The mathematics behind projections

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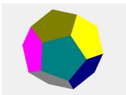

Ole Witt-Hansen 2005 (2015)

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1. The mathematics behind projections and 3-dim graphics

Doing computer graphics in the plane does usually not involve complex mathematics, as it does when working with projections into a plane from a 3-dim space.
In this article we shall only be concerned with the mathematical aspect of projection.
How to make a computer program, such that only the parts of the object facing the observer are drawn, are treated in my book (in Danish): "Programmering med Delphi" from 2004.

1.1 Central projection

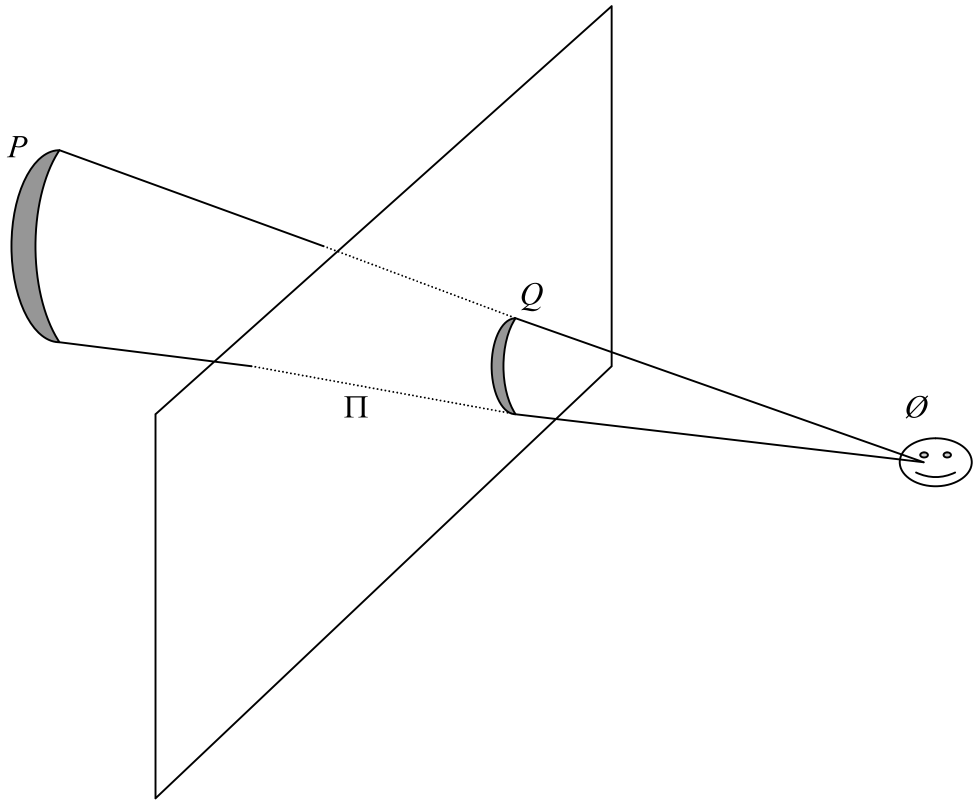
A central projection is a mapping of an object in space into a plane, as it is seen from an eye point
It could for example be a landscape seen through a rectangular opening in a wall corresponding to a painting frame.
In a central projection the space is seen, as it is the case for an observer, that is, perspectively.
In the perspective view remote objects (and the distances between them), will appear smaller than if they are positioned near to the image plane.

For painters the perspective has been a challenge since the medieval. It is a comprehensive part of the learning in every art school, how to draw a correct perspective. And it is certainly not a simple task. The basic methods for a painter is, however, entirely different, from the method used in computer graphics, since a painter initially draws a lot of lines according to certain rules, which all intersects in one point, called the disappearing point. This technique can for example be seen in sketches of Leonardo de Vinci and many others. The separation of the lines is then used as a scale for the objects in a distant position.

The painter perspective is founded on a complex technical graphical construction, whereas the perspective in computer graphics is based on mathematical calculations and geometrical constructions.
There is also another important difference. Whereas the painter perspective will result in one and only one painting, the computer based perspective (once the objects are determined mathematically), the objects may be seen from any angle and from any eye point.

Geometrically the central projection is simple. The observer is placed at the eye point *Ø*.
The image plane Π is placed in an arbitrary position in space. The only things that have to be determined are the shape of the image plane, and the distance from the eye point to the image plane. The object is placed on the opposite side of the image plane when seen from the eye point.

To determine the projection of a point P at the object a line is drawn from P to the eye point *Ø*.
The intersection point Q between the image plane Π and the line *PØ* is then the central projection of P on Π. This is thought illustrated in the figure below.



Figur 1.1 Central projection

Later we shall show how to calculate the coordinates to the projection *Q* , in a coordinate system in the projection plane, once the spatial coordinates to *Ø*, *P* is given together with the equation of the plane.

1.2 Parallel Projection

A parallel projection may be considered as a limiting case where the eye point is infinitely far from the projection plane. This implies that all lines from the object to the projection plane are parallel.

For practically purposes, this means that we may determine the parallel projection of a point *P* on the object simply as the usual orthogonal projection of a point into a line or into a plane.
In a parallel projection, the size of an object is independent of the distance from the image plane. The notion of a 3-dim object only comes about by viewing the angles under which the lines in the object intersects each other.

A parallel projection does not represent the surroundings as seen by an observer. This is anyway the case, when the objects in the image plane in the real world have very different distances from the image plane.
Even if the parallel projection is not lifelike, it is often use for objects which are not too far away. This has to do with the fact that we see with two eyes, and not one eye point, and two eyes perceptually compensate for the lack of perspective. Below the parallel perspective is illustrated.

Mathematically the parallel projection is easier to handle and requires less calculation as we shall see later.

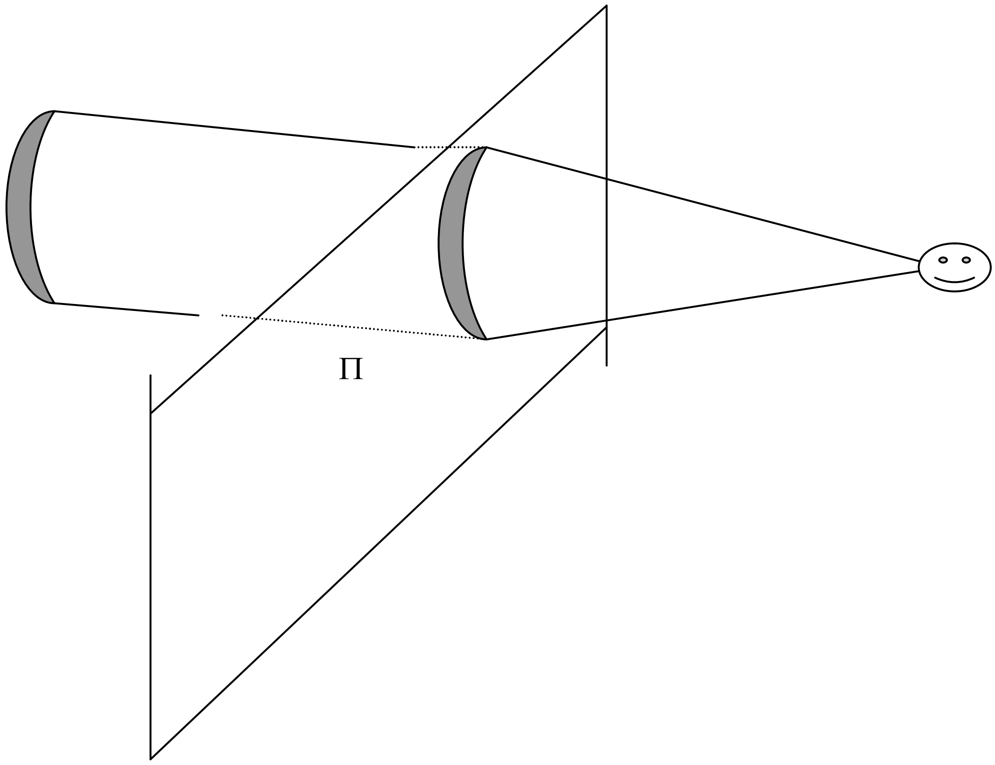


Figure 1.2 Parallel projection

1.3 Determination of the parallel projection of a point

The figure below is meant to illustrate how to determine the position of a point into the image plane.

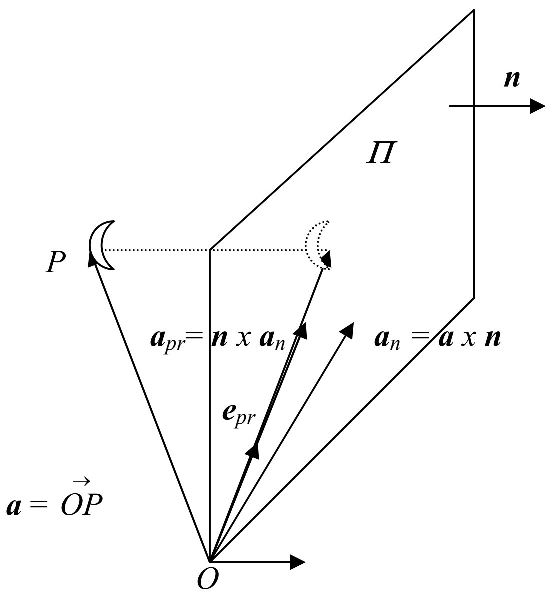


Figure 1.4 A parallel projection of a point into the image plane.

The determination of a projection of a point into a plane may in principle be done in to different ways, namely by analytical geometrics or by geometrical vectors.
Using analytic geometry it is straightforward. If P is the point, and $\boldsymbol{n}$ is a normal to the image plane Π, we only need to find the parametric for a line through P, that has $\boldsymbol{n}$ as its direction vector, and then find the intersection of this line with the image plane Π.

This is of course perfectly correct, but a problem arises, since the coordinate of the intersection with the plane are in a 3-dim space (x, y, z) coordinate system, but what we really want is the 2-dim "screen" coordinates, with its origin in the lower left corner of the screen.

The meaning of the vectors should almost be clear from the figure above.
$\boldsymbol{n}$ is a unit normal to the projection plane Π. P is the point in space, the projection of which we aim to determine. $\boldsymbol{a} = \overrightarrow{OP}$ is the position vector from the origin O to P.
We do the determination of the projection of P, using geometrical vectors, since then it is almost straightforward to find the coordinate of the projection in the projection plane.

First we determine $\boldsymbol{a}_n$, which is the *cross product* of $\boldsymbol{a}$ and $\boldsymbol{n}$. This vector is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{n}$, and it lies therefore in (parallel to) the plane Π. Subsequently we determine a vector $\boldsymbol{a}_{pr}$, which is the *cross product* of $\boldsymbol{n}$ with $\boldsymbol{a}_n$. This vector also lies in the plane Π, and it spans together with $\boldsymbol{a}$ the plane with is orthogonal to Π.
We may therefore find a position vector to the projection Q of P on Π, which is the projection of $\boldsymbol{a}$ on $\boldsymbol{a}_{pr}$.
We first find a unit vector $\boldsymbol{e}_{pr}$, which is aligned to $\boldsymbol{a}_{pr}$, and the projection $\boldsymbol{p}_{pr}$ is then found as demonstrated below. The coordinates (x,y) to Q in the coordinate system of the projection plane is then simply found by taking the scalar product of $\boldsymbol{p}_{pr}$ with the two base vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

$$\vec{a}_n = \vec{a} \times \vec{n} \qquad \vec{a}_{pr} = \vec{n} \times \vec{a}_n \qquad \vec{e}_{pr} = \frac{1}{\left|\vec{a}_{pr}\right|}\vec{a}_{pr} \qquad \overrightarrow{OQ} = \vec{p}_r = (\vec{a} \cdot \vec{e}_{pr})\vec{e}_{pr}$$

$$x = \vec{p}_r \cdot \vec{i} \qquad y = \vec{p}_r \cdot \vec{j}$$

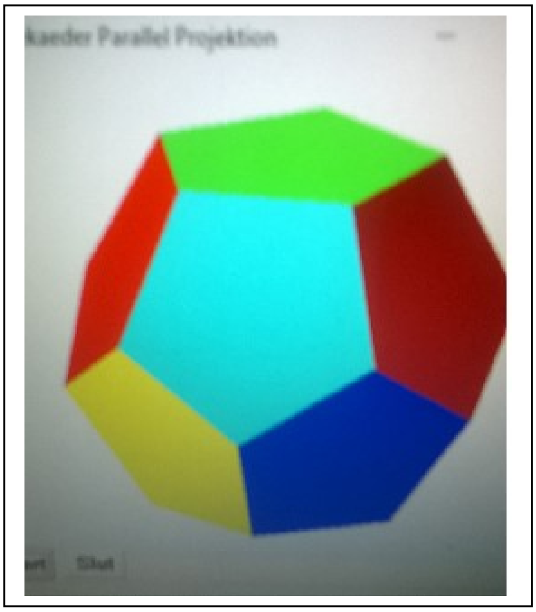
The determination of the parallel projection is different from, but not much simpler, than that of the central projection.
There is however a decisive mathematical difference, which makes parallel projection easier to cope with than central projections.
All the vector operations, which are applied to determine the parallel projection of a point are *linear* operations, which should be understood such: As if you want to determine the projection of a linear combination $\boldsymbol{c}$ of two vectors : $\boldsymbol{c} = s\cdot\boldsymbol{a} + t\cdot\boldsymbol{b}$, then it is equal to: $\boldsymbol{c}_{pr} = s\cdot\boldsymbol{a}_{pr} + t\cdot\boldsymbol{b}_{pr}$.

- This implies that doing a parallel projection, you only need to determine the projection of the three base vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ in the 3-dim coordinate system. The projection of any other vector $\boldsymbol{a}$ having coordinates (x,y,z). $\boldsymbol{a} = x\cdot\boldsymbol{e}_1 + y\cdot\boldsymbol{e}_2 + z\cdot\boldsymbol{e}_3$ can then be determined as the same linear combination of the projected bases vectors: $\boldsymbol{a}_{pr} = x\cdot\boldsymbol{e}_{1pr} + y\cdot\boldsymbol{e}_{2pr} + z\cdot\boldsymbol{e}_{3pr}$.

The projection of the base vectors $\boldsymbol{e}_{1pr}, \boldsymbol{e}_{2pr}, \boldsymbol{e}_{3pr}$, however, all lie in the image plane, naturally.

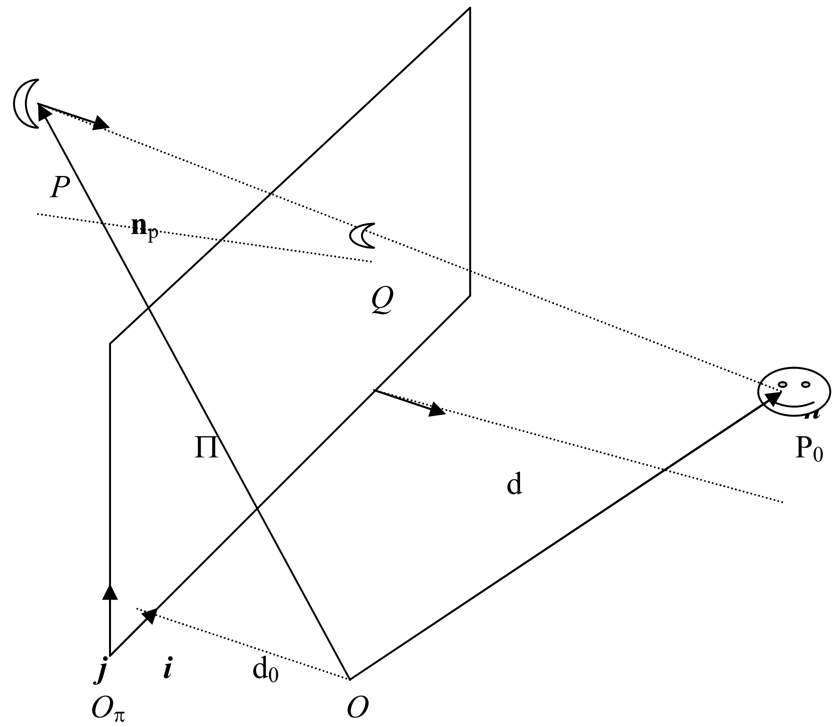
To make a parallel projection of a point it is thus only necessary to make six multiplications, which is much less than for a central projection.

Below is shown a computer generated parallel projection of a swim ball and a colorized dodecahedron
(The bad quality of the image is caused by that it is a photo taken from a computer screen).



1.4 Determination of the central projection of a point.

We shall then look at the geometrical method to find the central projection of a point on a screen. As it was the case with the parallel projection, the calculations are made using geometrical vectors, which makes the results independent of the coordinate system chosen.
The drawing below shows how the calculations are done. It is not entirely trivial.



Figue 1.5 The entral projection of a point on a screen.

Since the result, when expressed by vectors is independent of the coordinate system, we shall choose the 3-dim coordinate system to have the $x-y$ plane parallel to the image plane, and therefore share the same two base vectors $(\boldsymbol{i}, \boldsymbol{j})$.

The spatial coordinate system has its origin O together with the three base vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) =$ ((1,0,0), (0,1,0), (0,0,1)). The eye point lies on the positive z – axis at the distance d-d_0.
The image plane is perpendicular to the z-axis at the distance d_0 from O. The z-axis is directed towards the eye-point.

The coordinate system of the image plane has its origin at O_π and base vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.
The eye-point is at P_0.
We then wish to find the projection Q of a point P, which lies on the object in space.

$\boldsymbol{n}$ is a unit vector, which is perpendicular to the projection plane Π. The distance from the origin O to the plane Π is denoted d_0, and $\boldsymbol{n}_p$ is a unit vector parallel to $\overrightarrow{P_0P}$.

The determination of the coordinates (x,y) to the position vector $\overrightarrow{O_\pi Q}$ is the done in the following calculations.

$$\vec{PP_0} = \vec{OP_0} - \vec{OP} \qquad \vec{n}_p = \frac{1}{|\vec{PP_0}|}\vec{PP_0}$$

We wish to determine t_p, such that we may express the vector from the point P of the object to the intersection of line with the projection plane ∏ using $\boldsymbol{n_p}$:

$$\vec{PQ} = t_p \cdot \vec{n}_p$$

Taking the scalar product with $\boldsymbol{n}$ on both sides we find:

$$\vec{n} \cdot \vec{PQ} = t_p \, \vec{n} \cdot \vec{n}_p \quad and \quad \vec{n} \cdot \vec{PQ} = \vec{n} \cdot (\vec{OQ} - \vec{OP}) = d_0 - \vec{n} \cdot \vec{OP}$$

(Since the projection of $\vec{OQ}$ *on* $\vec{n}$ *is* d_0), From which follows:

$$t_p = \frac{(d_0 - \vec{n} \cdot \vec{OP})}{\vec{n} \cdot \vec{n}_p}$$

Furthermore we have:

$$\vec{OQ} = \vec{OP} + \vec{PQ} \qquad \vec{O_\pi Q} = \vec{OQ} - \vec{OO_\pi}$$

Where O_π is the origin of the coordinate system in the projection plane.
Then we can find the coordinates (x,y) to the projection Q of the point P in the coordinate system of the projection plane (O_π, $\boldsymbol{i}, \boldsymbol{j}$).

$$x = \vec{i} \cdot \vec{O_\pi Q} \quad and \quad y = \vec{j} \cdot \vec{O_\pi Q}$$

The signed projection of $\vec{OP}$ on $\boldsymbol{n}$ is $\boldsymbol{n} \cdot \vec{OP}$. (Positive in the figure above).

Cosine to the angle between $\boldsymbol{n}$ and $\boldsymbol{n_p}$ is determined as: $\boldsymbol{n} \cdot \boldsymbol{n_p}$. (in the figure a negative number).

What we want is to find the length $|PQ|$. But this length is precisely the distance from P to Π divided by $|cos(v)|$.

The distance from P to Π is then calculated as the length of the projection of $\vec{OP}$ on $\boldsymbol{n}$ minus d_0. (That the difference is inverted in the expression above for t_p ,is caused by the sign of $cos\ v$).

t_p is thus the length of $\vec{PQ}$, from which follows: $\vec{PQ} = t_p \cdot \boldsymbol{n_p}$.

Then it is only a minor problem to find the position vector for Q in the coordinate system of the projection plane, shown by the equations above.

Finally the coordinates to the projection point Q (x,y) by taking the scalar product with the two base vectors (O_π, $\boldsymbol{i}, \boldsymbol{j}$).

Since the calculation of the central projection involve non linear operations (the determination of t_p), it is not possible (as it was the case for the parallel projection) to settle for finding the projection of the three base vectors, and expanding the projection on these, using the same coordinates as for the point P in 3-dim space. The calculation must be performed for each projected point.

Below is shown a computer generated central projection. (A painters view of a tile walk, which prior was a mandatory craftsman test in all painting educational schools.
To document that the projection is made by using the formulas above and not just “painted by hand”. The tile walk is shown from two different eye points.
(The image quality is bad, yes because it is a photo of a screen)