The infinite sum of the reciprocals of n-square

The values of Zeta(2), Zeta(4) and Zeta(6)

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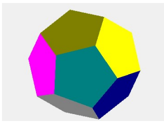

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1. The infinite sum of the reciprocals of *n*-square. Zeta(2).

The *Zeta* function is defined by the series:

(1.1) $$\varsigma(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}\ ,\qquad x>1$$

For $x = 1$ it is the harmonic series, which easily can be shown to diverge, but for $x > 1$ it is convergent. The aim of this note is to find the exact value of $\varsigma(2)$, $\varsigma(4)$ and $\varsigma(6)$

(1.2) $$\varsigma(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

There are in fact many exact values of the *Zeta* that have been found, but we shall be concerned with the series above for x =2, 4, 6, where the sum may be determined by applying Fourier series.

The coefficients of the Fourier series of a function $f(x)$ are in the interval $[-\pi,\pi]$ defined by the integral:

(1.3) $$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx \quad \text{where} \quad f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx}$$ is the Fourier series for $f(x)$

The norm of $f(x)$, which is written as $|f(x)|$, is defined by the expression:

(1.4) $$|f|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)f(x)^*dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx$$

We shall also use Parseval's theorem.

(1.5) $$|f|^2=\sum_{n=-\infty}^{\infty}c_nc_n^{\ *}=\sum_{n=-\infty}^{\infty}|c_n|^2$$

(For details see www.olewitthansen.dk: Fourier series and Fourier integrals)

We shall then apply the formulas (1.3) and (1.4) for the function $f(x) = x$. This gives:

(1.6) $$\sum_{n=-\infty}^{\infty}|c_n|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx \quad => \quad \sum_{n=-\infty}^{\infty}|c_n|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx$$

(1.6) $$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx \quad \Rightarrow \quad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}xe^{-inx}dx$$

We shall first do the coefficients c_n.

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}xe^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}x(\cos nx-i\sin nx)dx$$

And we must therefore do the two integrals:

$$\int_{-\pi}^{\pi} x\cos nx dx \qquad \text{and} \qquad \int_{-\pi}^{\pi} x\sin nx dx$$

Since $x\cos nx$ is an odd function, that is, $f(-x) = -f(x)$ the first integral is zero. However, the integrand in the second integral is even, since $(-x)\sin(-nx) = x\sin nx$.

That the integral of an odd function over a symmetric interval vanishes is obvious from a geometrical point of view, but it can also be proved directly: Let f be an odd function.

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx =$$

$$\int_{a}^{0} f(-x)d(-x) + \int_{0}^{a} f(x)dx = \int_{a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

In the first integral we have made the substitution: x -> $-x$, and changed sign for the limits:
Finally we have switched the upper and lower limits, resulting in a minus sign.

The integration is done by parts.

(1.7)
$$\int_{-\pi}^{\pi} x\sin nx dx = -\int_{-\pi}^{\pi} xd\cos nx = -\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\, dx$$
$$= -\frac{1}{n}(\pi\cos n\pi + \pi\cos n\pi) = \frac{2\pi}{n}(-1)^{n+1}$$

The last integral: $\int_{-\pi}^{\pi}\cos nx\, dx = 0$, because the integral of $\cos nx$ over a period of 2π is always zero for $n \neq 0$. So we have:

(1.8)
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(\cos nx - i\sin nx)dx = \frac{(-1)^n}{n}i \qquad \text{for } n \neq 0 \quad \text{and } c_0 = 0$$

since
$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$$

Therefore: $|c_n|^2 = \frac{1}{n^2}$. According to Parsevals theorem:

$$\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\, dx, \quad \text{So we find:}$$

(1.9)
$$\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2dx = \frac{1}{2\pi}\left[\tfrac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}, \text{ but also: } \sum_{n=-\infty}^{\infty}|c_n|^2 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

From (1.9) we therefore have: $2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3}$, which gives us the value of $\varsigma(2)$:

(1.10) $$\varsigma(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$$

2. The value of Zeta(4)

We shall now do the same line of calculations to determine

(2.1) $$\varsigma(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...$$

Applying the Fourier transformation formulas:

(2.2) $$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx \quad \text{where} \quad f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

And using Parseval's theorem: $\sum_{n=-\infty}^{\infty}|c_n|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx$ with the function $f(x)=x^2$.

We have:

(2.3) $$\sum_{n=-\infty}^{\infty}|c_n|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^4dx \quad \text{and} \quad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2e^{-inx}dx$$

We begin by evaluating c_n.

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2(\cos nx-i\sin nx)dx$$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2(\cos nx-i\sin nx)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\cos nxdx-\frac{i}{2\pi}\int_{-\pi}^{\pi}x^2\sin nxdx$$

The last integral is zero, because $x^2\sin nx$ is an odd function, so we shall only evaluate the first integral by parts.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\cos nxdx=\frac{1}{2\pi n}\int_{-\pi}^{\pi}x^2d\sin nx=\frac{1}{2\pi n}\left[x^2\sin nx\right]_{-\pi}^{\pi}-\frac{1}{2\pi n}\int_{-\pi}^{\pi}2x\sin nxdx$$

The first term vanishes because $\sin n\pi=0$. So we evaluate the second integral by parts.

$$-\frac{1}{\pi n}\int_{-\pi}^{\pi}x\sin nxdx=\frac{1}{\pi n^2}\int_{-\pi}^{\pi}xd\cos nx=\frac{1}{\pi n^2}\left[x\cos nx\right]_{-\pi}^{\pi}-\frac{1}{\pi n^2}\int_{-\pi}^{\pi}\cos nxdx$$

The second integral vanishes because the integral of cos nx over a period of 2 pi is always zero.

/2.5) $$\frac{1}{\pi n^2}\left[x\cos nx\right]_{-\pi}^{\pi}=\frac{1}{\pi n^2}(\pi\cos n\pi-(-\pi\cos n\pi))=\frac{2}{n^2}\cos n\pi=\frac{2}{n^2}(-1)^n$$

So we find: $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}dx = \frac{2}{n^2}(-1)^n$ for $n \neq 0$ and $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$

Thus

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 4\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{9} + 4\sum_{n=-\infty}^{-1}\frac{1}{n^4} + 4\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

On the other hand:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{1}{2\pi}\left[\frac{1}{5}x^5\right]_{-\pi}^{\pi} = \frac{\pi^4}{5}$$

So we find: $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx \iff \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5} \iff 8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \frac{4\pi^4}{45}$

(2.6) $$\varsigma(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

3. The value of Zeta(6)

As we did in the previous cases, we use the Fourier transformation formulas:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \qquad \text{where} \qquad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

And we apply also Parseval's theorem:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx$$

This time with the function $f(x) = x^3$. We then have:

(3.1) $$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^6 dx \qquad \text{and} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 e^{-inx}dx$$

We shall first evaluate c_n

(3.2) $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3(\cos nx - i\sin nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3\cos nx dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} x^3 \sin nx dx$$

The first term is zero, because $x^3 \cos nx$ is an odd function, so we shall only evaluate the second integral by parts.

$$c_n = -\frac{i}{2\pi}\int_{-\pi}^{\pi} x^3 \sin nx dx = \frac{i}{2\pi n}\int_{-\pi}^{\pi} x^3 d\cos nx = \frac{i}{2\pi n}\left[x^3 \cos nx\right]_{-\pi}^{\pi} - \frac{3i}{2\pi n}\int_{-\pi}^{\pi} x^2 \cos nx dx$$

$$c_n = \frac{i}{2\pi n}(\pi^3 \cos n\pi - (-\pi)^3 \cos n\pi) - \frac{3i}{2\pi n}\int_{-\pi}^{\pi} x^2 \cos nx dx$$

(3.3) $$c_n = \frac{i}{\pi n}\pi^3(-1)^n - \frac{3i}{2\pi n}\int_{-\pi}^{\pi} x^2 \cos nx dx$$

However, we have already evaluated the integral above, and the result was:

(3.4) $$\frac{3i}{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{3i}{n}\left(\frac{2}{n^2}(-1)^n\right)$$

So we get for c_n.

$$c_n = \frac{i}{\pi n}\pi^3(-1)^n - \frac{3i}{n}\left(\frac{2}{n^2}(-1)^n\right) = i\left(\frac{\pi^2}{n}(-1)^n - \frac{6}{n^3}(-1)^n\right) \quad \text{and} \quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 dx = 0$$

$$|c_n|^2 = \frac{\pi^4}{n^2} + \frac{36}{n^6} - \frac{12\pi^2}{n^4}$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{\infty} \frac{\pi^4}{n^2} + \sum_{n=-\infty}^{\infty} \frac{36}{n^6} - \sum_{n=-\infty}^{\infty} \frac{12\pi^2}{n^4} = \pi^4 \sum_{n=-\infty}^{\infty} \frac{1}{n^2} + 36\sum_{n=-\infty}^{\infty} \frac{1}{n^6} - 12\pi^2 \sum_{n=-\infty}^{\infty} \frac{1}{n^4} =$$

Changing the lower limit from minus infinity to 1 and multiplying by 2

(3.5) $$\sum_{n=-\infty}^{\infty} |c_n|^2 = 2\pi^4 \sum_{n=1}^{\infty} \frac{1}{n^2} + 72\sum_{n=1}^{\infty} \frac{1}{n^6} - 24\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

However, we have already evaluated the two sums: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, so

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^6}{3} + 72\sum_{n=1}^{\infty} \frac{1}{n^6} - \frac{4\pi^6}{15}$$

On the other hand: $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^6 dx = \frac{\pi^6}{7}$ so we have:

$$\frac{\pi^6}{3} + 72\sum_{n=1}^{\infty} \frac{1}{n^6} - \frac{4\pi^6}{15} = \frac{\pi^6}{7} \quad \Leftrightarrow \quad 72\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{4\pi^6}{15} + \frac{\pi^6}{7} - \frac{\pi^6}{3} = \frac{15\pi^6}{105} + \frac{28\pi^6}{105} - \frac{35\pi^6}{105} = \frac{8\pi^6}{105}$$

And we finally get:

(3.6) $$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$