Delta s}{\Delta t}$



14–21 A flow tube with changing cross-section area. The product Av is constant for an incompressible fluid.

In the figure we have drawn a (mathematical) ”tube”, which follows a streamline. I might also be a physical tube, in which the liquid flows, but this is without importance).
Presupposing that the liquid is incompressible, that is, that the density of the liquid ρ is overall the same, which is the case for liquids, but not for gasses.
For this reason it must be the same amount of liquid which passes the two cross sections, and therefore the velocity of the liquid is slower at the cross section at A_1 than A_2, since the tube is wider at that position.
In the time interval dt the liquid is staggered an amount ds_1 at A_1, and staggered an amount ds_2 at A_2.
The two liquid volumes $dV_1 = A_1ds_1$ and $dV_2 = A_2ds_2$ must be equal, (since ρ is unchanged), and since $ds_1=v_1dt$ and $ds_2=v_2dt$,we find that: $A_1v_1dt = A_2v_2dt$, or

(2.1) $A_1v_1 = A_2v_2$ (along a streamline)

However, if the liquid (or the gas) is compressible, it is rather the masses dm_1 and dm_2, which are equal in the two volumes dV_1 and dV_2.

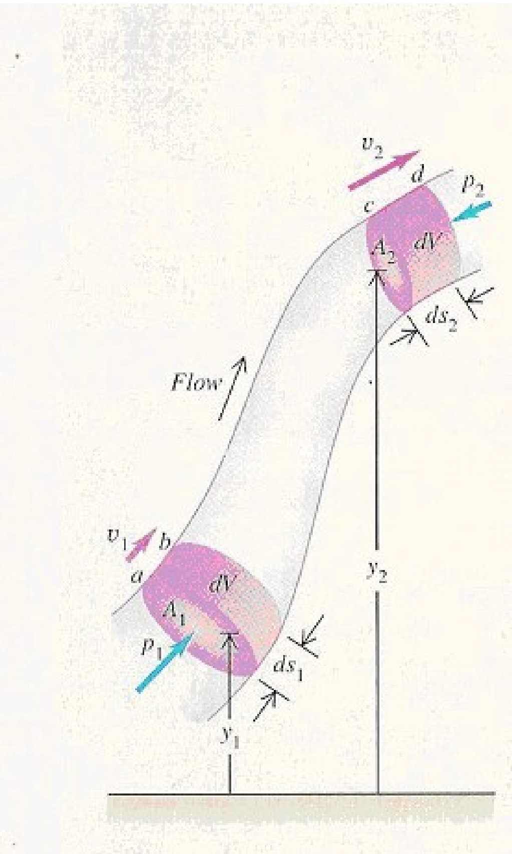
$$dm_1 = dm_2 \quad \Leftrightarrow \quad \rho_1 dV_1 = \rho_2 dV_2 \quad \Leftrightarrow \quad \rho_1 A_1 v_1 dt = \rho_2 A_2 v_2 dt$$

(2.2) $$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

Where ρ_1 and ρ_2 are the densities of the liquid at (1) and (2) respectively.

The equation (2.2) is an example of a continuation equation, and it has many other applications apart from flow of liquids.
For example it is the foundation of analysis of traffic, e.g. the creating of queues on high ways. In that context the density ρ means the density of cars, (which is not constant) and v is the velocity of the cars in a certain position.
If the number of cars is the same in a certain stretch, the equation tells us that, if the speed is lowered, then the density must go up. Since there is a physical limit to the density of cars in a high way, then the moving queue will go to a halt, when this density is reached.
In the same manner, if the number of lanes is diminished, then must either the density be increased or the speed increased! (But hardly the latter).

3. Bernoulli's law



14–22 The net work done by the pressure equals the change in the kinetic energy plus the change in

We shall now derive Bernoulli's law, which for a laminar i.e. non turbulent gives the relation between the pressure p and the streaming velocity v, at that point. Again we shall look at a tube, having the ends at (1) and (2).
When the liquid is displaced is displaced from (a) to (b) at (1), then it is displaced from (c) to (d) at (2).
All physical quantities having index (1) belong to the lower end and physical quantities having index (2) belong to the upper end.
It applies e.g. to the pressures p_1 and p_2.
The overall work done on the liquid in the tube is:

$$dW = F_1 ds_1 - F_2 ds_2$$

Since $F = pA$, the work done can be written.

(3.1) $$dW = p_1 A_1 ds_1 - p_2 A_2 ds_2 = p_1 dV_1 - p_2 dV_2$$

If the liquid is incompressible, then $dV_1 = dV_2$.
We shall then apply the general theorem, which is valid in the absence of viscosity:

Work done = change of energy.

(3.2) $$dW = dE_{kin} + dE_{pot}$$

The kinetic energy of the liquid between (b) og (c), is unchanged, since the liquid is in a stationary state. Therefore the change in kinetic energy between (2) and (1) becomes.

$$dE_{kin} = \tfrac{1}{2}(dm_2)\, v_2^{\,2} - \tfrac{1}{2}(dm_1)\, v_1^{\,2}$$

where $dm_1 = dm_2 = \rho dV$. From this we obtain:

(3.3) $$dE_{kin} = \tfrac{1}{2}\rho\, v_2^{\,2} dV - \tfrac{1}{2}\rho\, v_1^{\,2} dV$$

In the same manner, we have for the potential energy:

$$E_{pot} = mgy \text{ (the vertical axis is } y\text{)}$$

(3.4) $$dE_{pot} = (dm_2)g\, y_2 - (dm_1)g\, y_1 = \rho\, g\, y_2 dV - \rho\, g\, y_1 dV$$

The expressions (3.1), (3.3) and (3.4) are now inserted in (3.2).

$$p_1\, dV - p_2\, dV = \tfrac{1}{2}\rho\, v_2^{\,2} dV - \tfrac{1}{2}\rho\, v_1^{\,2} dV + \rho\, g\, y_2 dV - \rho\, g\, y_1 dV$$

Dividing by dV, and arranging after index (1) and (2) on the left and right side, we obtain:

$$p_1 + \tfrac{1}{2}\rho\, v_1^{\,2} + \rho\, g\, y_1 = p_2 + \tfrac{1}{2}\rho\, v_2^{\,2} + \rho\, g\, y_2$$

This equation, which is named after Bernoulli expresses that:

(3.5) $$p + \tfrac{1}{2}\rho\, v^2 + \rho\, g\, y = Constant \text{ (along a streamline).}$$

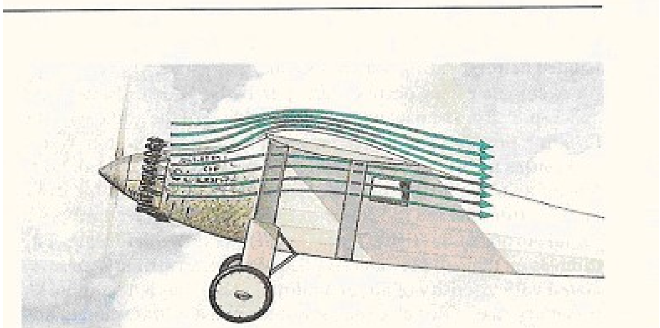
Bernoulli's law was derived under the three suppositions.

1) Laminar streaming (no turbulence/vortices)
2) The liquid/gas is free from viscosity, so there is no energy loss to heat.
3) The liquid is incompressible. (Does apply for liquids, but not for gasses).

However, it is almost never that all these three conditions are fulfilled, but never the less Bernoulli's law is the foundation of many theoretical calculations in hydrodynamics.
When it concerns aerodynamics one is usually referred to empirical data, air tunnel experiments and engineering ingenuity.

4. The art of flying

Nevertheless Bernoulli's law can deliver a qualitative explanation on many physical phenomenon in the real world. For example, why it is possible to fly (without barking with the wings), which remains still a mystery for many people outside the scientific community.



14–26 Flow lines around an airplane wing.

The figure demonstrates the principle of flying according to Bernoulli's law.
It shows the streamlines on a wing on an airplane.
Because of the shape of the wing, the air must travel a longer stretch above the wing than below the wing.
The air above the wing must therefore travel with a higher speed than below.

Following Bernoulli's law, it means that the pressure on the top side of the wing is diminished, compared to the pressure on the bottom side.
This difference in pressure causes the "buoyancy" counterpart to gravity, and that keeps the plane in the air.
The construction of airplanes is based on hundred years of experience, engineering ingenuity and wind tunnel experiments. Aerodynamics is a very complex science, especially when it comes to turbulence, and it certainly does, when it concerns flying.

3.6 Example:
To illustrate the application of Bernoulli's law to flying, we shall make a simple (but unrealistic) calculation.

We assume that the speed of the air flow on the upper side of the wing is 20% larger than on the lower side.
We can then find from Bernouli's law: (Since there is no contribution from potential energy)

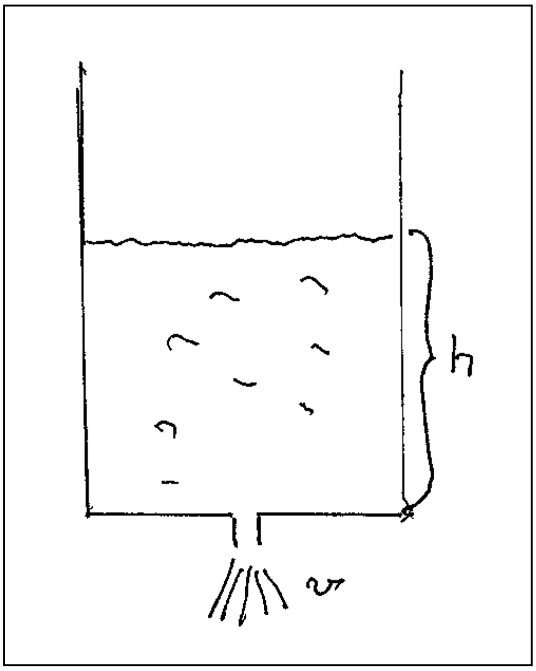
$$p_1 + ½\rho\, v^2 = p_2 + ½\, \rho\, (1.2v)^2 \quad => \quad \Delta p = p_1 - p_2 = ½\, \rho\, 0.44v^2.$$

Inserting $\rho = \rho_{air} = 1.29\ kg/m^3$, $v = 360\ km/h = 100\ m/s$, we get:

$\Delta p = 2.84 \cdot 10^3\ N/m^2 = 2.84 \cdot 10^{-2}\ atm = 284\ kp/m^2$. $1\ atm. \approx 10^5\ N/m^2 \approx 1\ kp/cm^2$.

Having a wingspan of 20 m^2 it corresponds to a buoyancy of: 284 kp/m^2 20 m^2 = 5680 kp.
This correspond roughly to the weight of 5,68 ton.

5. Emptying a barrel by a vessel at the bottom



We shall consider a barrel filled with a non viscous liquid, having an outlet at the bottom.
The liquid has density ρ.
We shall then apply Bernoulli's law, omitting the term with the pressure, since we assume that the outer pressure is the same at the top and bottom. We shall also replace y by h denoting the depth.

(4.1) $$½\rho\, v_1{}^2 + \rho\, g\, h_1 = ½\, \rho\, v_2{}^2 + \rho\, g\, h_2$$

At the surface $h = 0$, the velocity $v = 0$, and in the depth h the velocity is v.
Remarkedly but perhaps not surprisingly (if you think about it) giver Bernoullis law give the the same result as a free fall in the field of gravity.

(4.2) $$½\, \rho\, v^2 + \rho\, g(-h) = 0 \quad \Leftrightarrow \quad v = \sqrt{2gh}$$

v is the velocity in which the liquid leaves the outlet. If $m = m(t)$, denotes the mass of the liquid at time t , and if the pipe to the vessel has the cross section D, then the continuity equation applies for the amount of liquid dm, leaving the outlet in the time dt.

$$\frac{dm}{dt} = -\rho D v.$$ (Minus because m is decreasing).

If the cross section of the barrel is A, we have: $m = \rho Ah$, so that: $\frac{dm}{dt} = \rho A\frac{dh}{dt}$

By putting the two expressions for $\frac{dm}{dt}$ equal to each other, we get:

$$\rho A\frac{dh}{dt} = -\rho Dv$$

and inserting $v = \sqrt{2gh}$, we get a differential equation for h.

(4.3) $$\rho A\frac{dh}{dt} = -\rho D\sqrt{2gh} \quad \Leftrightarrow \quad \frac{dh}{dt} = -\sqrt{2g}\frac{D}{A}\sqrt{h}$$

The equation can be separated and integrated:

$$\frac{dh}{\sqrt{h}} = -\sqrt{2g}\frac{D}{A}dt \quad \Leftrightarrow \quad \int_{h_0}^{h}\frac{dh}{\sqrt{h}} = -\sqrt{2g}\frac{D}{A}\int_0^t dt \quad \Leftrightarrow$$

$$2\sqrt{h} - 2\sqrt{h_0} = -\sqrt{2g}\frac{D}{A}t \quad \Leftrightarrow \quad \sqrt{h} = \sqrt{h_0} - \sqrt{2g}\frac{D}{2A}t \quad \Leftrightarrow$$

(4.4) $$h = \left(\sqrt{h_0} - \frac{D\sqrt{2g}}{2A}t\right)^2$$

The barrel becomes empty when $h = 0$.

This happens at time $t = \frac{A}{D}\sqrt{\frac{2h_0}{g}}$ according to the equation above.

For a container with: $A = 50 \text{ x } 50 \text{ cm}^2$, $h_0 = 1.0\ m$, and $D = 2.0\ cm^2$, it results in a period for emptying the barrel equal to 564 s.