The complex number system

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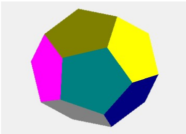

Ole Witt-Hansen 1981 (2018)

Contents

1. The complex number system

We shall begin by showing that the number pairs (a_1, a_2) where a_1 and a_2 are real numbers, when provided with two compositions for addition and multiplication form an algebra, which is called the complex numbers.

We shall also prefer to write a complex number with one letter such as: $a = (a_1, a_2)$, $b = (b_1, b_2)$, etc.
Some chose to write the number pair as (a, b), but that makes it awkward to identify it with a single letter.

By a composition $*$ within a set M, we should understand a mapping (a function) f: $M \times M \to M$.
But instead of writing $f(a,b) = c$, we write : $a * b = c$, e.g. $a + b = c$ or $a \cdot b = c$.

We denote the compositions "addition" and "multiplication" for number pairs with the usual symbols "+" and " ·", although they do not have the same significance as when used for real numbers.

Furthermore for all elements a,b,c belonging to M, we shall assume that the commutative and associative law apply for both compositions:

(1.1)

Commutative law: $a * b = b * a$

Associative law: $(a * b) * c = a * (b * c)$

The conditions that a set C form an algebra are the following:

1. Both compositions "addition" (+) and "multiplication" (·) form a *group* within C.
2. Multiplication is *distributive* with respect to addition, such that for any three elements a, b, c belonging to C applies:

(1.2) $$a \cdot (b + c) = a \cdot b + a \cdot c$$

Firstly we introduce a composition rule for addition, which is just conventionally addition of each component.

(1.3) $$a + b = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

Since the calculation rules for the real numbers apply for each of the components, this composition is both commutative and associative.
The neutral element is obviously $(0,0)$, and the opposite element to $a = (a_1, a_2)$ is $(-a_1, -a_2)$, as we also write as $-a = -(a_1, a_2)$.
The difference between two elements is: $a - b = (a_1, a_2) - (b_1, b_2) = (a_1 - b_1, a_2 - b_2)$

So while the expression for addition is the same as for the real numbers, the composition for multiplication is *not* $a \cdot b = (a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot b_1, a_2 \cdot b_2)$, but rather:

(1.4) $$a \cdot b = (a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$$

This composition is obviously commutative, but before we establish the associative and the distributive law, we shall introduce the imaginary unit *i*. We therefore calculate:

(1.4) $$(0,1) \cdot (0,1) = (-1,0) \text{ or } (0,1)^2 = (-1,0)$$

The complex number $(0,1)$ is denoted *i*, and it appears that $i^2 = -1$.
Since every complex number (a_1, a_2) may be written as: $(1,0)a_1 + (0,1)a_2 = a_1 + ia_2$, then from now on, we shall drop the notation $z = (x, y)$, and instead write:

(1.5) $$z = x + iy$$

For an arbitrary complex number.

Calculating the product of two complex numbers it is done in the same manner as for real numbers, as you just separate the real part from the imaginary part, remembering that $i^2 = -1$.

(1.6) $$\begin{aligned} &a \cdot b = (a_1 + ia_2) \cdot (b_1 + ib_2) = a_1 b_1 + ia_1 b_2 + ia_2 b_1 + i^2 a_2 b_2 = \\ &a_1 b_1 - a_2 b_2 + i(a_1 b_2 + a_2 b_1) \end{aligned}$$

This calculation rule is recognized as the composition, we already have introduced for multiplication of complex numbers. (since otherwise…)

It is straightforward but a bit lengthy to show the associative law for multiplication, so we shall settle for showing that multiplication of complex numbers is distributive with respect to addition.

$$\begin{aligned} &c(a+b) = ca + cb \quad \Leftrightarrow \\ &(c_1 + ic_2)(a_1 + b_1 + i(a_2 + b_2)) = (c_1 + ic_2)(a_1 + ia_2) + (c_1 + ic_2)(b_1 + ib_2) \end{aligned}$$

Evaluating the last expression gives, however:

$$\begin{aligned} &c_1 a_1 + ic_1 a_2 + ic_2 a_1 - c_2 a_2 + c_1 b_1 + ic_1 b_2 + ic_2 b_1 - c_2 b_2 = \\ &c_1(a_1 + b_1) + ic_2(a_1 + b_1) + ic_1(a_2 + b_2) + ic_2 i(a_2 + b_2) = \\ &(c_1 + ic_2)(a_1 + b_1 + i(a_2 + b_2)) \end{aligned}$$

The *complex conjugate* to a complex number $a = a_1 + ia_2$ is defined as $\overline{a} = a_1 - ia_2$.
We notice that:

(1.7) $$a\overline{a} = |a|^2 = (a_1 + ia_2)(a_1 - ia_2) = a_1^2 + a_2^2$$

Where $|a| = \sqrt{a_1^2 + a_2^2}$ is denoted the modulus, the numeric or absolute value of a complex number.

There apply some minor rules about the complex conjugate to a complex number.

(1.8) $$\overline{a+b} = \bar{a} + \bar{b}$$

(1.8) $$\overline{a-b} = \bar{a} - \bar{b}$$

$$\overline{a \cdot b} = \bar{a} \cdot \bar{b}$$

We settle for showing the last one:

$$\bar{a} \cdot \bar{b} = (a_1 - ia_2)(b_1 - ib_2) = a_1 b_1 - a_2 b_2 - i(a_1 b_2 + a_2 b_1)$$

$$\overline{ab} = \overline{(a_1 + ia_2)(b_1 + ib_2)} = \overline{a_1 b_1 - a_2 b_2 + i(a_1 b_2 + a_2 b_1)} = a_1 b_1 - a_2 b_2 - i(a_1 b_2 + a_2 b_1)$$

The inverse number to a non zero complex number a is:

(1.9) $$\frac{1}{a} = \frac{\bar{a}}{a\bar{a}} = \frac{a_1 - ia_2}{a_1^2 + a_2^2}$$

There is a tradition of never writing a complex number in the denominator, and always write a complex number in the form: $a + ib$, but this may always be accomplished by multiplying with the complex conjugate to the denominator in both the numerator and the denominator.

The equation: $ax = b$, where a and b are complex numbers have the solution:

(1.10) $$x = \frac{b}{a} = \frac{\bar{a}b}{\bar{a}a} = \frac{(a_1 - ia_2)(b_1 + ib_2)}{a_1^2 + a_2^2} = \frac{a_1 b_1 + a_2 b_2 + i(a_1 b_2 - a_2 b_1)}{a_1^2 + a_2^2}$$

1.1 de Moivre's formula. The exponential and the trigonometric functions

The complex number $a = a_1 + ia_2 = (a_1, a_2)$ may be represented as a point in an ordinary coordinate system. Where $r = |OA|$ is the distance from O to the point A.

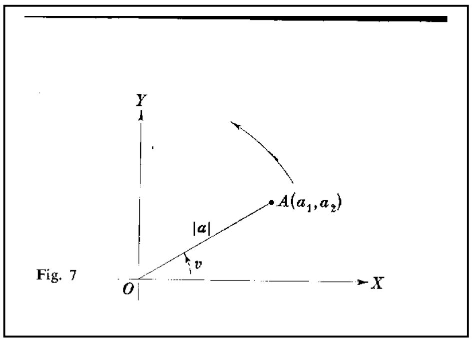

Fig. 7

(1.11) $$r = |OA| = |a| = \sqrt{a_1^2 + a_2^2}$$

Then $a = (a_1, a_2)$ can be written as:

$$a = |a|(\cos v, \sin v) = r_a(\cos v + i \sin v)$$

Where v is a direction angle for OA.

If: $b = |b|(\cos u, \sin u) = r_b(\cos u + i \sin u)$ is another complex number, then we may form the product of the two numbers:

$$a \cdot b = r_a r_b(\cos v + i \sin v)(\cos u + i \sin u) = r_a r_b(\cos u \cos v - \sin u \sin v + i(\cos u \sin v + \sin u \cos v))$$

If we recall the addition formulas for trigonometric numbers:

$$\cos(u\text{-}v) = \cos u \cdot\cos v + \sin u \cdot\sin v$$

$$\cos(u+v) = \cos u \cdot\cos v - \sin u \cdot\sin v$$

$$\sin(u\text{-}v) = \sin u \cdot\cos v - \cos u \cdot\sin v$$

$$\sin(u+v) = \sin u \cdot\cos v + \cos u \cdot\sin v$$

We can then see that:

(1.13) $$a \cdot b = r_a r_b(\cos(u+v) + i\sin(u+v))$$

Notably, if $a = b$ then $a^2 = r^2(\cos 2v + i\sin 2v)$

And by continuing the argument, we arrive at de Moivre's famous formula:

(1.14) $$a^n = r^n(\cos nv + i\sin nv)$$

Also from de Moivres formula follows *Eulers formula*, which is one of the most important formulas in mathematics. We define:

(1-15) $$e^{iy} = \cos y + i\sin y$$

Subsequently:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

However, the condition for this definition to make sense is that the complex exponential function must obey its *functional equation*.

(1.16) $$\begin{aligned} &e^{z_1}e^{z_2} = e^{z_1+z_2} \\ &e^{x_1+iy_1}e^{x_2+iy_2} = e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} = e^{x_1+x_2}e^{i(y_1+y_2)} \end{aligned}$$

But this property we have already shown above by using the additional formulas, but with other variable names, that is:

$$\begin{aligned} &a = r_a(\cos v + i\sin v) \text{ and } b = r_b(\cos u + i\sin u) \quad \Rightarrow \\ &a \cdot b = r_a r_b(\cos(u+v) + i\sin(u+v)) \end{aligned}$$

Or

$$e^{z_1} = e^{x_1+iy_1} = e^{x_1}(\cos y_1 + i\sin y_1) \quad \textit{and} \quad e^{z_2} = e^{x_2+iy_2} = e^{x_2}(\cos y_2 + i\sin y_2) \quad \Rightarrow$$

$$\begin{aligned} &e^{z_1}e^{z_2} = e^{x_1}e^{x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) = \\ &e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) = e^{x_1+x_2+i(y_1+y_2)} = e^{z_1+z_2} \end{aligned}$$

From (1.15), it follows immediately:

(1.17) $$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}) \quad \textit{and} \quad \sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix})$$

The trigonometric functions of a complex variable are therefore defined.

(1.18) $$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \quad and \quad \sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz})$$

The definitions of the hyperbolic functions are, on the other hand, the same.

(1.19) $$\cosh z = \tfrac{1}{2}(e^{z} + e^{-z}) \quad and \quad \sinh z = \tfrac{1}{2}(e^{z} - e^{-z})$$

From which we see that:

(1.20) $$\cosh z = \cos iz \quad and \quad i \sinh z = \sin iz$$

All the calculation rules and rules for differentiation of the trigonometric and hyperbolic functions can, however, be transferred directly to their complex analogous, without change.

1.21 Example. Harmonic oscillations.
In theoretical physics there are numerous differential equations, which lead to harmonic oscillations, that is, a solution of the general form.

$$x = A\cos(\omega t + \varphi)$$

To solve such equations it is often advantageous to use the complex exponential function because differentiation means multiplying with a (complex) constant.
Evidently only the real part can be the solution, but that offers no problem as long as the differential equations are linear.
If we look at the generic differential equation for a mass *m* suspended in a spring having spring constant *k*.

(1.22) $$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If we try with the solution

$$x = Ae^{i\omega t}$$

We get: $$-\omega^2 A e^{i\omega t} = -\frac{k}{m}Ae^{i\omega t} \quad \Rightarrow \quad \omega^2 = \frac{k}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$$

For this simple equation the advantages are moderate, since it can be easily solved by guessing the solution $x = A\cos(\omega t + \varphi)$.
Otherwise, when it is related to the equation for a damped harmonic oscillation. Such an equation can be reduced to:

(1.23) $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c \cdot x = 0 \qquad \text{where} \quad b > 0 \text{ and } c > 0$$

Solving this equation by traditional methods is by no means trivial.

So we try with a complex exponential of the form: $x = e^{z \cdot t}$, where *z* in general is a complex number.
It then follows:

$$\frac{dx}{dt} = z \cdot e^{z \cdot t} \quad and \quad \frac{d^2x}{dt^2} = z^2 \cdot e^{z \cdot t}$$

Inserting in (1.23) and reducing by $e^{z \cdot t}$ we get a quadratic equation.

$$z^2 + b \cdot z + c = 0$$

The discriminator is: $d = b^2 - 4 \cdot c$. If $d > 0$, then the quadratic equation has two real solutions.

(1.20) $$z = -\frac{b}{2} + \frac{\sqrt{b^2 - 4 \cdot c}}{2} \vee z = -\frac{b}{2} - \frac{\sqrt{b^2 - 4 \cdot c}}{2}$$

On the other hand if $d < 0$, then the quadratic equation has no real solution, but rather two complex solutions.

(1.21) $$z = -\frac{b}{2} + i\frac{\sqrt{4 \cdot c - b^2}}{2} \vee z = -\frac{b}{2} - i\frac{\sqrt{4 \cdot c - b^2}}{2} ,$$

If we put $\omega = \frac{\sqrt{4 \cdot c - b^2}}{2}$ then we get from the first equation:

$$x = Ae^{zt} = Ae^{-\frac{b}{2}t + i\omega t} = Ae^{-\frac{b}{2}t}(\cos \omega t + i \sin \omega t)$$

The physical solution is the real part, where we have added an initial phase φ , such that we have to integration constants A and φ , as we should have..

.(1.22) $$x = Ae^{-\frac{b}{2}t} \cos(\omega t + \varphi)$$

From obvious reasons (1.22) is called to solution for the damped harmonic oscillation.

2. Equations with complex numbers. The binome equation

We shall demonstrate the general solution to the equation:

(2.1) $$z^n = a \quad \Leftrightarrow \quad |z|^n (\cos nx + i \sin nx) = |a| (\cos v + i \sin v)$$

Where, $z = |z| (\cos x + i \sin x)$ and a are complex numbers

The direction angles for a are: $v + p2\pi, \quad p = 0,1,2...$,which immediately gives:

$$|z| = \sqrt[n]{|a|} \quad \text{and} \quad nx = v, v + 2\pi, v + 4\pi, v + (n-1)2\pi \quad \Leftrightarrow$$

$$|z| = \sqrt[n]{|a|} \quad \text{and} \quad x = \frac{v}{n} + p\frac{2\pi}{n}, \quad p = 0,1,2..., (n-1)$$

So the complete solution is:

(2.2) $$|z| = \sqrt[n]{|a|}(\cos(\frac{v}{n} + p\frac{2\pi}{n}) + i \sin(\frac{v}{n} + p\frac{2\pi}{n})), \ p = 0,1,2..., (n-1)$$

The equation: $z^n = a$ therefore has (for nonzero a) always n solutions.

2.1 The general solution to the quadratic equation, having real coefficients

Within the body of the real numbers the quadratic equation:

(2.3) $$ax^2 + bx + c = 0$$

Has none, one or two solution, depending on whether $d < 0$, $d = 0$ or $d > 0$, where $d = b^2 - 4ac$ is the discriminator

We shall then show, that within the body of the complex numbers, that the quadratic equation:

(2.4) $$az^2 + bz + c = 0$$

Always has at least one and at most two solution.

For non zero a, we shall do the same rewriting, as we did for the real quadratic equation, initiated by dividing by a.

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \quad \Leftrightarrow$$

$$\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\left(z + \frac{b}{2a}\right)^2 = \frac{d}{4a^2}$$

If $d \geq 0$, the equation has one or two solutions, which are determined by the solution formula.

$$z = \frac{-b \pm \sqrt{d}}{2a} \quad , \quad where \quad d = b^2 - 4ac$$

In the case $d < 0$, there are two complex solutions.

$$\left(z + \frac{b}{2a}\right)^2 = \frac{d}{4a^2} \quad \Leftrightarrow \quad \left(z + \frac{b}{2a}\right) = \frac{\pm i\sqrt{-d}}{2a} \quad \Leftrightarrow$$

$$z = \frac{-b \pm i\sqrt{-d}}{2a} \text{ , } where \quad d = b^2 - 4ac$$

If a, b, c are complex numbers it is not possible to give a general solutions formula.

The rewriting

$$az^2 + bz + c = 0 \quad \Leftrightarrow \quad \left(z + \frac{b}{2a}\right)^2 = \frac{d}{4a^2}$$

Is still valid, however, but the equation must in each case be solved as a binome equation.

3. The fundamental theorem of algebra.

We consider a n-degree complex polynomial :

(3.1) $$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0$$

Where n = 1, 2..., and the coefficients $a_0, a_1, a_2, \ldots a_{n-1}$ are complex numbers , where at least one is non zero. The fundamental theorem of algebra then states:

(3.2)	A n-degree polynomial has exactly n roots.

This theorem is not entirely simple to prove, but we shall conduct an adapted "proof", by first showing that a polynomial of degree $n > 0$ always has at least 1 root.
We remind you that for any complex number.

$$a = (a_1, a_2) = a_1 + ia_2$$

Where

$$|a| = \sqrt{a_1^2 + a_2^2}$$

Which is also the distance |OA|, where $A = (a_1, a_2)$.

Correspondingly for two complex numbers $a = (a_1, a_2)$ and $b = (b_1, b_2)$, representing two points A and B, in a coordinate system

$$|AB| = |a - b| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

Is equal to the distance $|AB|$.

If $a_0 = 0$ then the polynomial has the root $z = 0$, so we shall assume that $a_0 \neq 0$.

If $z = x + iy$, then the equation $|z| = r$ displays a circle having centre in O in the complex plane since: $|z| = r \Leftrightarrow x^2 + y^2 = r^2$

As z passes through a circle C_r with radius r, then $P(z)$ will run through a curve C_P.

Our aim is to demonstrate that $P(z)$ has a root, by showing that there must be a value of r, where C_P passes through O. Let M be the number:

$$M = \max\{1, |a_{n-1}|, |a_{n-2}|, \ldots, |a_0|\}$$

Then we have for $|z| > 1$:

$$\begin{aligned} &|P(z)-z^n| = |a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\ldots+a_1z+a_0| \le \\ &|a_{n-1}||z|^{n-1}+|a_{n-2}||z|^{n-2}+\ldots+|a_1||z|+|a_0| \le \\ &M(|z|^{n-1}+|z|^{n-2}+\ldots+|z|+1) \le Mn|z|^{n-1} \end{aligned}$$

Thus, we get:

(3.3) $$|z|>1 \;\Rightarrow\; |P(z)-z^n| \le Mn|z|^{n-1}$$

In quite a similar manner we have:

(3.4) $$|z|<1 \;\Rightarrow\; |P(z)-a_0| \le Mn|z|$$

Now we let z run through a circle, having centre O and radius R, where $R > 2Mn$.
Inserting $|z| = R$ in the first of the inequalities (3.3), we get:

(3.5) $$|z|>1 \;\Rightarrow\; |P(z)-z^n| \le Mn|z|^{n-1} < \tfrac{1}{2}RR^{n-1} = \tfrac{1}{2}R^n$$

As z runs through a circle with radius R once, then z^n will run n times through a circle with R^n.
This follows, since: $z = R\,e^{iv}$ implies $z^n = R^n e^{inv}$.

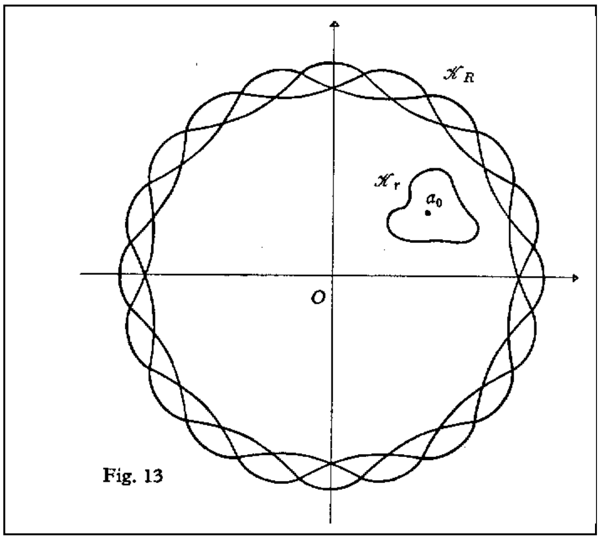

Fig. 13

In the figure is illustrated an example of the be behaviour of $P(z)$, when z runs through a circle with radius R^n.
The inequality (3,3) shows that the distance between z^n and $P(z)$ is less than $\frac{1}{2}R^n$, as long as $|z|>1$. Therefore the curve that $P(z)$ follows , will enclose the point O. On the other hand if :

$$|z| < 1 \;\wedge\; |z| < \frac{|a_0|}{2Mn}$$

it follows from the inequality (3.4):

$$|z|<1 \;\Rightarrow\; |P(z)-a_0| \le Mn|z|$$, that

(3.6) $$|P(z)-a_0| \le \tfrac{1}{2}a_0$$

$P(z)$ will therefore, for small radii $|z - a_0| = r$, lie on a curve, which does not enclose O.

The curve that $P(z)$ describes in the two limits, corresponding to the inequalities (3.3) and (3.4) must somewhere (because of continuity) in between have passed through O , so there must be at least one value, where $P(z)$ =0.

When we have proved that any polynomial of degree $n > 0$ has at least one root, the *fundamental theorem of algebra* follows from the theorems we know from polynomials of a real variable, which, without restriction, can be taken over by the complex polynomials.

1. If α is a root in a polynomial $P(z) \Leftrightarrow P(\alpha)=0$, then $P(z)$ is divisible by $(z-\alpha)$.

(3.7) $$P(z) = (z - \alpha)Q(z)$$

2. If $P(z)$ has degree $n > 0$, then $Q(z)$ has degree n-1.

Since we have already shown that any polynomial with degree $n > 0$ has at least one root α, we may write:

(3.8) $$P(z) = (z - \alpha_1)Q_{n-1}(z)$$

Where the degree of $Q_{n-1}(z)$ is $n - 1$. If the degree of $Q_{n-1}(z)$ is greater than 0 then $Q_{n-1}(z)$ has at least one root, such that:

$$Q_{n-1}(z) = (z - \alpha_2)Q_{n-2}(z)$$

And thus:

$$P(z) = (z - \alpha_1)(z - \alpha_2)Q_{n-2}(z)$$

This line of argument can be continued until the degree of the Q-polynomial is 0. We may then write $P(z)$ as:

(3.9) $$P(z) = (z - \alpha_1)(z - \alpha_2)...(z - \alpha_n)Q_0$$

Where Q_0 is a constant.

From the equation (3.9) we conclude (from the zero-rule) that a polynomial of degree n has the n roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, and that it can have no other roots, which is the contents of :

T*he fundamental theorem of algebra*.

As it is the case for real polynomials the roots in first and second degree polynomials are easily found, but somewhat more complicated for 3. degree polynomials, and as you probably know, there are no general methods to determine the roots in polynomials having degree larger than 4.