

Solution of the third degree equation



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1. Polynomials of the third degree

Our purpose is to derive a solution formula for the roots of a third degree polynomial.

$$(1.1) \quad P(z) = a_3z^3 + a_2z^2 + a_1z + a_0$$

Where the coefficients are real numbers and a_3 is non-zero, while z may be a complex number.

We know that a third degree polynomial always has one real root, due to the fact that a third degree polynomial is a continuous function, which has both positive and negative values, and therefore also must have the value 0.

A third degree polynomial can have 1, 2 or 3 *real* roots, but according to the *fundamental theorem of algebra*, it has always 3 (complex) roots.

Our aim is to derive a formula for finding the 3 roots.

To begin with, we do a simple rewriting, as we remove the second order term, by setting $z = w + a$.

$$P(w) = a_3(w+a)^3 + a_2(w+a)^2 + a_1(w+a) + a_0$$

$$(1.2) \quad \begin{aligned} P(w) &= a_3(w^3 + 3w^2a + 3wa^2 + a^3) + a_2(w^2 + 2aw + a^2) + a_1(w+a) + a_0 = \\ &= a_3w^3 + (3aa_3 + a_2)w^2 + (\dots)w + (\dots) \end{aligned}$$

If we choose $a = -\frac{a_2}{3a_3}$, the second order term disappears, resulting in a third degree polynomial having the form:

$$(1.3) \quad P(w) = a_3(w^3 + pw + q) = a_3Q(w)$$

Where p and q may be calculated from (1.2). So if $Q(w)$ has the roots w_1, w_2, w_3 , then $Q(w)$ may be written as:

$$Q(w) = (w - w_1)(w - w_2)(w - w_3)$$

And thereby

$$P(z) = a_3(z - w_1 - a)(z - w_2 - a)(z - w_3 - a)$$

To determine the roots in an arbitrary third degree polynomial, we may settle for determining the roots in a polynomial of the form:

$$(1.4) \quad P(z) = z^3 + pz + q$$

We can assume that both p and q are non-zero, since otherwise the solution becomes trivial. Making the substitution: $z = u + v$, we get:

$$(1.5) \quad \begin{aligned} P(z) &= (u+v)^3 + p(u+v) + q = u^3 + v^3 + 3u^2v + 3uv^2 + p(u+v) + q \\ P(z) &= u^3 + v^3 + (3uv + p)(u+v) + q \end{aligned}$$

Now for each z we may choose u and v , such that: $u+v = z \wedge uv = -\frac{p}{3}$, since these equations just imply that u and v are roots in the quadratic equation: $x^2 - zx - \frac{p}{3} = 0$.

So as you probably remember from high school:

In the ordered and reduces quadratic equation, $x^2 + bx + c = 0$ the sum of the roots is equal to minus the coefficient to x , and their product is equal to the last term in the equation.

For this specific choice of u and v , we find:

$$(1.6) \quad P(z) = u^3 + v^3 + q$$

And the condition that $z = u + v$ is a root is therefore $u^3 + v^3 = -q$.

When two numbers which fulfil the conditions: $u^3 + v^3 = -q$ and $uv = -\frac{p}{3}$ it is equivalent to fulfilling the conditions:

$$(1.7) \quad u^3 + v^3 = -q \quad \text{and} \quad u^3v^3 = -\frac{p^3}{27}$$

According to the theorem about the sum and the product of the roots in a quadratic equation, this means that u^3 and v^3 are roots in the quadratic equation:

$$(1.8) \quad x^2 + qx - \frac{p^3}{27} = 0 \Leftrightarrow x = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

If $\frac{q^2}{4} + \frac{p^3}{27} < 0$, however, the two roots should be replaced by:

$$(1.9) \quad x = -\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}$$

However we shall continue our writing as if:

$$\frac{q^2}{4} + \frac{p^3}{27} \geq 0, \text{ but with the significance above, if } \frac{q^2}{4} + \frac{p^3}{27} < 0.$$

We therefore proceed taking the third root of the solution (1.8), as if we were dealing with real numbers.

$$(1.10) \quad \left. \begin{array}{l} u \\ v \end{array} \right\} = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

And thus finding the (possible complex) solutions: $z = u + v$

$$(1.11) \quad z = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

To determine the other (possible complex) solutions, we shall recall the binome equation:

$$(1.12) \quad z^n = a$$

Where $z = |z|(\cos x + i \sin x)$ and $a = |a|(\cos v + i \sin v)$

$$(1.13) \quad z^n = a \Leftrightarrow |z|^n (\cos nx + i \sin nx) = |a|(\cos v + i \sin v)$$

If the direction angles for a are: $v + p2\pi$, $p = 0, 1, 2, \dots$ we get immediately:

$$|z| = \sqrt[n]{|a|} \quad \text{and} \quad nx = v, v + 2\pi, v + 4\pi, v + (n-1)2\pi \quad \Leftrightarrow$$

$$|z| = \sqrt[n]{|a|} \quad \text{and} \quad x = \frac{v}{n} + p \frac{2\pi}{n}, \quad p = 0, 1, 2, \dots, (n-1)$$

So the complete solution becomes:

$$(1.14) \quad |z| = \sqrt[n]{|a|} \left(\cos\left(\frac{v}{n} + p \frac{2\pi}{n}\right) + i \sin\left(\frac{v}{n} + p \frac{2\pi}{n}\right) \right), \quad p = 0, 1, 2, \dots, (n-1)$$

For the equation $z^3 = a$, where a is a real number, we therefore find:

$$(1.15) \quad z = \sqrt[3]{a}, \quad z = \sqrt[3]{a} \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right), \quad z = \left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right)$$

Or

$$z = \sqrt[3]{a}, \quad z = \sqrt[3]{a} \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right), \quad z = \sqrt[3]{a} \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2} \right)$$

It is seen that: $\left(-\frac{1}{2} - i \frac{\sqrt{3}}{2}\right) = \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)^2$, so we put $\alpha = \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$ and $\alpha^2 = \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)$

The three roots in the equation then becomes, (since $\alpha^3 = 1$):

$$(1.16) \quad z = \sqrt[3]{a}, \quad z = \alpha \cdot \sqrt[3]{a}, \quad z = \alpha^2 \cdot \sqrt[3]{a}$$

If we return to the equation: $x = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$, which is equivalent with the two binome equations:

$$(1.17) \quad u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad \text{and} \quad v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Then if both right hand sides are real, we find according to (1.16).

$$u_0 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad u_1 = \alpha \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad u_2 = \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

(1.18)

$$v_0 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad v_1 = \alpha \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \quad v_2 = \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

One might think that it should result in six solutions to the equation $z = u + v$, but we recall that the solutions u and v must obey the equation: $uv = -\frac{p}{3}$. That this applies to u_0v_0 is seen from:

$$\begin{aligned} (1.19) \quad u_0v_0 &= \left(\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right) \left(\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \right) = \\ &= \sqrt[3]{\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right) \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)} = \sqrt[3]{-\frac{p^3}{27}} = -\frac{p}{3} \end{aligned}$$

A simple calculation shows that the three solutions, which comply with the claim: $uv = -\frac{p}{3}$ are:

$$(1.20) \quad u_0 + v_0, \alpha u_0 + \alpha^2 v_0 \text{ og } \alpha^2 u_0 + \alpha v_0, \text{ where } \alpha = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

This follows from $\alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$ and $\alpha^3 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + \frac{3}{4} = 1$.

Finally the solutions to the equation:

$$z^3 + pz + q = 0$$

Are then given by:

$$\begin{aligned} (1.21) \quad z &= \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \\ z &= \alpha \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \vee \\ z &= \alpha^2 \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \alpha \cdot \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \end{aligned}$$

Since every third degree equation: $a_3z^3 + a_2z^2 + a_1z + a_0 = 0$ can be transformed into the equation

$z^3 + pz + q = 0$ by the substitution $z \rightarrow z + a$ with $a = -\frac{a_2}{3a_3}$, we have proven that every third

degree equation has the solutions shown above.

Notice, however, we have used the square root, as if the argument was a positive number, but that is not necessarily the case, which we shall look into now.

We shall then consider the 3 cases:

- 1) $\frac{q^2}{4} + \frac{p^3}{27} = 0$. Here we must have $p < 0$: As u_0 and v_0 we may apply the real value of $\sqrt[3]{-\frac{q}{2}}$, and we therefore find, according to the solution formula above.

$$(1.22) \quad z_1 = u_0 + v_0 = 2\sqrt[3]{-\frac{q}{2}}, \text{ together with the double root: } z_2 = z_3 = -\sqrt[3]{-\frac{q}{2}}$$

- 2) $\frac{q^2}{4} + \frac{p^3}{27} > 0$. The numbers $-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ are then real, and we may apply the real cubic roots for u_0 and v_0 . We then have the one real root $z_1 = u_0 + v_0$ and the two complex conjugate roots.

$$(1.23) \quad \left. \begin{matrix} z_2 \\ z_3 \end{matrix} \right\} = \left. \begin{matrix} \alpha u_0 + \alpha^2 v_0 \\ \alpha^2 u_0 + \alpha v_0 \end{matrix} \right\} = -\frac{u_0 + v_0}{2} \pm i\sqrt{3} \frac{u_0 - v_0}{2}$$

- 3) $\frac{q^2}{4} + \frac{p^3}{27} < 0$. Here we must have $p < 0$. For u and v , we apply:

$$\left. \begin{matrix} u^3 \\ v^3 \end{matrix} \right\} = -\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}.$$

It appears that the two roots are complex conjugates, and therefore have the same modulus.

$$(1.24) \quad \sqrt{\left(-\frac{q}{2}\right)^2 + \left(\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}\right)^2} = \sqrt{-\frac{p^3}{27}}.$$

We then get the solutions to u and v , as we notice that:

$$\sqrt[3]{\sqrt{-\frac{p^3}{27}}} = \sqrt{-\frac{p}{3}} = |u| = |v|.$$

If we put:

$$-\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}} = \sqrt{-\frac{p^3}{27}}(\cos \varphi \pm i \sin \varphi), \quad \text{where} \quad \cos \varphi = \frac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}}$$

Then the applicable solutions to the equations becomes.

$$(1.25) \quad \left. \begin{matrix} u^3 \\ v^3 \end{matrix} \right\} = -\frac{q}{2} \pm i\sqrt{-\frac{q^2}{4} - \frac{p^3}{27}}$$

Equal to:

$$(1.26) \quad \left. \begin{array}{l} u_0 \\ v_0 \end{array} \right\} = \sqrt{-\frac{p}{3}} (\cos \frac{\varphi}{3} \pm i \sin \frac{\varphi}{3})$$

$$\left. \begin{array}{l} \alpha u_0 \\ \alpha^2 v_0 \end{array} \right\} = \sqrt{-\frac{p}{3}} (\cos \frac{\varphi+2\pi}{3} \pm i \sin \frac{\varphi+2\pi}{3})$$

$$\left. \begin{array}{l} \alpha^2 u_0 \\ \alpha v_0 \end{array} \right\} = \sqrt{-\frac{p}{3}} (\cos \frac{\varphi+4\pi}{3} \pm i \sin \frac{\varphi+4\pi}{3})$$

So in this case we get three *real* roots!

$$(1.27) \quad z_1 = 2\sqrt{-\frac{p}{3}} \cos \frac{\varphi}{3}, \quad z_2 = 2\sqrt{-\frac{p}{3}} \cos \frac{\varphi+2\pi}{3}, \quad z_3 = 2\sqrt{-\frac{p}{3}} \cos \frac{\varphi+4\pi}{3}$$

Determining the roots in a concrete example

To determine the numeric solutions to a specific cubic equation e.g. $z^3 - 2z^2 - 5z + 6 = 0$, (which is seen to have the roots $\{-2, 1, 3\}$), is rarely possible analytically, mostly because of the factor $\cos \frac{\varphi}{3}$.

To express $\cos \frac{\varphi}{3}$ by $\cos \varphi$ one has namely to solve a 3. degree equation in $\cos \frac{\varphi}{3}$.

For the equation: $z^3 - 2z^2 - 5z + 6 = 0$ we find:

$$p = 3(\frac{2}{3})^2 - 4(\frac{2}{3}) - 5 = \frac{4}{3} - \frac{8}{3} - 5 = -\frac{19}{3}$$

$$q = (\frac{2}{3})^3 - 2(\frac{2}{3})^2 - 5(\frac{2}{3}) + 6 = \frac{8}{27} - \frac{8}{9} - \frac{10}{3} + 6 = \frac{8-24-90}{27} + 6 = \frac{-106}{27} + 6 = \frac{56}{27}$$

$$\cos \varphi = \frac{-\frac{q}{2}}{\sqrt{-\frac{p^3}{27}}} = \frac{-\frac{56}{54}}{\sqrt{\frac{(\frac{19}{3})^3}{27}}}$$

Therefore there seem to be only one way to get the solutions, namely by numerical calculations.

Reference: Børge Jessen. Lectures on complex numbers. Mat 2. 1965 -1966. (Handwritten notes).