Taylor's Formula
and
Series expansions

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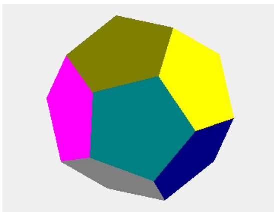

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1. Taylor's formula

To derive Taylor's formula for series expansions of real functions, we shall apply some results from analysis. Firstly the definition of the differential *dy* of a real function $y = f(x)$.

$$dy = df(x) = f'(x)dx$$

Then the formula for integration by parts:

$$\int f(x)g'(x)dx = f(x)\cdot g(x) - \int g(x)f'(x)dx$$

Which, may be written in a more compact way by means of differentials .

$$\int f(x)dg(x) = f(x)\cdot g(x) - \int g(x)df(x)$$

For any infinitely often differentiable function defined in an interval [a,b] Taylor's formula reads:

$$f(b) = f(a) + \frac{f'(a)(b-a)}{1!} + \frac{f''(a)(b-a)^2}{2!} + \frac{f^{(3)}(a)(b-a)^3}{3!} + \ldots + \frac{f^{(n)}(a)(b-a)^n}{n!} + \int_a^b \frac{f^{(n+1)}(x)(b-x)^n}{n!}dx$$

The formula is proven by successive applications of partial integration on the function $f'(x)$ (supplied with some simple tricks). According to the definition of the definite integral, we have:

$$f(b) - f(a) = \int_a^b f'(x)dx$$

We rewrite it a bit, so that the integration variable is now *b - x*.

$$f(b) - f(a) = -\int_a^b f'(x)d(b-x)$$

The reason for this is that it makes the contribution from the upper limit to vanish. when we do the partial integration.

$$f(b) - f(a) = -\int_a^b f'(x)d(b-x) = -\left(\left[f'(x)(b-x)\right]_a^b - \int_a^b (b-x)df'(x)\right) = f'(a)\cdot(b-a) + \int_a^b (b-x)f''(x)dx$$

Notice that the contribution from the upper limit vanishes in the first term.

The last integral is then remodelled in the following way:

$$\int_a^b (b-x)f''(x)dx = -\int_a^b f''(x)d\frac{(b-x)^2}{2!}$$

Again we apply partial integration on the right hand integral, and the calculations go entirely as before. We just state the result.

$$-\int_a^b f''(x)d\frac{(b-x)^2}{2!} = f''(a)\cdot\frac{(b-a)^2}{2!} - \int_a^b f^{(3)}(x)d\frac{(b-x)^3}{3!}$$

More generally it can be shown in the same manner:

$$-\int_a^b f^n(x)d\frac{(b-x)^n}{n!} = f^{(n)}(a)\cdot\frac{(b-a)^n}{n!} - \int_a^b f^{(n+1)}(x)d\frac{(b-x)^{n+1}}{(n+1)!}$$

If we continue applying the formula above, we arrive at Taylor's formula for a finite number of terms.

$$f(b) = f(a) + \frac{f'(a)(b-a)}{1!} + \frac{f''(a)(b-a)^2}{2!} + \frac{f^{(3)}(a)(b-a)^3}{3!} + ... + \frac{f^{(n)}(a)(b-a)^n}{n!} + \int_a^b \frac{f^{(n+1)}(x)(b-x)^n}{n!}dx$$

2. Maclaurin series

Most frequently one meets the formula applied with $b = x$ and $a = 0$.
To avoid misconceptions the integration variable x is replaced by t. This gives the series.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n + \int_0^x \frac{f^{(n+1)}(t)}{n!}t^n dt$$

Taylor's formula is also denoted as the *Taylor expansion* of $f(x)$ from $x = 0$, and the last integral is called the remainder term.

The formula is mostly applied, if we let. $n \to \infty$ as the remainder goes to zero. In that case, we obtain a infinite series expansion of the function, which is called the Maclaurin series.

If the remainder vanishes for $n \to \infty$, it follows that the series is uniform convergent, that is, the infinite sum has a finite value for any x.

Concerning the estimate of the remainder integral, we shall make some fairly reasonable assumptions:

We presuppose that $f^{(n)}(x)$ is limited in the interval $[a, b]$ for all n, so that there exists a number M, such that $|f^{(n)}(x)| < M$ in $[a, b]$ for all n. The remainder integral is then estimated in the following way.

$$\int_0^x \frac{f^{(n+1)}(t)}{n!} t^n dt \le \int_0^x \frac{\left|f^{(n+1)}(t)\right|}{n!} \left|t^n\right| dt \le M \cdot \int_0^x \frac{\left|t^n\right|}{n!} dt \le M \cdot \frac{\left|x^{n+1}\right|}{n!} = M \cdot |x| \frac{|x| \cdot |x| \cdots |x| \cdot |x|}{1 \cdot 2 \cdots (n-1)n} \cdot$$

In the last expression $\left|\frac{x}{k}\right| < 1$ for a certain $n = k$. If we put the factors to the left of k equal to K, we obtain the estimate:

$$M \cdot |x| \frac{|x| \cdot |x| \cdots |x| \cdot |x|}{1 \cdot 2 \cdots n} = K \cdot \left|\frac{x}{k}\right| \cdot \left|\frac{x}{k+1}\right| \cdots \left|\frac{x}{n}\right|$$

In this product any of the factors $\left|\frac{x}{p}\right| < 1$, where $p = k, k+1, \ldots n+1$, and the product will converge to 0, when for $n \to \infty$,since the product is less than the last factor, which goes to 0 for $n \to \infty$.

The result of these considerations are, provided that $f^{(n)}(x)$ is limited in [a, b] for all n, that the Maclaurin-series for $f(x)$ is convergent.

Before we demonstrate the Maclaurin series for various standard functions, we shall show another frequently applied variant of Taylor's formula, where we expand $f(x_0 + h)$ from x_0.

Here we put $a = x_0$ and $b = x_0 + h$, and thus $b - a = h$.

$$f(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!} h + \frac{f''(x_0)}{2!} h^2 + \frac{f^{(3)}(x_0)}{3!} h^3 + \ldots + \frac{f^{(n)}(x_0)}{n!} h^n + \int_0^h \frac{f^{(n+1)}(x_0 + t)}{n!} t^n dt$$

For the remainder, we find:

$$\left|\int_0^h \frac{f^{(n+1)}(x_0 + t)}{n!} t^n dt\right| \le M \frac{h^n}{n!} h = M \frac{h^{n+1}}{n!}$$

Often the expression above is used to approximate $f(x_0 + h)$, when h is small. Especially the formula is applied in numerical integration, where only the first two terms are involved.

$f(x_0 + h) \approx f(x_0) + f'(x_0)h$ when h is a small increment.

This formula is often referred to as the linear approximation to $f(x)$ in the vicinity of x_0.

If $|h| < 1$ is a small number, e.g. $h = 0.01$, then $h^2 = 0.0001$ is considered as vanishing small compared to h, and that justifies that we discard terms of magnitude h^2, since such terms will go to 0 after division by h in the limit $h \to 0$.
The same reason can be applied for h^2 and h^3 . If the formula is used as a n'th order approximation to a function then the remainder term is of order h^{n+1}, and therefore is small when h is small.

3. Expansion series for e^x

Below we shall derive the Mclaurin series for e^x, cos x, sin x, ln(1+x) and $(1+x)^{\alpha}$.
Firstly the exponential function.

$$f(x)=e^x \quad \Rightarrow \quad f^{(n)}(x) = e^x \,,$$

From which it follows that $f^{(n)}(x)$ has limited variations in every limited interval. Furthermore

$$f^{(n)}(0)=1.$$

So the remainder integral term will tend to 0 for $n \to \propto$ for any x in the interval.
The Mclaurin-series becomes.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ... + \frac{x^n}{n!} + ...$$

Or written with the summation symbol

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

4. Maclaurin series for *sin x* og *cos x*

For sine and cosine $|f^{(n)}(x)| < 1$ for all x, so the series will automatically converge.
Futhermore for *cos x*:

$f(0) = \cos(0)=1,\ f'(0) = -\sin(0) = 0,\ f''(0) = -\cos(0) = -1\ ,f^{(3)}(0) = \sin(0) = 0,\ f^{(4)}(0) = \cos(0) = 1$, and so on.

Correspondingly for sin x:

$f(0) = \sin(0) = 0,\ f'(0) = \cos(0) = 1,\ f''(0) = -\sin 0 = 0\ ,\ f^{(3)}(0) = -\cos 0 = -1,\ f^{(4)}(0) = \sin(0) = 0,$ and so on..

This leads to the series expansions

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \quad \text{or} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \quad \text{or} \quad \sin x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$$

5. A series expansion for *ln(1+ x)*

We shall the look at the series expansion of $f(x)$ = ln(1+x) from $x = 0$. We find::

$$f(0) = \ln(1) = 0$$
$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$
$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$$
$$f^{(3)}(x) = 2\frac{1}{(1+x)^3} \Rightarrow f^{(3)}(0) = 2$$
$$f^{(4)}(x) = -2\cdot 3\frac{1}{(1+x)^4} \Rightarrow f^{(4)}(0) = -2\cdot 3$$
$$f^{(n)}(x) = (-1)^{n-1}\cdot 1\cdot 2\cdot 3\cdots(n-1)\frac{1}{(1+x)^n} \qquad \Rightarrow$$
$$f^{(n)}(0) = (-1)^{n-1}\cdot 1\cdot 2\cdot 3\cdots(n-1) = (-1)^{n-1}\cdot(n-1)!$$

In this case, however, $f^{(n)}(x)$ is not limited for any n, so we have to make an individual estimate of the integration remainder. We shall show that the Maclaurin series is convergent only for $|x| \leq 1$.

The estimate of the remainder $R(x)$ is a bit more complicated than in the previous cases.:

$$R_n(x) = \int_0^x \frac{f^{(n+1)}(t)}{n!}t^n dt \leq \left|\int_0^x \frac{(-1)^n n!}{(1+t)^{n+1} n!}t^n dt\right| \leq \int_0^x \frac{|t|^n}{|1+t|^{n+1}}dt$$

If we determine $k = min(|1+t|)$, where $t \in [0, x]$, we get the estimate:

$$R_n(x) \leq \int_0^x \frac{|t|^n}{|1+t|^{n+1}}dt \leq \frac{1}{k^{n+1}}\int_0^x |t|^n\ dt = \frac{1}{k^{n+1}}\frac{|x|^{n+1}}{n+1} = \frac{1}{n+1}\frac{|x|^{n+1}}{k^{n+1}}$$

As we can see, the last expression will tend to 0 for $n \to \infty$, when $-1 < x \leq 1$, what had to be shown. We therefore find

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{5} + ...$$
$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

The series

$$\sum_{k=1}^{n}\frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$$

Is named the harmonic series. It is not convergent, since it is an upper sum for $f(x) = \frac{1}{x}$, and

.
$$\int_1^n \frac{1}{x}dx = \ln n \quad \to \infty \quad for \quad n \quad \to \infty$$

The alternating harmonic series is, however, convergent, since it is the series expansion for ln 2.

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

6. Series expansion for $(1+x)^{\alpha}$

Next we look at the series expansion for $f(x) = (1+x)^{\alpha}$, where $x > -1$ and α is a real number different from 0. Not because the series is very significant in it self, but as shown below, it can be used to derive series expansions for some of the inverse trigonometric functions.

$$f(0) = 1$$
$$f'(x) = \alpha(1+x)^{\alpha-1} \quad \Rightarrow \quad f'(0) = \alpha$$
$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \Rightarrow f''(0) = \alpha(\alpha-1)$$

$$f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)\cdot(1+x)^{\alpha-n} \Rightarrow f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1)$$

We shall use a generalized form of the combinatorial symbol.

$$\binom{\alpha}{n} = \frac{\alpha\cdot(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n} \quad \text{for n > 0} \quad \text{and where} \quad \binom{\alpha}{0} = 1$$

In almost the same manner, as we did when treating ln(1+ x), one can show that the remainder goes to 0, when $n \to \infty$, and $|x| < 1$.

$$(1+x)^{\alpha} = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{2}x^3 + \ldots = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$$

7. Series expansions for *$\sin^{-1}x$* og *$\tan^{-1}x$*

If we replace x with x^2 and put $\alpha = -1$ or replace x with $-x^2$ and put $\alpha = -\frac{1}{2}$, we arrive at the differential quotients for $\tan^{-1}(x)$ and $\sin^{-1}(x)$. More precisely:

$$\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}dt \qquad \text{for } |x| < 1$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}dt$$

Applying the expansion for $(1+x)^{\alpha} = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{2}x^3 +$

With $x = -t^2$ and $\alpha = -½$, we get: $\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 +$, and consequently:

$$\sin^{-1}(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}dt \int_0^x (1 + \tfrac{1}{2}t^2 + \tfrac{3}{8}t^4 + ...)dt = x + \frac{x^3}{6} + \frac{3x^5}{40} + ...$$

Giving

$$\sin^{-1}(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + ... \quad \text{for } |x| < 1$$

With $x = t^2$ and $\alpha = -1$, we get: $\frac{1}{1+t^2} = 1 - t^2 + t^4 +$, and consequently:

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}dt = \int_0^x (1 - t^2 + t^4 + ...)dt = x - \frac{x^3}{3} + \frac{x^5}{5} - ...$$

Giving

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - ... \quad \text{for } |x| < 1$$

These to series may among other things be applied to establish two series expansions for π , since $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$ and $\tan^{-1}(1) = \frac{\pi}{4}$

$$\frac{\pi}{6} = \sin^{-1}(\frac{1}{2}) = \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + ... \quad \text{and} \quad \frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - ...$$

The last series however converges all to slow for any practical purpose of calculating π