The peculiar Fibonacci numbers

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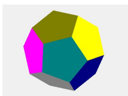

Ole Witt-Hansen 2008

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1. The Fibonacci numbers

The Fibonacci numbers originates from an Italian mathematician Leonardo Fibonacci.
The are usually denoted by $a_0, a_1, a_2, \ldots, a_n$, such that $a_0 = 0$ and $a_1 = 1$, and the obey the very simple recursion relation:

$$a_{n+2} = a_{n+1} + a_n$$

Using $a_0 = 0$ and $a_1 = 1$ one easily finds:

$$\begin{aligned}
a_2 &= 0 + 1 = 1, \\
a_3 &= 1 + 1 = 2 \\
a_4 &= 2 + 1 = 3 \\
a_5 &= 3 + 2 = 5 \\
a_6 &= 5+3 = 8 \\
a_7 &= 8 + 5 = 13 \\
a_8 &= 13 + 8 = 21 \\
a_9 &= 21 + 13 = 34 \\
a_{10} &= 34+ 21 = 55
\end{aligned}$$

According to the mathematical mythology the Fibonacci numbers first appear in a small thesis on rabbit breeding!

2. Rabbit breeding

As a model we shall assume that there is a constant period T from a rabbit is born until it is sexually mature, and furthermore there is the same period T from a rabbit pair are sexually mature until they give birth. Finally it assumed that a rabbit pair always get exactly get two cubs! And this goes on and on with the same period.

To determine the number of rabbit pairs after n periods ($t = nT$), we may reason as follows:
a_n is the number of rabbit pairs after n - 1 periods. We start out with just one pair, so $a_1 = 1$.
The rabbit pair become sexual mature after one period T, but they have not yet given birth, so $a_2 = 1$.

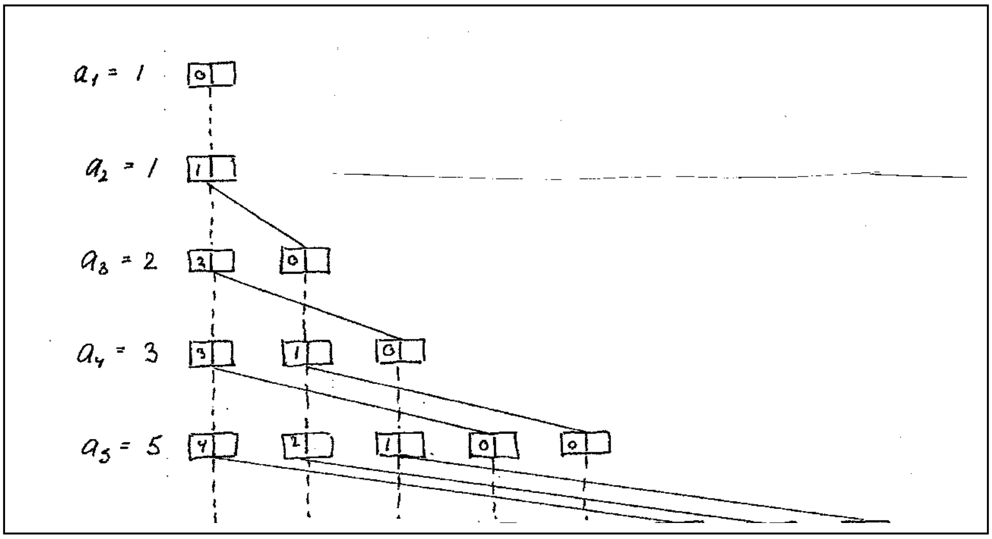
After 2 periods the first rabbit pair has born one pair of cubs, so $a_3 = 2$. After yet another period, the first rabbit pair have had another pair of cubs, so $a_4 = 3$.

Generally we will reason as follows. The number of rabbit pairs after n +1 periods, a_{n+2} is equal to the number of rabbit pairs after n periods a_{n+1} in additionto the numbers of new pairs of cubs, which is according to the premises, is the same as the number of rabbit pairs to periods earlier a_n.
We must therefore have:

$$a_{n+2} = a_{n+1} + a_n$$

Which is exactly the recursion equation for the Fibonacci numbers.

Below is a graphic illustration of the number of rabbit pairs until the eighth level.



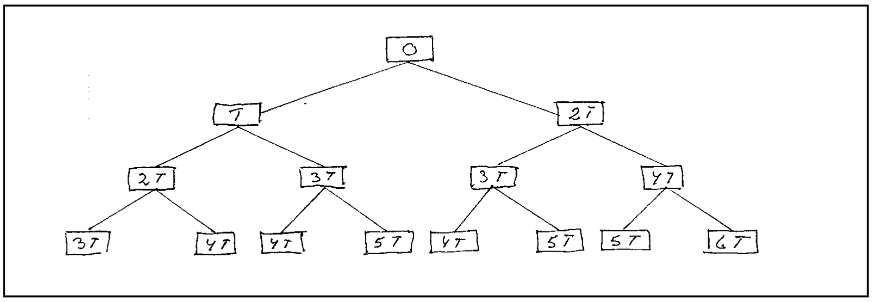
3. Telephone alerting chains

The Fibonacci numbers seem to pop up in many connections. For example, if a group of people shall receive a certain message by telephone in the fastest manner. To optimize the chain means that everybody should have the message in the shortest possible time. Telephone chains are almost always organized in a manner, so a person who receives a call, makes calls to two persons in the next level of the hierarchy.

To model this situation, we do some reasonable assumptions, namely that there passes an interval T, (e.g. 1 minute), from you receive a call until you can make the first call, and another interval T, until you can make the second call.

Such a traditional chain is shown graphically below having 4 levels. The obvious disadvantage is that the persons in the same level receive the message at rather different times. Most flagrant it is seen in the forth level, where the person to the left (along the line of the first call) receives the message at time $3T$, whereas the person far to the right receives the message at time $6T$.

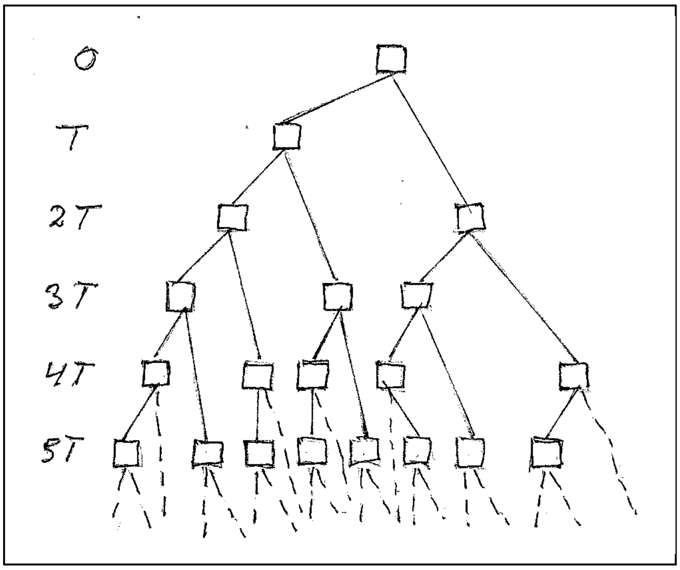
Generally the largest time difference in receiving a call in the nth level is $(n-1)T$ from the first to the last person.



The relation between the number of persons N in the chain and the number of levels n.

$$N = 2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1 \quad \text{Thus} \quad N = 2^n - 1 \ \text{ or } n = \log_2(N+1)$$

In the example above: $n = 4$, and $N = 15$.



We now ask ourselves the task to design a telephone chain, so that everyone in the same level gets the message simultaneously.
On the figure to the left every square marks a person in the chain, and the lines going to and from the chains connect the persons which either receive a call or make a call.
Notice that opposite of what is the case in the traditional chain described above, a person in the *n*th level first makes a call to someone in the n+1'th level, (time$T$) and next to one in the n +2'th level. (time $2T$).
So everybody in the n+1'th level receives a call T later, than everyone in the *n*th level, and everyone in the n +2'th level receives a call $2T$ later than everyone in the *n*th level.
Furthermore it is seen, that the number of persons in the *n*th level is precisely the Fibonacci number a_n. This can be verified by the following reason:

The persons who receive the message in the n+2'th level a_{n+2}, are the ones who have received the message in the n+1'th level a_{n+1}, (because they have passed the message on to the n+2'th level) plus them in the *n*th level, (because they have also passed the message on to the n+2'th level), having the same delay. So we must have:

$$a_{n+2} = a_{n+1} + a_n$$

That is, the recursion relation for the Fibonacci numbers.
With only 6 levels the advantage compared to the traditional telephone chain are of no real importance ($6T$ is reduced to $5T$). The telephone chain described above is therefore of more theoretical interest, but it is applied in computer science, where the number of participants in the chains are huge.
The number of elements in the n+1'th row is close to doubling of the n'th row, since:

$$2\,a_{n+1} > a_{n+1} + a_n > 2\,a_n \quad \Leftrightarrow \quad 2\,a_{n+1} > a_{n+2} > 2\,a_n$$

4. A formula for the Fibonacci numbers

We shall now show that the Fibonacci number a_n can be expressed as:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

It is easy to show that the formula is valid for n =0,1,2. For n>2, it is necessary to introduce the notion of difference equations, which is nothing but a functional equation with integer variables. For the Fibonacci numbers it may be written as:

$$f(n+2) = f(n+1) + f(n)$$

Where we have replaced a_n with $f(n)$. Our task is then to determine $f(n)$. If we extend the range of the variable n from integers to real numbers x we have:

$$f(x+2) = f(x+1) + f(x) \quad \Leftrightarrow \quad f(x+2) - f(x+1) - f(x) = 0$$

A (qualified) guess to the solution of the equation could be: $f(x) = b \cdot a^x$, and inserting this in the recursion equation gives:

$$b \cdot a^{x+2} - b \cdot a^{x+1} - b \cdot a^x = 0$$

Dividing by $b \cdot a^x$ we obtain the characteristic equation:

$$a^2 - a - 1 = 0 \text{ , which has the solutions } a = \frac{1 \pm \sqrt{5}}{2}$$

It is possible to show that the complete solution to the difference equation has the form:

$$f(x) = c_1 \cdot f_1(x) + c_2 \cdot f_2(x),$$

where $f_1(x)$ and $f_2(x)$, correspond to each of the two solutions a of the characteristic equation, and the constants c_1 and c_2 arte to be determined from the Fibonacci numbers $a_0 = 0$ and $a_1 = 1$

$$f(0) = 0 \quad \text{and} \quad f(1) = 1$$

where

$$f(x) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^x + c_2\left(\frac{1-\sqrt{5}}{2}\right)^x$$

$$f(0) = 0 \quad \Leftrightarrow \quad c_1 + c_2 = 0 \quad \text{and} \quad f(1) = 1 \quad \Leftrightarrow \quad c_1\left(\tfrac{1+\sqrt{5}}{2}\right) + c_2\left(\tfrac{1-\sqrt{5}}{2}\right) = 1$$

They are easily solved to give: $c_2 = -c_1 \ \wedge \ c_1 = \dfrac{1}{\sqrt{5}}$

Yielding the expression for the nth Fibonacci number:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

As you may have observed $\dfrac{1+\sqrt{5}}{2} \approx 1{,}618$ is what is named "the golden cut".

In fact one may show that:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}$$

If we put

$$b = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad c = \frac{1-\sqrt{5}}{2}.$$

Then it follows, since $c < b$.

$$\frac{a_{n+1}}{a_n} = \frac{b^{n+1} - c^{n+1}}{b^n - c^n} = \frac{b - c\left(\dfrac{c}{b}\right)^n}{1 - \left(\dfrac{c}{b}\right)^n} \to b = \frac{1+\sqrt{5}}{2} \quad for \;\; n \to \infty \qquad \text{since} \;\; \left(\frac{c}{b}\right)^n \to 0 \quad for \;\; n \to \infty$$