Weierstrass Approximation theorem

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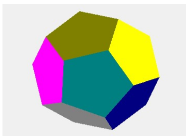

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Contents

1. Weierstrass' approximation theorem (1885) 1
1.1 It is sufficient to prove the theorem in the interval [0,1] 1
1.2 Establishing an appropriate polynomial $p(x)$ from the binomial formula 1
1.3 Separating the binomial sum into two sums 2
1.4 Remarks on Chebychev’s inequality 4

1. Weierstrass' approximation theorem (1885)

If $f(x)$ is a continuous function in the closed interval $[a,b]$, and $\varepsilon > 0$, then there exist a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \ ,$$

such that

$$| f(x) - p(x)| < \varepsilon \ \ for \ \ all \ \ x \in [a,b]$$

There are various proofs of this theorem. The following proof is due to S. N. Bernstein.
The proof is completed in various steps.

1.1 It is sufficient to prove the theorem in the interval [0,1]

First we show that if the theorem is proven in the interval $[0,1]$, then it can be proven for an arbitrary interval $[a,b]$ in the following manner. If $f(x)$ is continuous in $[a,b]$ then

$$g(\xi) = f(a + (b-a)\xi)$$

is a continuous function in the interval $[0,1]$, (since $x = a + (b-a)\xi$ is a mapping of $[0,1]$ into $[a,b]$). Accordingly there exists a polynomial

$$q(\xi) = b_0 + b_1\xi + b_2\xi^2 + ... + b_n\xi^n$$

That for any $\varepsilon > 0$ applies:

$$| g(\xi) - q(\xi)| < \varepsilon \ \ for \ all \ \xi \in [0,1]$$

We therefore have:

$$| g(\frac{x-a}{b-a}) - q(\frac{x-a}{b-a})| < \varepsilon \ \ for \ all \ x \in [a,b],$$

Where

$$f(x) = g(\frac{x-a}{b-a}) \quad for \ x \in [a,b] \quad and$$

$$p(x) = q(\frac{x-a}{b-a}) = b_0 + b_1(\frac{x-a}{b-a}) + b_2(\frac{x-a}{b-a})^2 + ... + b_n(\frac{x-a}{b-a})^n \quad for \ x \in [a,b]$$

$p(x)$ is then a polynomial of degree *n,* such that:

$$| g(\frac{x-a}{b-a}) - q(\frac{x-a}{b-a})| < \varepsilon \quad \Leftrightarrow \quad | f(x) - p(x)| < \varepsilon \quad for \ all \ x \in [a,b]$$

1.2 Establishing an appropriate polynomial *p*(*x*) from the binomial formula

From now on we assume that the interval $[a,b]$ is $[0,1]$. Using the binomial formula:

$1 = (x + (1-x))^n = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$, we form, for an arbitrary positive integer, the polynomial:

$$p_n(x) = \sum_{\nu=0}^{n} f(\frac{\nu}{n}) \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

We shall then show that for $\varepsilon > 0$ there exists an n_ε, such that for $n > n_\varepsilon$ it applies.

$$| f(x) - p_n(x) | < \varepsilon \quad \textit{for all } x \in [0,1]$$

According to the binomial formula, we have

$$1 = (x + (1-x))^n = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

Multiplying the equation by $f(x)$, we get:

$$f(x) = \sum_{\nu=0}^{n} f(x) \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \quad \text{for all } x \in [0,1]$$

From which we get:

$$| f(x) - p_n(x) | = | \sum_{\nu=0}^{n} (f(x) - f(\frac{\nu}{n})) \binom{n}{\nu} x^\nu (1-x)^{n-\nu} |$$

And consequently, since $\binom{n}{\nu} x^\nu (1-x)^{n-\nu} \geq 0$ *for all* $\nu = 0,1....,n$ *and all* $x \in [0,1]$, we can write

$$| f(x) - p_n(x) | \leq \sum_{\nu=0}^{n} | f(x) - f(\frac{\nu}{n}) | \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

1.3 Separating the binomial sum into two sums

For a given *x* and *n,* the sum is divided in two, since for a given $\delta > 0$, we take in the first sum all terms where $| x - \frac{\nu}{n} | < \delta$ and in the second sum the terms, where $| x - \frac{\nu}{n} | \geq \delta$. We then have:

$$| f(x) - p_n(x) | \leq \sum_{\substack{\nu=0 \\ | x - \frac{\nu}{n} | < \delta}}^{n} | f(x) - f(\frac{\nu}{n}) | \binom{n}{\nu} x^\nu (1-x)^{n-\nu} + \sum_{\substack{\nu=0 \\ | x - \frac{\nu}{n} | \geq \delta}}^{n} | f(x) - f(\frac{\nu}{n}) | \binom{n}{\nu} x^\nu (1-x)^{n-\nu} = \sum_1 ... + \sum_2 ...$$

We put $M = \sup_{0 \leq x \leq 1} | f(x) |$ and $\omega(\delta) = \sup_{|x_2 - x_1| < \delta} | f(x_2) - f(x_1) |$

Since $f(x)$ is continuous in the closed interval $[0,1]$ $f(x)$ is bounded and uniform continuous. So we have: $M < \infty$ *and* $\omega(\delta) \to 0$ *for* $\delta \to 0$. And consequently:

$$\sum_1 ... \leq \omega(\delta) \sum_{\substack{\nu=0 \\ |x - \frac{\nu}{n}| < \delta}}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \leq \omega(\delta) \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} = \omega(\delta)$$

$$\sum_{2} \ldots \le \; 2M \sum_{\substack{\nu=0 \\ |x-\frac{\nu}{n}|\ge\delta}}^{n} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$$

The crucial point in the proof is the estimate of the right side of the last inequality.
We take as a starting point the binomial formula.

$$(x+y)^{n} = \sum_{\nu=0}^{n} \binom{n}{\nu} x^{\nu} y^{n-\nu}$$

Differentiating after x, followed by multiplication by x, we obtain:

$$nx(x+y)^{n-1} = \sum_{\nu=0}^{n} \nu \binom{n}{\nu} x^{\nu} y^{n-\nu}$$

Repeating the process:

$$nx(x+y)^{n-1} + n(n-1)x^{2}(x+y)^{n-2} = \sum_{\nu=0}^{n} \nu \binom{n}{\nu} x^{\nu} y^{n-\nu}$$

If we put $y = 1-x$, we get the three equations:

$$\sum_{\nu=0}^{n} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} = 1 \quad , \quad nx = \sum_{\nu=0}^{n} \nu \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} \quad , \quad nx + n(n-1)x^{2} = \sum_{\nu=0}^{n} \nu^{2} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$$

From these three expressions we can obtain the expression: $\sum_{\nu=0}^{n} (x-\frac{\nu}{n})^{2} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$

Since: $$(x-\frac{\nu}{n})^{2} = x^{2} - 2\frac{x}{n}\nu + \frac{1}{n^{2}}\nu^{2}$$

So we get the expression $\sum_{\nu=0}^{n} (x-\frac{\nu}{n})^{2} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$ by multiplying the three equations by,

$x^{2}, -2\frac{x}{n} \; \textit{and} \; \frac{1}{n^{2}}$ respectively and add these three terms. The left hand side becomes:

$$x^{2} - 2\frac{\nu}{n}x + \frac{1}{n^{2}}(nx + n(n-1)x^{2}) = \frac{x(1-x)}{n}$$

We therefore arrive at the equation:

$$\sum_{\nu=0}^{n} (x-\frac{\nu}{n})^{2} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} = \frac{x(1-x)}{n}$$

The term $x(1-x)$ has its max $\frac{1}{4}$ for $x = \frac{1}{2}$ in the interval $[0,1]$, and we therefore have:

$$\sum_{\nu=0}^{n} (x-\frac{\nu}{n})^{2} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} \le \frac{1}{4n}$$

We notice that all of the terms are non negative in the interval $[0,1]$. We throw away the terms where $|x-\frac{\nu}{n}|<\delta$ and replace the other terms $(x-\frac{\nu}{n})^2$ by δ^2 to obtain Chebychevs inequality.

$$\sum_{\substack{\nu=0 \\ |x-\frac{\nu}{n}|\geq\delta}}^{n}\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}\leq\frac{1}{4n\delta^2}\quad \textit{for all}\ \ x\in[0,1]$$

We summarize:

$$|f(x)-p_n(x)|\ \leq\omega(\delta)+\frac{M}{2n\delta^2}\leq\ \frac{1}{4n\delta^2}\quad \textit{for all}\ \ x\in[0,1]$$

To an $\varepsilon>0$ we first choose $\delta(\varepsilon)$, such that $\omega(\delta)\leq\frac{\varepsilon}{2}$, then we determine an $n_0=n_0(\varepsilon)$, such that

$$\frac{M}{2n_0\delta(\varepsilon)^2}\leq\frac{\varepsilon}{2}\quad\Leftrightarrow\quad\frac{M}{\varepsilon\delta(\varepsilon)^2}\leq n_0$$

Then we have for $n\geq n_0$:

$$|f(x)-p_n(x)|\leq\varepsilon$$

And thus, we have proven the Weierstrass' approximation theorem.

1.4 Remarks on Chebychev's inequality

From Chebychevs inequality it follows in particular that for a fixed $x\in[0,1]$ and fixed $\delta>0$ applies:

$$\sum_{\substack{\nu=0 \\ |x-\frac{\nu}{n}|\geq\delta}}^{n}\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}\to 0\ \ \textit{for}\ \ n\to\infty$$

If we consider a game with a stochastic outcome, which has the win probability x , and if the game is played n times independently of each other, then the expression $\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}$ is the probability of winning exactly ν games, and the expression on the left side of

$$\sum_{\substack{\nu=0 \\ |x-\frac{\nu}{n}|\geq\delta}}^{n}\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}\leq\frac{1}{4n\delta^2}$$

is therefore the probability of won games divided by *n*.
It deviates at least δ from the probability *x*. The limit equation above secures that this probability goes to zero as $n\to\infty$.
This conjecture was first proven by Jacob Bernoulli in 1713, and has later been known as

The law of big numbers

Reference: Hand written Lectures notes from professor Børge Jessen. University of Copenhagen 1965