Vector Analysis

The gradient, curl and Laplace operators.
Stokes and Greens theorems.
Polar coordinates.

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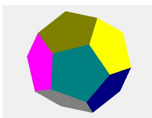

Ole Witt-Hansen 2018

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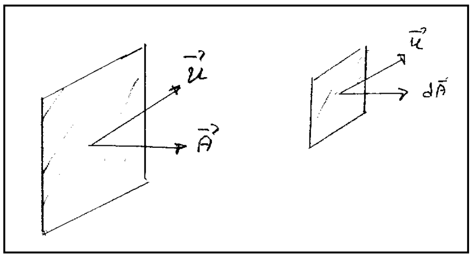
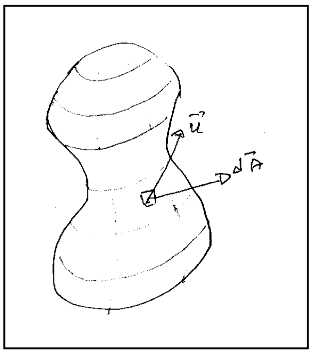


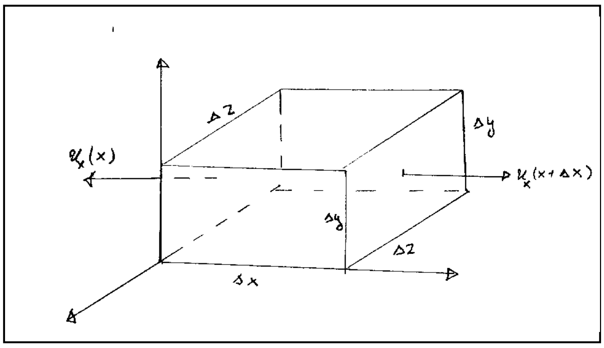
If we have a plane surface characterized by a vector $\vec{A}$, which is a vector normal to the surface, and the length of which is the area of the surface $A = |\vec{A}|$, and its direction is such that it together with the orientation of the surface forms a right hand screw. (The orientation of the squares in the figures is thus counter clock wise).

The flux Φ of a constant vector field $\vec{u}$ through the surface is then defined as u_n the normal component of $\vec{u}$ times A. If the surface is not flat or the vector field is not constant, the relation is written in differential form.

(1.1) $$\Phi = \vec{u} \cdot \vec{A} \qquad\qquad d\Phi = \vec{u} \cdot d\vec{A}$$

We shall be interested in finding an expression for the flux of a vector field through a closed surface.





The figure to the left shows a closed surface, where we have also drawn a surface element $d\vec{A}$. The total flux through the surface is found by the surface integral.

(1.2) $$\Phi = \int_{surface\ A} \vec{u} \cdot d\vec{A}$$

To find an alternative expression for the flux through a closed surface, we shall first consider the flux through a small rectangular box, as shown in the figure to the right.

The component of the vector field perpendicular to the side having lengths Δy and Δz is u_x.
The opposite directed normals to the flat at x and $x+\Delta x$ are also shown.
We shall then calculate the flux of the vector field through the box in the x direction.

$$\Delta\Phi_x = u_x(x+\Delta x)\Delta y\Delta z - u_x(x)\Delta y\Delta z = (u_x(x+\Delta x) - u_x(x))\Delta y\Delta z \approx \frac{\partial u_x}{\partial x}\Delta x\Delta y\Delta z = \frac{\partial u_x}{\partial x}\Delta V$$

Where ΔV is the volume of the box. Correspondingly we find for the two other sides:

$$\Delta\Phi_y = u_y(y+\Delta y)\Delta x\Delta z - u_y(y)\Delta x\Delta z = (u_y(y+\Delta y) - u_y(y))\Delta x\Delta z \approx \frac{\partial u_y}{\partial y}\Delta y\Delta x\Delta z = \frac{\partial u_y}{\partial y}\Delta V$$

$$\Delta\Phi_z = u_z(z+\Delta z)\Delta x\Delta y - u_z(z)\Delta x\Delta y = (u_z(z+\Delta z) - u_z(z))\Delta x\Delta y \approx \frac{\partial u_z}{\partial z}\Delta y\Delta x\Delta z = \frac{\partial u_z}{\partial z}\Delta V$$

For the total flux through the box, we thus find:

$$\Delta\Phi = \Delta\Phi_x + \Delta\Phi_y + \Delta\Phi_z = (\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z})\Delta x\Delta y\Delta z$$

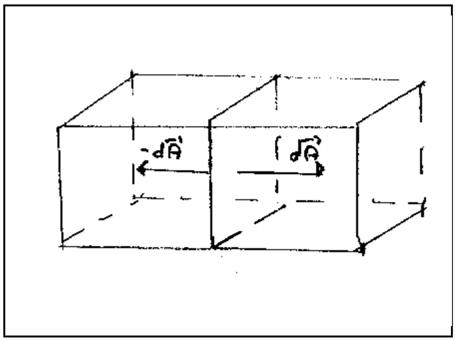
Or writing it differential form

(1.3) $$d\Phi == (\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z})dV$$

The expression $(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z})$ is called the *divergence* of the vector field, and is written symbolically:

(1.4) $$\vec{\nabla} \cdot \vec{u} \quad or \quad div\,\vec{u}$$

We cannot immediately transform (1.3) to integral form, since we must first show that (1.3) is valid for any volume. We do this by considering two adjacent small boxes having the one *x-y* side in common.



Since the area normal vectors $d\vec{A}$, are opposite on the common side of the two boxes, while the vector field is the same, there will be no contribution to the flux from this side. In the same manner we may put adjacent boxes on the *y-z* and *x-z* sides, but the flux from the inner adjacent flats will cancel. This allows us to integrate (1.3)

(1.4) $$\int_{surface} \vec{u} \cdot dA = \int \vec{\nabla} \cdot \vec{u}\, dV$$

(1.4) is usually called Stokes first theorem, or Gauss law:

The flux of a vector field through a closed surface is equal to of the divergence of the vector field integrated over the volume within the surface.

Stokes theorem is best illustrated by considering the flow of an incompressible fluid with density ρ.

The flow of the fluid through the area element $d\vec{A}$ is $\rho \vec{v} \cdot d\vec{A}$, where $\vec{v}$ is the velocity of the fluid. The flow out of a closed surface is therefore:

(1.5) $$\int_{surface\ A} \rho \vec{v} \cdot d\vec{A}$$

According to the continuity equation, (which states that the net flux out of a closed surface is equal to the production of matter inside the closed surface), we must have.

(1.6)
$$\int_{surface\ A} \rho \vec{v} \cdot d\vec{A} = \frac{\partial}{\partial t} \int_{volumen} \rho dV$$

On the other hand, according to stokes theorem:

$$\int_{surface\ A} \rho \vec{v} \cdot d\vec{A} = \int_{volume\ V} \vec{\nabla} \cdot (\rho \vec{v}) dV$$

So we must have:

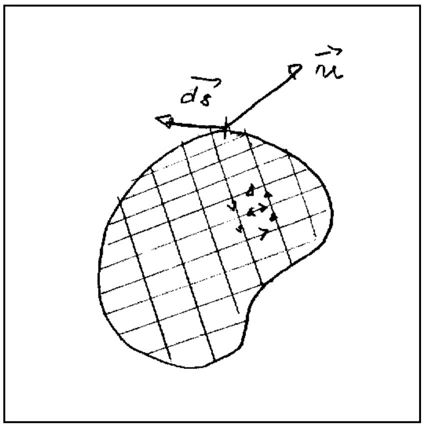
(1.7)
$$\int \vec{\nabla} \cdot (\rho \vec{v}) dV = \frac{\partial}{\partial t} \int_{volume} \rho dV$$

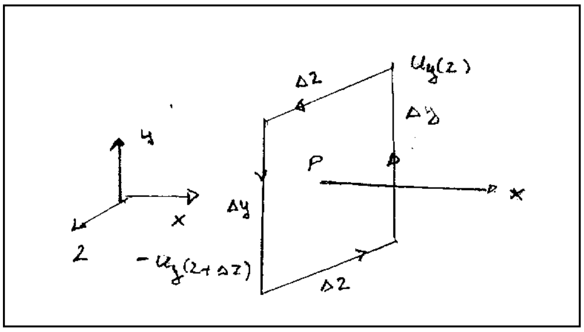
Since this equation must be valid for any volume, the two integrands must be equal, and the following equation must be valid.

(1.8)
$$\vec{\nabla} \cdot (\rho \vec{v}) = \frac{\partial \rho}{\partial t}$$

(1.8) represents the continuity equation in differential form.

2. The circulation of a vector field. Stokes second theorem





The circulation of a vector field $\vec{u}$ is defined as the line integral along a closed curve, as illustrated in the figure to the left.

(2.1)
$$\oint_{closed\ curve} \vec{u} \cdot d\vec{s}$$

To establish another expression for the circulation, we shall first evaluate the circulation of a flat small rectangular curve in the *y-z* plane, illustrated in the figure to the right.
First we do the two integrals along the *y*-axis at *z*.

$$\int \vec{u}\cdot d\vec{s} = \int u_y(z)dy \quad \text{and} \quad \int \vec{u}\cdot d\vec{s} = -\int u_y(z+\Delta z)\cdot dy$$

Taking the sum of the two terms we find:

$$\int \vec{u}\cdot d\vec{s} = -\int (u_y(z+\Delta z) - u_y(z))dy = -\frac{\partial u_y}{\partial z}\Delta z\Delta y$$

For the other two sides (parallel to the *z*-axis) of the rectangle we find in a similar way:

$$\int \vec{u}\cdot d\vec{s} = \int (u_z(y+\Delta y) - u_z(y))dy = \frac{\partial u_z}{\partial y}\Delta y\Delta z$$

Holding the two expressions together, we find the *x*-component of the *curl*, where the *curl* is written $(\vec{\nabla}\times\vec{u})$ in symbolic form.

(2.2) $$curl_x\vec{u} = (\vec{\nabla}\times\vec{u})_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}$$

In a quite similar manner we find:

$$curl_y\vec{u} = (\vec{\nabla}\times\vec{u})_y = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \qquad curl_z\vec{u} = (\vec{\nabla}\times\vec{u})_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$$

So that we have:

(2.3) $$curl\,\vec{u} = (\vec{\nabla}\times\vec{u}) = (\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}, \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}, \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y})$$

What we have obtained is:

(2.4) $$\oint_{rectangle:\,y-z} \vec{u}\cdot d\vec{s} = (\vec{\nabla}\times\vec{u})_x\Delta y\Delta z = \int (\vec{\nabla}\times\vec{u})_x dydz = \int (\vec{\nabla}\times\vec{u})\cdot d\vec{A}$$

Since this is now written as a vector equation, which is independent of the choice of coordinates, we may generalize to a rectangle with an arbitrary orientation:

(2.5) $$\oint_{small\ rectangle:} \vec{u}\cdot d\vec{s} = \int (\vec{\nabla}\times\vec{u})\cdot d\vec{A}$$

To extend this to an arbitrary curve, we should look at the figure above to the left, where we have separated the area bounded by the curve into small rectangles.
It is then easy to see that the contribution to the circulation from the sides of two adjacent rectangles must be zero, since the direction of integration is opposite, while the vector field is the same. So the contribution from all the internal rectangles will cancel and the circulation will only have contribution from the border of the curve. This is the content of Stokes second theorem.

(2.6) $$\oint \vec{u}\cdot d\vec{s} = \int (\vec{\nabla}\times\vec{u})\cdot d\vec{A} \qquad \text{meaning} \qquad \oint_{closed\ curve} \vec{u}\cdot d\vec{s} = \int_{area\ bounded\ by\ the\ curve} (\vec{\nabla}\times\vec{u})\cdot d\vec{A}$$

3. The operators of vector analysis

Below we list the most common functions and operators of vector analysis:

A **scalar field** is a function of time and position: $\varphi = \varphi(x,y,z,t)$.

A **vector field** i.e. $\boldsymbol{v} = (v_x, v_y, v_z)$ consists of three spatial components each being a function of position and time. The vector analysis uses the following mathematical symbols:

The **gradient** of a scalar field φ:

(3.1) $$\vec{\nabla}\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$ (Also written as *grad* φ)

Divergence of a vector field $\boldsymbol{v}$:

(3.2) $$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$ (also written as *div* $\boldsymbol{v}$)

The **Laplace operator**:

(3.3) $$\nabla^2\varphi = \vec{\nabla}\cdot\vec{\nabla}\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$ (Sometimes also written as $\Delta\,\varphi$)

Curl of a vector field:

(3.4) $$\vec{\nabla}\times\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$ (Also written as rot $\boldsymbol{v}$)

For any vector field the following identities are valid:

(3.5) $$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0 \quad \text{and} \quad \vec{\nabla}\times(\vec{\nabla}\varphi) = \vec{0}$$

These identities may be "relatively" easy proven, by writing the expressions in coordinates.
For the first relation we have:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

The two terms : $\frac{\partial}{\partial x}\frac{\partial v_z}{\partial y}$ *and* $-\frac{\partial}{\partial y}\frac{\partial v_z}{\partial x}$ cancel each other and similarly for the other pairs.

For the second relation, we settle for the x – component, if we put $(v_x, v_y, v_z) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$

and calculate: $$(\vec{\nabla}\times\vec{v})_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = \frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y} = 0.$$

The second rule (3.5) has the consequence that, if a vector field $\vec{v}$ is rotational free i.e. $\vec{\nabla}\times\vec{v}=0$, then the field can be written as the gradient of a scalar field $\vec{v}=\vec{\nabla}\varphi$, since only then $\vec{\nabla}\times(\vec{\nabla}\varphi)=\vec{0}$. This is for example the case for a conservative field of force: $\vec{F}$, since if F is conservative then

$$\oint_{\text{closed curve}} \vec{F}\cdot d\vec{s}=0 \quad \Leftrightarrow \quad \vec{\nabla}\times\vec{F}=0, \quad \text{so that } \vec{F}=\vec{\nabla}\varphi\text{, as it is well known.}$$

This also leads to the equation for the potential, if we use Gauss law. $\int_{\text{surface}}\vec{u}\cdot dA=\int_{\text{volume}}\vec{\nabla}\cdot\vec{u}\,dV$.

$$\int_{\text{surface}}\vec{F}\cdot dA=\int\vec{\nabla}\cdot\vec{F}\,dV=\int\vec{\nabla}\cdot(\vec{\nabla}\varphi)\,dV=\int\nabla^2\varphi dV$$

Especially if the left hand side is zero we find the well known potential equation:

(3.7) $$\nabla^2\varphi=0$$

3.1 Stokes first theorem

This equation represents the continuity equation in differential form.

The flux of a vector field through a closed surface is equal to of the divergence of the vector field integrated over the volume within the surface.

(3.8) $$\int_{\text{closed surface } A}\vec{v}\cdot d\vec{A}=\int_{\text{volume } V}\vec{\nabla}\cdot\vec{v}\,dV$$

3.2 Stokes second theorem

The curve integral of a vector field along a closed curve is equal to the surface integral of the curl of any open surface which has the curve as its border curve.

(3.9) $$\oint_{\text{closed curve}}\vec{v}\cdot d\vec{s}=\int_{\text{surface}}\vec{\nabla}\times\vec{v}\cdot d\vec{A}$$

3.3 Greens theorem

If U and V are two scalar fields, then Greens first theorem states:

(3.10) $$\int_{\text{closed surface}}(U\vec{\nabla}V-V\vec{\nabla}U)\cdot d\vec{\sigma}=\int_{\text{volume}}(U\nabla^2V-V\nabla^2U)d\Omega$$

Where $d\vec{\sigma}$ and $d\Omega$ are the area and the volume element respectively.

The theorem may be proven by applying Stokes first theorem (or Gauss' law).

$$\int_{closed\ surface\ \sigma} \vec{v} \cdot d\vec{\sigma} = \int_{volume\ \Omega} \vec{\nabla} \cdot \vec{v}\, d\Omega$$

If we in this equation put: $\vec{v} = U\vec{\nabla}V - V\vec{\nabla}U$, we find:

$$\int_{surface\ \sigma} (U\vec{\nabla}V - V\vec{\nabla}U) \cdot d\vec{\sigma} = \int_{volume\ \Omega} \vec{\nabla} \cdot (U\vec{\nabla}V - V\vec{\nabla}U) \cdot d\Omega$$

We then evaluate:

$$\begin{aligned}
&\vec{\nabla} \cdot (U\vec{\nabla}V - V\vec{\nabla}U) = \\
&\frac{\partial}{\partial x}(U\frac{\partial V}{\partial x}) + \frac{\partial}{\partial y}(U\frac{\partial V}{\partial y}) + \frac{\partial}{\partial z}(U\frac{\partial V}{\partial z}) - (\frac{\partial}{\partial x}(V\frac{\partial U}{\partial x}) + \frac{\partial}{\partial y}(V\frac{\partial U}{\partial y}) + \frac{\partial}{\partial z}(V\frac{\partial U}{\partial z})) = \\
&(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + U\frac{\partial^2 V}{\partial x^2}) + (\frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + U\frac{\partial^2 V}{\partial y^2}) + (\frac{\partial U}{\partial z}\frac{\partial V}{\partial z} + U\frac{\partial^2 V}{\partial z^2}) - \\
&((\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} + V\frac{\partial^2 U}{\partial x^2}) + (\frac{\partial V}{\partial y}\frac{\partial U}{\partial y} + V\frac{\partial^2 U}{\partial y^2}) + (\frac{\partial V}{\partial z}\frac{\partial U}{\partial z} + V\frac{\partial^2 U}{\partial z^2}))
\end{aligned}$$

We can see that all the mixed partial derivatives cancel each other, and we are left with:

$$(U\frac{\partial^2 V}{\partial x^2} + U\frac{\partial^2 V}{\partial y^2} + U\frac{\partial^2 V}{\partial z^2}) - (V\frac{\partial^2 U}{\partial x^2} + V\frac{\partial^2 U}{\partial y^2} + V\frac{\partial^2 U}{\partial z^2}) = U\nabla^2 V - V\nabla^2 U$$

And this completes the proof of Greens first theorem:

(3.10) $$\int_{closed\ surface} (U\vec{\nabla}V - V\vec{\nabla}U) \cdot d\vec{\sigma} = \int_{volume} (U\nabla^2 V - V\nabla^2 U) d\Omega$$

If we replace $U\vec{\nabla}V - V\vec{\nabla}U$ by $U\vec{\nabla}V$ we have:

$$\begin{aligned}
&\vec{\nabla} \cdot (U\vec{\nabla}V) = \\
&\frac{\partial}{\partial x}(U\frac{\partial V}{\partial x}) + \frac{\partial}{\partial y}(U\frac{\partial V}{\partial y}) + \frac{\partial}{\partial z}(U\frac{\partial V}{\partial z}) = \\
&(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + U\frac{\partial^2 V}{\partial x^2}) + (\frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + U\frac{\partial^2 V}{\partial y^2}) + (\frac{\partial U}{\partial z}\frac{\partial V}{\partial z} + U\frac{\partial^2 V}{\partial z^2}) = \vec{\nabla}U \cdot \vec{\nabla}V + U\nabla^2 V
\end{aligned}$$

And therefore:

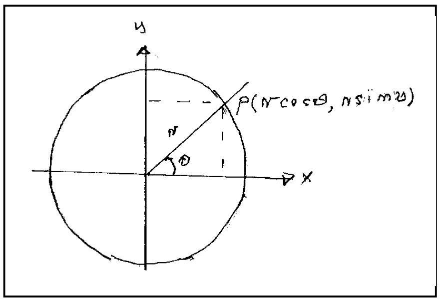
(3.11) $$\int_{closed\ surface} (U\vec{\nabla}V) \cdot d\vec{\sigma} = \int_{volume} \vec{\nabla}U \cdot \vec{\nabla}V d\Omega + \int_{volume} U\nabla^2 V d\Omega$$

If we specially put $U = V$, we find:

(3.12) $$\int_{closed\ surface} (U\vec{\nabla}U)\cdot d\vec{\sigma} = \int_{volume} (\vec{\nabla}U)^2 d\Omega + \int_{volume} U\nabla^2 U d\Omega$$

The equations (3.11) and (3.12) are often referred to as Greens second theorem.

4. Polar coordinates in the plane



The polar coordinates in the plane are (r,θ) as shown in the figure: We have:

(4.1) $$x = r\cos\theta \quad and \quad y = r\sin\theta$$

The infinitesimal distance vector is: $d\vec{s} = (rd\theta, dr)$
Since the direction of $rd\theta$ *and* dr are orthogonal the square of the distance element is.

(4.2) $$ds^2 = r^2 d\theta^2 + dr^2$$

And the area element dA is:

$$dA = rd\theta dr$$

These results obtained geometrically, may also be formally proven, by making a coordinate transformation.

$$dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial r}dr = -r\sin\theta d\theta + \cos\theta dr$$
$$dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial r}dr = r\cos\theta d\theta + \sin\theta dr$$

$$ds^2 = dx^2 + dy^2 = (-r\sin\theta d\theta + \cos\theta dr)^2 + (r\cos\theta d\theta + \sin\theta dr)^2 \quad \Rightarrow$$

$$ds^2 = r^2 d\theta^2 + dr^2$$

Since the double products cancel each other, and we have used the relation: $\cos^2\theta + \sin^2\theta = 1$.

The transformation of the area element is given by the Jacoby determinant:

$$dV = dxdy = \left| \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial r} \end{vmatrix} \right| d\theta dr = \left| \begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} \right| d\theta dr = |-r\sin^2\theta - r\cos^2\theta|\, rd\theta dr = rd\theta dr$$

The **gradient** of a scalar field $u = u(x, y)$ in the plane:

(4.3) $$\vec{\nabla} u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$$ (Also written as *grad* u)

The meaning of the gradient is that: $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \vec{\nabla} u \cdot d\vec{s}$

In polar coordinates the infinitesimal displacements in the directions (r, θ) are $(dr, rd\theta)$, so that:

$$\frac{\partial u}{\partial x} \to \frac{\partial u}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial y} \to \frac{\partial u}{r\partial \theta} \quad \text{such that } du = \frac{\partial u}{\partial r} dr + \frac{\partial u}{r\partial \theta} rd\theta$$

Divergence of a vector field ***v***:

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$ (also written as *div* ***v***)

The **Laplace operator**:

$$\nabla^2 \varphi = \vec{\nabla} \cdot \vec{\nabla} \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}$$ (Sometimes also written as Δ)

Solving equations in the plane involving the Laplace operator have the advantage that the equation

$$\nabla^2 \varphi = 0$$

is satisfied by the real as well as the complex part of any complex (holomorf) differentiable function $w = f(z)$, where $z = x + iy$. This is a consequence of the *Cauchy-Riemann differential equations*.

If $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ then we can establish two equations for the differential of *f*.

$$df(z) = f'(z)dz = f'(z)(dx + idy) = f'(z)dx + if'(z)dy$$

And

$$df(z) = (\frac{\partial u}{dx} + i\frac{\partial v}{dx})dx + (\frac{\partial u}{dy} + i\frac{\partial v}{dy})dy$$

Comparing the two expressions we see that: $f'(z) = (\frac{\partial u}{dx} + i\frac{\partial v}{dx}) \quad and \quad f'(z) = \frac{1}{i}(\frac{\partial u}{dy} + i\frac{\partial v}{dy})$ so

$$f'(z) = (\frac{\partial u}{dx} + i\frac{\partial v}{dx}) = \frac{1}{i}(\frac{\partial u}{dy} + i\frac{\partial v}{dy})$$

It then follows:

(4.5) $$\frac{\partial u}{dx} = \frac{\partial v}{dy} \quad and \quad \frac{\partial v}{dx} = -\frac{\partial u}{dy}$$

And consequently:

(4.4) $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}(-\frac{\partial v}{\partial x}) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

To establish an expression for the gradient in polar coordinates is only a bit more complex.

We take as the starting point the formulas: $x = r\cos\theta \quad and \quad y = r\sin\theta$.
And we then calculate:

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

By multiplying the first equation by $\cos\theta$ and the second by $r\sin\theta$, and adding the two equations we find after some reduction.

$$r\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \theta}\cos\theta + \frac{\partial u}{\partial r} r\sin\theta \quad \Leftrightarrow \quad \frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta + \frac{\partial u}{\partial r}\sin\theta$$

And subsequently multiplying the first equation by $\sin\theta$ and the second by $-r\cos\theta$, adding the two equations we find after some reduction.

$$r\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \theta}\sin\theta - \frac{\partial u}{\partial r} r\cos\theta \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta - \frac{\partial u}{\partial r}\cos\theta$$

From this we find the gradient in polar coordinates

(4.6) $$\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (\frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta - \frac{\partial u}{\partial r}\cos\theta, \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta + \frac{\partial u}{\partial r}\sin\theta)$$

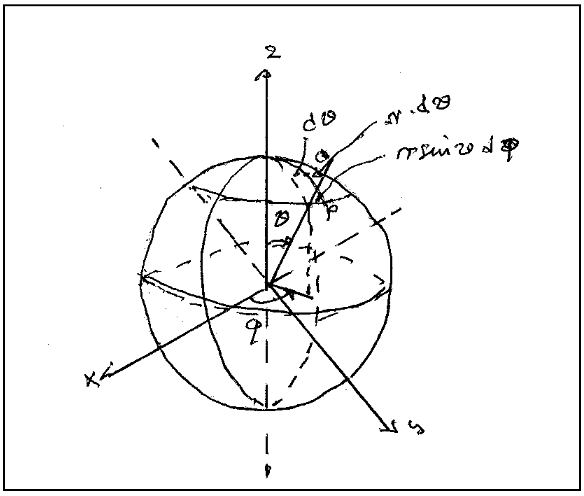
To try to find an expression for the Laplace operator

$$\nabla^2 u = (\frac{\partial^2 u_x}{\partial x^2}, \frac{\partial^2 u_y}{\partial y^2})$$

in polar coordinates is however not “recommendable”.

5. Polar coordinates in space

A point P(x, y, z) in space has the polar coordinates (r, θ, φ) See figure below. θ is the polar angle and φ is the azimuth angle.
The coordinates are, when expressed by the polar angles.



(5.1) $x = r\sin\theta\cos\varphi,\ \ y = r\sin\theta\sin\varphi, z = r\cos\theta$

To come from a point P (r,θ,φ) to $Q(r+dr,\ \theta + d\theta,\ \varphi + d\varphi)$ you first go "horizontally" $r\sin\theta\ d\varphi$ then "vertically" the distance $rd\theta$, and then the radial distance dr.
The infinitesimal distance vector becomes

(5.2) $d\vec{s} = (r\sin\theta d\phi, rd\theta, dr)$

Since the three "steps" are perpendicular to each other, the square of the distance is:

(5.3) $ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 + dr^2$

Likewise the infinitesimal volume element is the product of the three orthogonal steps.

(5.4) $dV = r^2\sin\theta d\theta d\varphi dr$

The two formulas derived geometrically above, can also be shown more formally, by evaluating: dx, dy, dz in polar coordinates and using:

$$d\vec{s}^{\,2} = dx^2 + dy^2 + dz^2, \text{ where } x = r\sin\theta\cos\varphi,\ \ y = r\sin\theta\sin\varphi, z = r\cos\theta$$

$$dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\varphi}d\varphi + \frac{\partial x}{\partial r}dr = r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi + \sin\theta\cos\varphi dr$$
$$dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\varphi}d\varphi + \frac{\partial y}{\partial r}dr = r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi + \sin\theta\sin\varphi dr$$
$$dz = \frac{\partial z}{\partial\theta}d\theta + \frac{\partial z}{\partial\varphi}d\varphi + \frac{\partial z}{\partial r}dr = -r\sin\theta d\theta + \cos\theta dr$$

Evaluating $ds^2 = dx^2 + dy^2 + dz^2$, we shall after some cumbersome calculations (15 terms) arrive at the same expression as (2.5). Well! The sums of the first two quadratic terms in the expression for dx and dy give: $r^2\cos^2\theta d\theta^2$ together with the first quadratic term i dz: $r^2\sin^2\theta d\theta^2$ gives: $r^2 d\theta^2$.
The sums of the next two quadratic terms in the expression for dx and dy give: $r^2\sin^2\theta d\varphi^2$

And sums of the last two quadratic terms in the expression for dx and dy give: $\sin^2\theta dr^2$ together with the second quadratic term i dz: $\cos^2\theta dr^2$ give: dr^2.
The double product of the first two terms in dx and dy cancel, and we are left with

$$
\begin{aligned}
&2r\cos\theta\cos\varphi\sin\theta\cos\varphi drd\theta - 2r\sin\theta\sin\varphi\sin\theta\cos\varphi drd\varphi + \\
&2r\cos\theta\sin\varphi\sin\theta\sin\varphi drd\theta + 2r\sin\theta\cos\varphi\sin\theta\sin\varphi drd\varphi = \\
&2r\cos\theta\sin\theta\cos^2\varphi drd\theta - 2r\sin^2\theta\sin\varphi\cos\varphi drd\varphi + \\
&2r\cos\theta\sin\theta\sin^2\varphi drd\theta + 2r\sin^2\theta\cos\varphi\sin\varphi drd\varphi - \\
&2r\sin\theta\cos\theta drd\theta
\end{aligned}
$$

$$
\begin{aligned}
&2r\cos\theta\sin\theta\cos^2\varphi drd\theta + 2r\cos\theta\sin\theta\sin^2\varphi drd\theta - 2r\sin\theta\cos\theta drd\theta = \\
&2r\cos\theta\sin\theta drd\theta - 2r\sin\theta\cos\theta drd\theta = 0
\end{aligned}
$$

Bringing it all together:

$$ds^2 = dx^2 + dy^2 + dz^2 = r^2\sin^2\theta d\theta^2 + r^2 d\theta^2 + dr^2$$

As stipulated geometrically.

Unfortunately the formal calculation of the volume element (although geometrically simple) becomes almost insuperable, since it involves the evaluation of a 3 x 3 determinant.

We want to prove that: $dV = r^2\sin\theta d\theta d\varphi dr$ from:

$$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi,\ z = r\cos\theta :$$

$$dV = dxdydz = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial r} \end{vmatrix} d\theta d\varphi dr \qquad (5.4)$$

$$dV = dxdydz = \begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi & \sin\theta\cos\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi & \sin\theta\sin\varphi \\ -r\sin\theta & 0 & \cos\theta \end{vmatrix} d\theta d\varphi dr$$

There are only four terms, so it rather straightforward to evaluate the determinant to.

$$r^2\cos\theta\cos\varphi\sin\theta\cos\varphi\cos\theta + r^2\cos\theta\sin\varphi\sin\theta\sin\varphi\cos\theta +$$
$$r^2\sin\theta\sin\theta\cos\varphi\sin\theta\cos\varphi + r^2\sin\theta\sin\theta\sin\varphi\sin\theta\sin\varphi =$$

$$r^2(\cos^2\theta\cos^2\varphi\sin\theta + \cos^2\theta\sin^2\varphi\sin\theta + \sin^2\theta\cos^2\varphi\sin\theta + \sin^2\theta\sin^2\varphi\sin\theta) =$$
$$r^2(\cos^2\theta\sin\theta(\cos^2\varphi + \sin^2\varphi) + \sin^2\theta\sin\theta(\cos^2\varphi + \sin^2\varphi) =$$
$$r^2(\cos^2\theta\sin\theta + \sin^2\theta\sin\theta) = r^2\sin\theta$$

To find the gradient of a scalar field in space is considerably more intriguing, since we must solve three equations with the unknowns: $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})$ to express them by $(\frac{\partial u}{\partial \theta}, \frac{\partial u}{\partial \varphi}, \frac{\partial u}{\partial r})$.

Let u be a **scalar field** which is a function of time and position: $u = u(x,y,z,t)$, and we recall that

$$x = r\sin\theta\cos\varphi,\ \ y = r\sin\theta\sin\varphi, z = r\cos\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = \frac{\partial u}{\partial x} r\cos\theta\cos\varphi + \frac{\partial u}{\partial y} r\cos\theta\sin\varphi + \frac{\partial u}{\partial z}(-r\sin\theta)$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \varphi} = \frac{\partial u}{\partial x}(-r\cos\theta\sin\varphi) + \frac{\partial u}{\partial y} r\sin\theta\cos\varphi$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial u}{\partial x}\sin\theta\cos\varphi + \frac{\partial u}{\partial y}\sin\theta\sin\varphi + \frac{\partial u}{\partial z}\cos\theta$$

We eliminate $\frac{\partial u}{\partial x}$ from the two first equations by multiplying the first equation by $\sin\varphi$ and the second equation by $\cos\varphi$. Then we have the two equations:

$$\frac{\partial u}{\partial \theta}\sin\varphi = \frac{\partial u}{\partial x} r\cos\theta\cos\varphi\sin\varphi + \frac{\partial u}{\partial y} r\cos\theta\sin^2\varphi + \frac{\partial u}{\partial z}(-r\sin\theta)\sin\varphi$$
$$\frac{\partial u}{\partial \varphi}\cos\varphi = \frac{\partial u}{\partial x}(-r\cos\theta\sin\varphi\cos\varphi) + \frac{\partial u}{\partial y} r\sin\theta\cos^2\varphi$$

Adding the two equations then gives:

$$\frac{\partial u}{\partial \theta}\sin\varphi + \frac{\partial u}{\partial \varphi}\cos\varphi = \frac{\partial u}{\partial y}(r\sin\theta\cos^2\varphi + r\cos\theta\sin^2\varphi) + \frac{\partial u}{\partial z}(-r\sin\theta)\sin\varphi$$

Then we eliminate $\frac{\partial u}{\partial x}$ from the last two equations by multiplying the first equation by $\sin\theta\cos\varphi$ and the second equation by $r\cos\theta\sin\varphi$. Then we have the two equations:

$$\frac{\partial u}{\partial \varphi}\sin\theta\cos\varphi = \frac{\partial u}{\partial x}(-r\cos\theta\sin\varphi\sin\theta\cos\varphi) + \frac{\partial u}{\partial y}r\sin\theta\cos\varphi\sin\theta\cos\varphi$$

$$\frac{\partial u}{\partial r}r\cos\theta\sin\varphi = \frac{\partial u}{\partial x}r\sin\theta\cos\varphi\cos\theta\sin\varphi + \frac{\partial u}{\partial y}r\sin\theta\sin\varphi\cos\theta\sin\varphi + \frac{\partial u}{\partial z}r\cos^2\theta\sin\varphi$$

Adding the two equations gives:

(V): $\frac{\partial u}{\partial \varphi}\sin\theta\cos\varphi + \frac{\partial u}{\partial r}r\cos\theta\sin\varphi = \frac{\partial u}{\partial y}(r\sin\theta\sin^2\varphi\cos\theta + r\sin^2\theta\cos^2\varphi) + \frac{\partial u}{\partial z}r\cos^2\theta\sin\varphi$

Together with the first equation, where we eliminated the term with: $\frac{\partial u}{\partial x}$.

(IV): $\frac{\partial u}{\partial \theta}\sin\varphi + \frac{\partial u}{\partial \varphi}\cos\varphi = \frac{\partial u}{\partial y}(r\sin\theta\cos^2\varphi + r\cos\theta\sin^2\varphi) + \frac{\partial u}{\partial z}(-r\sin\theta)\sin\varphi$

Multiplying (V) by $\sin\theta$ and multiplying (IV) by $\cos^2\theta$. We find:

(V):

$$\frac{\partial u}{\partial \varphi}\sin^2\theta\cos\varphi + \frac{\partial u}{\partial r}r\cos\theta\sin\theta\sin\varphi = \frac{\partial u}{\partial y}(r\sin^2\theta\sin^2\varphi\cos\theta + r\sin^3\theta\cos^2\varphi) + \frac{\partial u}{\partial z}r\cos^2\theta\sin\theta\sin\varphi$$

(IV):

$$\frac{\partial u}{\partial \theta}\sin\varphi\cos^2\theta + \frac{\partial u}{\partial \varphi}\cos^2\theta\cos\varphi = \frac{\partial u}{\partial y}(r\sin\theta\cos^2\varphi\cos^2\theta + r\cos\theta\sin^2\varphi\cos^2\theta) + \frac{\partial u}{\partial z}(-r\sin\theta)\sin\varphi\cos^2\theta$$

Adding the equations then gives:

$$\frac{\partial u}{\partial \varphi}\sin^2\theta\cos\varphi + \frac{\partial u}{\partial r}r\cos\theta\sin\theta\sin\varphi + \frac{\partial u}{\partial \theta}\sin\varphi\cos^2\theta + \frac{\partial u}{\partial \varphi}\cos^2\theta\cos\varphi =$$
$$\frac{\partial u}{\partial y}(r\sin^2\theta\sin^2\varphi\cos\theta + r\sin^3\theta\cos^2\varphi + r\sin\theta\cos^2\varphi\cos^2\theta + r\cos\theta\sin^2\varphi\cos^2\theta)$$

And finally we get the expression for *du/dy*:

$$\frac{\partial u}{\partial \varphi}\sin^2\theta\cos\varphi + \frac{\partial u}{\partial r}r\cos\theta\sin\theta\sin\varphi + \frac{\partial u}{\partial \theta}\sin\varphi\cos^2\theta + \frac{\partial u}{\partial \varphi}\cos^2\theta\cos\varphi =$$
$$\frac{\partial u}{\partial y}r(\sin^2\theta\sin^2\varphi\cos\theta + \sin^3\theta\cos^2\varphi + \sin\theta\cos^2\varphi\cos^2\theta + \cos\theta\sin^2\varphi\cos^2\theta)$$

(5.6) $$\frac{\partial u}{\partial y} = \frac{\frac{\partial u}{\partial \varphi}\sin^2\theta\cos\varphi + \frac{\partial u}{\partial r}r\cos\theta\sin\theta\sin\varphi + \frac{\partial u}{\partial \theta}\sin\varphi\cos^2\theta + \frac{\partial u}{\partial \varphi}\cos^2\theta\cos\varphi}{r(\sin^2\theta\sin^2\varphi\cos\theta + \sin^3\theta\cos^2\varphi + \sin\theta\cos^2\varphi\cos^2\theta + \cos\theta\sin^2\varphi\cos^2\theta)}$$

At this stage however it has become meaningless to evaluate $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{dz}$, although it can be done by inserting $\frac{\partial u}{\partial y}$ in (IV) and (V).

6. Using curvilinear orthogonal coordinates

For general orthogonal curvilinear coordinates in the plane (p_1, p_2) or in space (p_1, p_2, p_3), the distance element in the plane is $d\vec{s} = (g_1 dp_1, g_2 dp_2)$ and the distance element in space is $d\vec{s} = (g_1 dp_1, g_2 dp_2, g_3 dp_3)$, where the *g's* are functions of the coordinates (p_1, p_2) or (p_1, p_2, p_3). Since the curvilinear coordinates are orthogonal we have:

(6.1) $\quad ds^2 = g_1{}^2 dp_1{}^2 + g_2{}^2 dp_2{}^2 \quad$ and $\quad ds^2 = g_1{}^2 dp_1{}^2 + g_2{}^2 dp_2{}^2 + g_3{}^2 dp_3{}^2$

As we have seen earlier the polar coordinates in the plane are (r, θ) where

$$x = r\cos\theta \quad and \quad y = r\sin\theta$$

And the infinitesimal distance vector is: $d\vec{s} = (rd\theta, dr)$

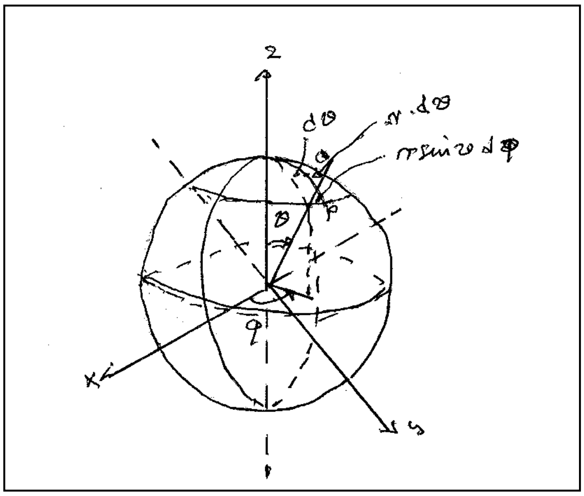
Since the direction of $rd\theta$ *and* dr are orthogonal, the square of the distance element is:

$$ds^2 = r^2 d\theta^2 + dr^2$$

And the area element dA is:

$$dA = rd\theta dr$$

The polar coordinates in space is (r, θ, φ) See figure below. θ is the polar angle and φ is the azimuth angle. The coordinates are when expressed by the polar angles.



$$x = r\sin\theta\cos\varphi,\ \ y = r\sin\theta\sin\varphi,\ \ z = r\cos\theta$$

$$d\vec{s} = (r\sin\theta d\phi, rd\theta, dr)$$

$$ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 + dr^2$$

$$dV = r^2\sin\theta d\theta d\varphi dr$$

The infinitesimal displacement along the curvilinear coordinates is in the plane $d\vec{s} = (g_1 dp_1, g_2 dp_2)$ and in space: $d\vec{s} = (g_1 dp_1, g_2 dp_2, g_3 dp_3)$, dx is replaced by $g_1 dp_1$ and so on, and therefore the gradient in curvilinear becomes:

(6.2) $$\vec{\nabla} U = (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}) \quad \to \quad \vec{\nabla} U = (\frac{1}{g_1}\frac{\partial U}{\partial p_1}, \frac{1}{g_2}\frac{\partial U}{\partial p_2}, \frac{1}{g_3}\frac{\partial U}{\partial p_3})$$

In the plane the gradient becomes in polar coordinates: $d\vec{s} = (rd\theta, dr)$:

(6.3) $$\vec{\nabla} U = (\frac{1}{r}\frac{\partial U}{\partial \theta}, \frac{\partial U}{\partial r})$$

And in polar coordinates in space: $d\vec{s} = (r\sin\theta d\varphi, rd\theta, dr)$

(6.4) $$\vec{\nabla} U = (\frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}, \frac{1}{r}\frac{\partial U}{\partial d\theta}, \frac{\partial U}{\partial r})$$

Concerning the divergence things become somewhat more complicated: This is because, when have a "rectangular" box in curvilinear coordinates the area of the front end and the back end are not necessarily equal.

The areas in a box with sides (1-2) (1-3) and (2-3) are $\Delta A_{23} = g_2 g_3 \Delta p_2 \Delta p_3$, $\Delta A_{13} = g_1 g_3 \Delta p_1 \Delta p_3$, $\Delta A_{12} = g_1 g_2 \Delta p_1 \Delta p_2$,so in the calculation of the change of flux, along one of the coordinates in a Cartesian coordinate system: $\frac{\partial u_x}{\partial x}\Delta x \Delta y \Delta z$ is replaced by
$\frac{1}{g_1}\frac{\partial}{\partial p_1}(g_2 g_3 u_1) g_1 \Delta p_1 \Delta p_2 \Delta p_3 = \frac{\partial}{\partial p_1}(g_2 g_3 u_1)\Delta p_1 \Delta p_2 \Delta p_3$, because $g_2 g_3$ may depend on p_1

The divergence however is defined by the flux out of the volume dV: $\int \vec{\nabla}\cdot \vec{u} dV$
The flux calculated in this manner from a "rectangular" box is, however:

(6.5) $$\begin{aligned}&\int (\frac{\partial}{\partial p_1}(g_2 g_3 u_1) + \frac{\partial}{\partial p_2}(g_1 g_3 u_2) + \frac{\partial}{\partial p_1}(g_1 g_2 u_3)) dp_1 dp_2 dp_3 = \\ &\int \frac{1}{g_1 g_2 g_3}(\frac{\partial}{\partial p_1}(g_2 g_3 u_1) + \frac{\partial}{\partial p_2}(g_1 g_3 u_2) + \frac{\partial}{\partial p_1}(g_1 g_2 u_3)) g_1 dp_1 g_2 dp_2 g_3 dp_3\end{aligned}$$

Since $dV = g_1 dp_1 g_2 dp_2 g_3 dp_3$ the correct expression for the divergence becomes.

(6.6) $$\vec{\nabla}\cdot\vec{u} = \frac{1}{g_1 g_2 g_3}\left(\frac{\partial}{\partial p_1}(g_2 g_3 u_1) + \frac{\partial}{\partial p_2}(g_1 g_3 u_2) + \frac{\partial}{\partial p_1}(g_1 g_2 u_3)\right)$$

In the *plane* we get similarly:

(6.7) $$\vec{\nabla}\cdot\vec{u} = \frac{1}{g_1 g_2}\left(\frac{\partial}{\partial p_1}(g_2 u_1) + \frac{\partial}{\partial p_2}(g_1 u_2)\right)$$

In polar coordinates in the plane $d\vec{s} = (rd\theta, dr)$, we find:

(6.8) $$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\left(\frac{\partial u_1}{\partial\theta} + \frac{\partial}{\partial r}(ru_2)\right) = \frac{1}{r}\frac{\partial u_1}{\partial\theta} + \frac{1}{r}\frac{\partial(ru_2)}{\partial r}$$

For the Laplace operator in the plane we have:

$$\nabla^2 U = \vec{\nabla}\cdot\vec{\nabla}U = \vec{\nabla}\cdot\left(\frac{1}{r}\frac{\partial U}{\partial\theta}, \frac{\partial U}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial U}{\partial\theta}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)$$

(6.9) $$\nabla^2 U = \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)$$

In *space* it is only somewhat more complicated:

For the gradient we have: $\vec{\nabla}U = \left(\frac{1}{r\sin\theta}\frac{\partial U}{\partial\varphi}, \frac{1}{r}\frac{\partial U}{\partial\theta}, \frac{\partial U}{\partial r}\right)$

To evaluate the divergence:

(6.10) $$\vec{\nabla}\cdot\vec{u} = \frac{1}{g_1 g_2 g_3}\left(\frac{\partial}{\partial p_1}(g_2 g_3 u_1) + \frac{\partial}{\partial p_2}(g_1 g_3 u_2) + \frac{\partial}{\partial p_1}(g_1 g_2 u_3)\right),$$

We notice that: $d\vec{s} = (g_1 dp_1, g_2 dp_2, g_3 dp_3) = (r\sin\theta d\varphi, rd\theta, dr)$, so

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\varphi}(ru_1) + \frac{\partial}{\partial\theta}(r\sin\theta\, u_2) + \frac{\partial}{\partial r}(r^2\sin\theta\, u_3)\right)$$

And for the Laplace operator in space

$$\nabla^2 U = \vec{\nabla}\cdot\vec{\nabla}U = \vec{\nabla}\cdot\left(\frac{1}{r\sin\theta}\frac{\partial U}{\partial\varphi}, \frac{1}{r}\frac{\partial U}{\partial\theta}, \frac{\partial U}{\partial r}\right) =$$
$$\frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\varphi}\left(r\frac{1}{r\sin\theta}\frac{\partial U}{\partial\varphi}\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\frac{1}{r}\frac{\partial U}{\partial\theta}\right) + \frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right)\right)$$

$$\nabla^2 U = \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)$$

The expression above is probably the most commonly applied for the Laplace operator in polar coordinates.
If the potential does not depends on φ the equation becomes somewhat more tangible (But still)

(6.11) $$\nabla^2 U = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial U}{\partial\theta}) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial U}{\partial r})$$

If the potential depends neither on φ nor θ it certainly gets simpler:

(6.12) $$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial U}{\partial r})$$

As you can see the potential $U(r) = \frac{\alpha}{r}$ is a solution to the equation: $\nabla^2 U = 0$, as we know is the equation for the potential in a rotational symmetric field.