Spherical Geometry

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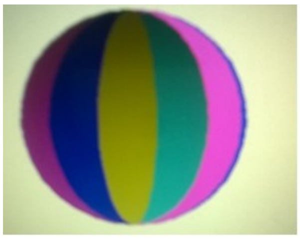

Ole Witt-Hansen nov. 2016

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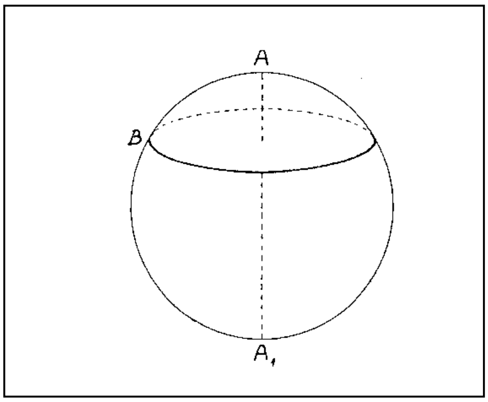
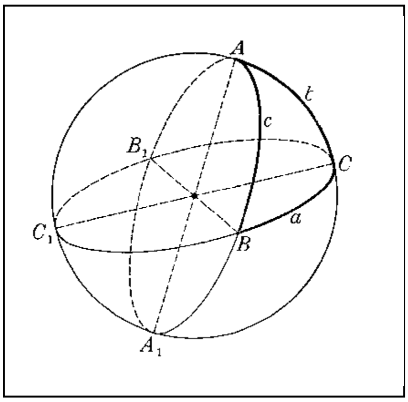
1. Geometry on a sphere

It is fundamental fact in Euclidian geometry that the shortest path between to points lies on at straight line between the two points.
In an arbitrary two dimensional surface, things become more complex.
The part of mathematics that treats this area is called differential geometry.[1]
It is, however, far more complicated than ordinary geometry in a plane.
If the surface is given by a parameter representation $P(u_1,u_2) = (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$, then using the theory of differential geometry, one can derive a (rather complicated) differential equation, where the solution is a parameter representation of the shortest path between two points on the surface.
Such a curve is called a geodetic. As "geo" means earth in Greek, the concept of a geodetic refers to the shortest path between two points on the surface of the earth.
If one ranks the two dimensional surfaces after increasing complexity, the sphere comes in as number two after the plane.

Figur (1)



Figur (2)



If you cut a sphere with a plane, the intersecting curve is a circle. If the cut goes through the centre of the sphere the intersection is a great circle, that is, having a diameter equal to the diameter of the sphere. Both cases are shown in the figures above.

In the differential geometry, one may show[1], (but it is far from simple), that the shortest path between two points on a sphere is part of the great circle through the two points.

In the figure to the right, the two great circles are e.g. the shortest distance between *A* and *B*, and *B* and *C*.
In the plane most geometrical figures consists of straight line segments, and since the great circles, represents "the straight lines", (the geodetics), we shall only be concerned with the geometry of figures consisting of great circle segments.

A great circle is uniquely determined by to points on the sphere *A* and *B,* which do not lie diametrically opposite.

[1] www.olewitthansen.dk Differential Geometry 1

The intersections between two different great circles are two diametrically opposite points, and the line connecting them is a diameter, which is also the intersection line between the planes that generates the two great circles.

The angle between two great circles is defined (it is) the angle between the two corresponding planes. From the figure (2), it is obvious that the arc $A_1B = 180^0 - AB$.

The diameter (the axis) (that is perpendicular to a plane that belongs to a great circle) intersects the sphere in two diametrically opposite points A and A_1, which are named the poles belonging to the great circle.

When one of two great circles intersects the other in its poles, the two great circles are perpendicular to each other, as illustrated on the figure below.

Figur (3)

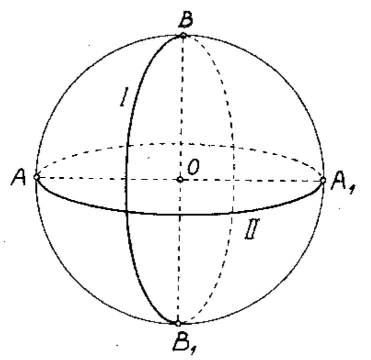


Fig. 49.

Figur (4)

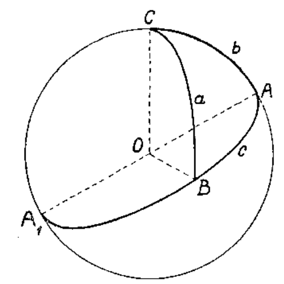


Fig. 52.

2. Spherical triangles

A spherical triangle is a part of the sphere that is confined by the arcs of three great circles, and they are called the sides of the spherical triangle.
The sides are measured in degrees, or alternatively in radians. The length *a,* of a side on a sphere with radius *R* is found by multiplying its radian number α by *R.*

$$a = \alpha R.$$

The labels *a, b, c* for the sides in a spherical triangle, may denote the degrees, the radians or the length of the sides dependent on the context.

The intersection angles between the great circles that form the spherical triangle are the angles of the triangle.
The angles lying opposite to the sides *a, b, c* are denoted *A, B, C,* as is the case in plane geometry.

If lines are drawn form the centre of the sphere to the three vertices of the triangle, it will form a triangular corner. Sides and angles in the triangular corner are in pairs equal to the sides and angle in the spherical triangle, as depicted in figure (4) above.

The concepts of isosceles, height, bisecting lines and bisector normal are the same as in the geometry of the plane.

Figur (5)

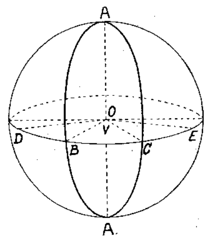


Fig. 51.

A right-angled triangle is a spherical triangle, having (at least) one angle equal to 90^0. A spherical triangle might have one, two or three right-angles. Cathetus and hypotenuse have the same meaning as in a plane right-angled triangle, (if the triangle has only one right-angle).

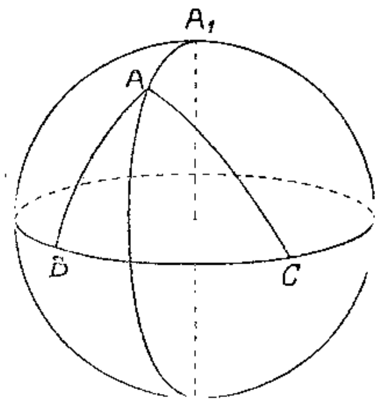
The lune (half moon) ABA_1C shown in figure (5) is called a spherical double edge.
It has only one angle, which is the angle between the two planes that form the double edge.

As we shall see the sum of the three angles in a spherical triangle are always greater than 180^0.
In figure (5) is shown a triangle having two right-angles. If also $A = 90^0$, the triangle has three right-angles, which is possible.
However, as a spherical angle cannot exceed 180^0, the sum of the angles is always less than 540^0.

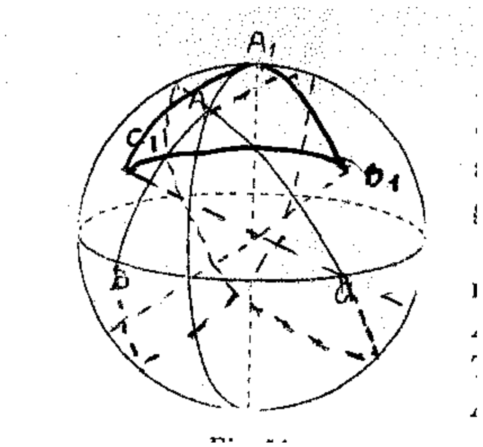
2.1 Polar triangles

By the polar triangle to a spherical triangle, we understand the triangles created by the poles of the great circles that the sides of the given triangle are part of. The angles in the polar triangle are often denoted with an index 1. So A_1, B_1, C_1, are the poles belonging to the sides *a, b, c*.

Figur (*6*)



Figur(*7*)



In the figure to the left, the pole A_1 belonging to the side *a,* is constructed. In the figure to the right the whole polar triangle A_1 B_1 C_1 is shown.
But obviously it is difficult to visualize a spatial figure, even when it is drawn in perspective, and especially, when it is not drawn professionally. For example, in figure (7) both B_1 and C_1 lie on the back side of the sphere, while A lies in the plan of the paper.

A simple but rather non transparent theorem states:

The polar triangle to a given spherical triangles polar triangle is the triangle itself.

If we consider the fact that when one of two great circles passes through the poles of the other great circle, then the two great circles are perpendicular to each other, as is depicted in figure (3), we reason as follows

Let the given spherical triangle be *A, B, C* and its polar triangle A_1 B_1 C_1.
We seek the pole for the side $a_1 = B_1$ C_1.

As B_1 C_1 intersects a pole for each of the great circles *AB* and *AC*, we can conclude that *AB* and *AC* pass through the poles for B_1 C_1. Consequently one of the intersection points must be *A*. Furthermore since $AA_1 < 90^0$, *A* lies on the same side of B_1 C_1 as A_1 does, then *A* must be the pole to be used when constructing the polar triangle to the spherical triangle A_1 B_1 C_1.
A similar reasoning may be used to construct the poles *C* and *B* to A_1 B_1 and A_1 C_1.

This is supplemented by a somewhat surprising theorem.

The sides (a_1, b_1, c_1) *and angles* (A_1, B_1, C_1) *in the polar triangle are equal to the complementary angles* of *the angles* (*A, B, C*) *and sides* (*a, b, c*) *in the original spherical triangle respectively.*

Since B_1 and C_1 lie on the normal to the planes corresponding to the sides *AC* and *AB*, and as the angle between the normal to two planes is $180^0 - v$, where *v* is the angle between the planes themselves, and since *A* is the angle between the planes corresponding to the sides *AC* and *AB*, then $a_1 = B_1C_1 = 180 - A$.
A quite similar argument can be applied for the other two sides of the polar triangle.

Applying the above theorem, that the polar triangle to a polar triangle is the triangle itself, we find $AB = c = 180^0 - C_1$ or $C_1 = 180^0 - c$, and similarly for the two other angles.
Thus the theorem is proved.

Applying the above theorem about the sides and the angles in the polar triangle to a spherical triangle, we can prove that the sum of the three angles in a spherical triangle is always bigger than 180^0 and less than 540^0.

We assume that we have constructed the polar triangle to a spherical triangle *ABC*.
According to the above theorem the sides in the polar triangle are $180^0 - A$, $180^0 - B$, $180^0 - C$.
since the sum of the three sides must be less than 360^0, the following inequality is valid:

(2.1) $$180^0 - A + 180^0 - B + 180^0 - C < 360^0 \quad \Leftrightarrow \quad A + B + C > 180^0$$

Furthermore the sum of the sides in the polar triangle must be greater than 0

(2.2) $$180^0 - A + 180^0 - B + 180^0 - C > 0 \quad \Leftrightarrow \quad A + B + C < 540^0$$

The difference $E = A + B + C - 180^0$ is called the *spherical excess*.

3. The right-angle spherical triangle

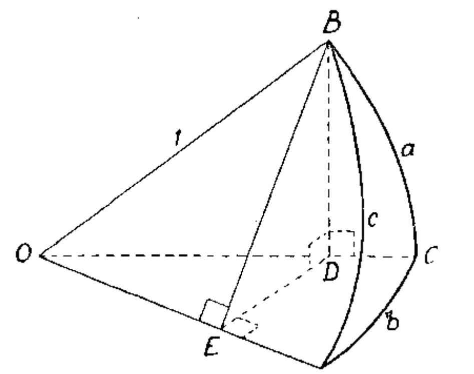


Fig. 56.

Let the triangle be ABC, where $C = 90^0$. O is the centre of the sphere, and we assume that the radius is 1, so that the line segments OA, OB and OC all have the length 1.

D is the projection of B on OC, and E is the projection of B on OA.
We assume that the two cathetus' a and b are acute, so that D lies between O and C, and E lies between O and A. DE is perpendicular to OA, since ED is the projection of BE on the plane OAC.

As BE and ED both are perpendicular to OA, then $\angle\ BED$ is an angle between the planes AOB and AOC, and thus $\angle BED = \angle A$. At the same time the hypotenuse $c = \angle BOE$.

We are frequently going to apply the formulas for the plane right-angle triangle, so we write them below, both as formulas and as statements.

For any *plane* right-angle triangle, the following holds:

(3.1) $$\sin A = \frac{a}{c} \qquad \cos A = \frac{b}{c} \qquad \tan A = \frac{a}{b}$$

Sinus to an angle is equal to the opposite cathetus divided by the hypotenuse.

Cosine to an angle is equal to the adjacent cathetus divided by the hypotenuse.

Tangent to an angle is equal to the opposite cathetus divided by the adjacent cathetus

For the triangular corner in figure (8) is thus seen:

$$OD = OB\cos a = \cos a \quad \text{and} \quad OE = OD\cos b = \cos a \cos b$$

From the right-angle triangle OEB we further get: $OE = OB\cos c = \cos c$, thus

(3.2) $$\cos c = \cos a \cos b$$

From the right-angle triangle OEB we get: $OE = OB\cos c$ and from ΔBED we find:

$$\sin A = \frac{BD}{BE}, \text{ but } BD = \sin a \text{ and } BE = \sin c, \text{ so}$$

(3.3) $$\sin A = \frac{\sin a}{\sin c}$$

Similarly to this we have: $\sin B = \dfrac{ED}{BE} = \dfrac{\sin b}{\sin c}$, so

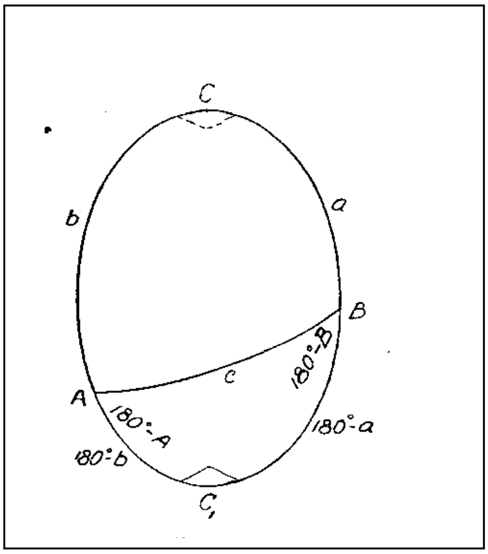
(3.4) $$\sin B = \frac{\sin b}{\sin c}$$

Also we find from ΔBDE that $\cos A = \dfrac{DE}{BE}$, and since $DE = OD \sin b = \cos a \sin b$, we get

(3.5) $$\cos A = \frac{\cos a \sin b}{\sin c} \qquad \text{correspondingly} \qquad \cos B = \frac{\cos b \sin a}{\sin c}$$

We have derived the formulas above under the condition that both angles A and B are acute.
If $C = 90^0$, and one cathetus e.g. $a = 90^0$, then $A = 90^0$, which also appears from the formulas.

Figur (9)



Figur (10)

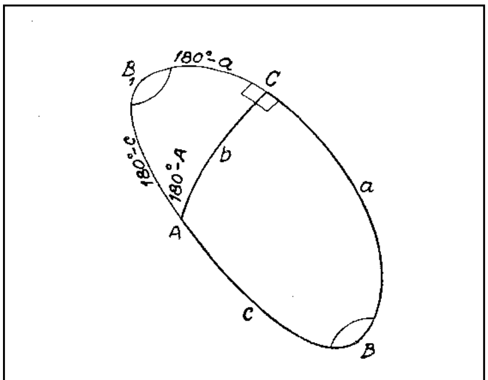


Fig. 58.

A spherical triangle can be considered as a crescent (a half moon) connecting two poles C and C_1 where the angles $C = C_1$, intersected by a great circle, thereby creating two spherical triangles ACB and AC_1B. They are called *neighbouring triangles*. $\angle C_1AB = 180 - CAB$, and $\angle C_1BA = 180 - CBA$. Every spherical triangle has three neighbouring triangles, one for each side.

If both cathesus' are obtuse angles (meaning greater than 90^0) what is shown in figure (9) we focus on the neighbour triangle $A_1B_1C_1$, with sides a_1 , b_1, c_1 . The points $A = A_1$, $B = B_1$, C and C_1 are opposite poles. $a_1 = 180 - a$, $b_1 = 180 - b$, $c_1 = c$.

In the triangle $A_1\ B_1\ C_1$ both cathetus' are acute, so we may apply the formulas (3.2) - (3.5).

$$\sin A_1 = \frac{\sin a_1}{\sin c_1} \quad \Leftrightarrow \quad \sin(180 - A) = \frac{\sin(180 - a)}{\sin(180 - c)} \quad \Leftrightarrow \quad \sin A = \frac{\sin a}{\sin c}$$

Since sin(180 – v) = sin$v$, we arrive at the same formulas as before

$$\cos A_1 = \frac{\cos a_1 \sin b_1}{\sin c_1} \quad \Leftrightarrow \quad \cos(180 - A) = \frac{\cos(180-a)\sin(180-b)}{\sin(180-c)} \quad \Leftrightarrow \quad \cos A = \frac{\cos a \sin b}{\sin c}$$

$$\cos c_1 = \cos c = \cos(180-a)\cos(180-b) = \cos a \cos b$$

Since cos(180 – v) = - cos$v$, we also find the same formulas as for the triangle with acute cathetus'.

If $a > 90^0$ and $b < 90^0$, we may focus on the neighbour triangle $A_1\ B_1\ C_1$, to the side b, (the figure to the right), which has an angle 90^0 and a acute angle, and according to the precedents, the following formulas are valid.

$$\cos c_1 = \cos(180-c) = \cos(180-a)\cos b \quad \Leftrightarrow \quad \cos c = \cos a \cos b$$

$$\sin A_1 = \frac{\sin a_1}{\sin c_1} \quad \Leftrightarrow \quad \sin(180-A) = \frac{\sin(180-a)}{\sin(180-c)} \quad \Leftrightarrow \quad \sin A = \frac{\sin a}{\sin c}$$

$$\cos A_1 = \frac{\cos a_1 \sin b_1}{\sin c_1} \quad \Leftrightarrow \quad \cos(180-A) = \frac{\cos(180-a)\sin b}{\sin(180-c)} \quad \Leftrightarrow \quad \cos A = \frac{\cos a \sin b}{\sin c}$$

The formulas derived above are therefore valid for any right-angle spherical triangle.

4. Calculation of sides and angles in the right-angle spherical triangle.

From the formulas (3.2) - (3.5) we may further derive some formulas involving tan.

From:

$$\cos c = \cos a \cos b \quad \Leftrightarrow \quad \cos a = \frac{\cos c}{\cos b} \quad \text{and} \quad \cos A = \frac{\cos a \sin b}{\sin c} = \frac{\cos c \sin b}{\cos b \sin c} = \frac{\tan b}{\tan c}$$

(4.1) $$\cos A = \frac{\tan b}{\tan c}$$

By dividing $\sin A = \frac{\sin a}{\sin c}$ with $\cos A = \frac{\cos a \sin b}{\sin c}$, we get, $\tan A = \frac{\frac{\sin a}{\sin c}}{\frac{\cos a \sin b}{\sin c}} = \frac{\tan a}{\sin b}$

(4.2) $$\tan A = \frac{\tan a}{\sin b}$$

And its analogous formula

$$\tan B = \frac{\tan b}{\sin a}$$

By multiplication of (4.1) and (4.2) we get:

$$\tan A \tan B = \frac{\tan a}{\sin b}\frac{\tan b}{\sin a} = \frac{\sin a \sin b}{\cos a \sin b \cos b \sin a} = \frac{1}{\cos a \cos b} = \frac{1}{\cos c}$$

(4.3) $$\tan A \tan B = \frac{1}{\cos c} \quad \Leftrightarrow \quad \cos c = \frac{1}{\tan A \tan B}$$

Below we have collected all the formulas belonging the right-angle spherical triangle.

(4.4) $$\cos c = \cos a \cos b$$

(4.5) $$\sin A = \frac{\sin a}{\sin c} \quad \text{and} \quad \sin B = \frac{\sin b}{\sin c}$$

(4.6) $$\cos A = \frac{\cos a \sin b}{\sin c} \quad \text{and} \quad \cos B = \frac{\cos b \sin a}{\sin c}$$

(4.7) $$\cos A = \frac{\tan b}{\tan c} \quad \text{and} \quad \cos B = \frac{\tan a}{\tan c}$$

(4.8) $$\tan A \tan B = \frac{1}{\cos c} \quad \Leftrightarrow \quad \cos c = \frac{1}{\tan A \tan B}$$

4.9 Example:

$$a = 35^0 \ , b = 60^0.$$

$$\cos c = \cos a \cos b = \cos 35 \cos 60 = 0.4095 \Rightarrow c = 66^0.82$$

$$\sin A = \frac{\sin a}{\sin c} = \frac{\sin 35}{\sin 65.82} = 0.6287 \Rightarrow A = 38^0.96$$

$$\sin B = \frac{\sin b}{\sin c} = \frac{\sin 60}{\sin 65.82} = 0{,}9493 \Rightarrow B = 71^0.68$$

4.10 Example:

$$a = 40^0\text{-}25 \ , c = 100^0.56$$

$$\cos c = \cos a \cos b \Rightarrow \cos b = \frac{\cos c}{\cos a} = -0.2401 \Rightarrow b = 103^0.89$$

$$\sin A = \frac{\sin a}{\sin c} = 0.6372 \Rightarrow A = 41^0.09$$

$$\sin B = \frac{\sin b}{\sin c} = 0 - 9875 \Rightarrow B = 80^0.93$$

4.11 Example:

$$a = 36^0.70 \ , A = 50^0.83$$

$$\sin A = \frac{\sin a}{\sin c} \Rightarrow \sin c = \frac{\sin a}{\sin A} = 0.77^0,64 \quad \Rightarrow \quad c = 50^0.93$$

$$\cos c = \cos a \cos b \Rightarrow \cos b = \frac{\cos c}{\cos a} = 0.7861 \Rightarrow b = 38^0.18$$

$$\sin B = \frac{\sin b}{\sin c} = 0.7962 \Rightarrow B = 52^0.77$$

4.12 Example:

$$\mathrm{A} = 43^0.28 \text{ , } B = 80^0.59$$

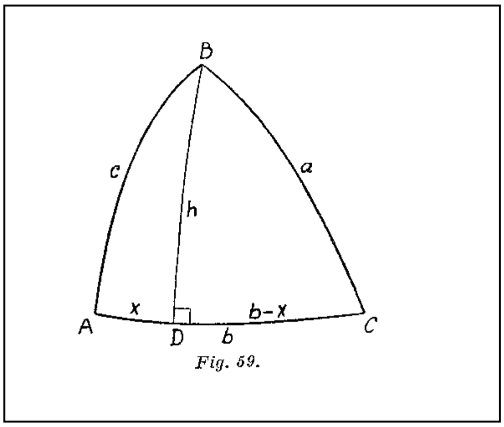
$$\cos c = \frac{1}{\tan A \tan B} = 0.1760 \quad \Rightarrow \quad c = 79^0.86$$

$$\sin a = \sin A \sin c = 0.6760 \quad \Rightarrow \quad a = 42^0.45$$

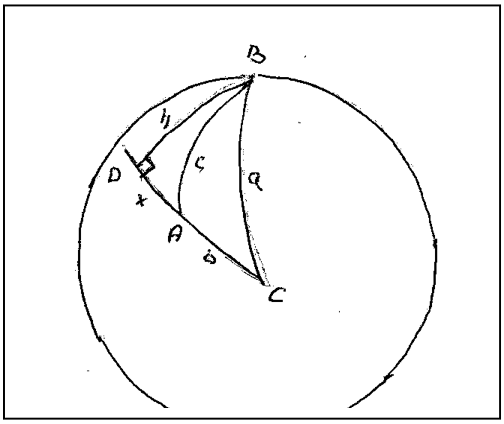
$$\sin b = \sin B \sin c = 0.9711 \quad \Rightarrow \quad a = 76^0.20$$

5. The general spherical triangle. Cosine- and sine relations

Figur(11)



Figur(12)



5.1 The cosine relations

We shall then consider the general spherical triangle as shown from figure (11).
We draw the height from one of the angles e.g. from *B*. The low end of the height is *D*.
Firstly we assume that *D* lies between *A* and *C*. *AD* is denoted *x*, so that *DC* becomes $b - x$.

We then apply formula (4.4) on ΔABD and ΔBDC.

$$\cos c = \cos x \cos h \qquad \text{and} \qquad \cos a = \cos h \cos(b - x)$$

Then we shall use the addition formulas for cosine: $\cos(u - v) \cos u \cos v + \sin u \sin v$ in the last equation above.

$$\cos a = \cos h \cos(b - x) = \cos h \cos b \cos x + \cos h \sin b \sin x$$

And with help from (4.4) $\cos c = \cos x \cos h$ the above equation can be written:

$$\cos a = \cos b \cos c + \cos h \sin b \sin x$$

Using (4.6) $\cos A = \dfrac{\cos a \sin b}{\sin c}$ on ΔABD

we get

$$\cos A = \frac{\cos h \sin x}{\sin c} \Leftrightarrow \cos h \sin x = \cos A \sin c$$

We thus end with the *cosine relation* , valid for the general spherical triangle.

(5.1)	$\cos a = \cos b \cos c + \sin b \sin c \cos A$

(5.1) has (as is the case for the cosine relation for the plane triangle) two analogous expressions, which comes about, by permuting the letters.

(5.2)	$\cos b = \cos a \cos c + \sin a \sin c \cos B$
(5.3)	$\cos c = \cos a \cos b + \sin a \sin b \cos C$

The cosine relations are suitable to find the angles in a spherical triangle, when the three sides are given.

If the low point of the height lies outside *AC*, as shown in figure (12) the calculations are almost identical to that of the former case.

From ΔABD it follows as before, from (4.4) : $\cos c = \cos h \cos x$, and from ΔBDC we get:

$$\cos a = \cos h \cos(x + b) = \cos h(\cos x \cos b - \sin x \sin b) = \cos h \cos x \cos b - \cos h \sin x \sin b$$

When we insert the expression for cos *c*, we get:

$$\cos a = \cos b \cos c - \cos h \sin x \sin b$$

As we did before, using (4.6) $\cos A = \dfrac{\cos a \sin b}{\sin c}$ on ΔABD, we find

$$\cos(180 - A) \sin c = \cos h \sin x$$

When inserted in the expression for cos *a* , we end with the same expression for the cosine relation as previously.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

In contrast to what holds true for the plane triangle, a spherical triangle is complete determined by its three angles. To try to solve the three cosine relations to determine *a, b, c* is not algebraically palatable, although there are three independent algebraic equations with three unknowns.

Instead of trying one should recall the theorem mentioned above:

In a polar triangle, which is formed by the poles of the three great circles, (of which the three sides *a,b,c* belong to), the angles A_1, B_1, C_1 of the polar triangle are complementary angles to the sides *a,b,c* in original spherical triangle *ABC,* and the sides a_1, b_1, c_1 in the polar triangle are complementary angles to *A,B,C* in the original spherical triangle. e.g. $a_1 = 180 - A$

Writing the three cosine relations belonging to of the sides the polar triangle, we get three equations to determine the sides in the original triangle.

(5.4) $$\cos a_1 = \cos b_1 \cos c_1 + \sin b_1 \sin c_1 \cos A_1$$

Which according to what is stated above is equal to

$$\cos(180 - A) = \cos(180 - B)\cos(180 - C) + \sin(180 - B)\sin(180 - C)\cos(180 - a)$$

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a) \quad \Leftrightarrow$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

From which one can determine *a*.
Formulas for the two sides *b* and *c* can be obtained, by permuting the letters cyclically.

5.2 The sine relations

If we apply the formula (4.5): $\sin A = \dfrac{\sin a}{\sin c}$ to the two right-angle triangles ΔABD and ΔBDC in figure (11), then it holds true, whether or not the height has it low end between *B* and *C,* that:

$$\sin A = \frac{\sin h}{\sin c} \quad \textit{and} \quad \sin C = \frac{\sin h}{\sin a} \quad \Rightarrow \quad \sin h = \sin A \sin c \quad \textit{and} \quad \sin h = \sin C \sin a$$

From which it follows:

$$\sin A \sin c = \sin C \sin a$$

Resulting in the *sine relations* for any spherical triangle.

(5.5) $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Where the term in the middle is the result of permuting the letters.

You should however contemplate that, when using the sine relations to find an angle, the equation $\sin v = x$ has two solution v and $180 - v$. Which one belongs to the triangle or the neighbouring triangle may be resolved by drawing a test triangle, or using the cosine relations.

6. The Area (the surface) of a spherical triangle

Figur(13)

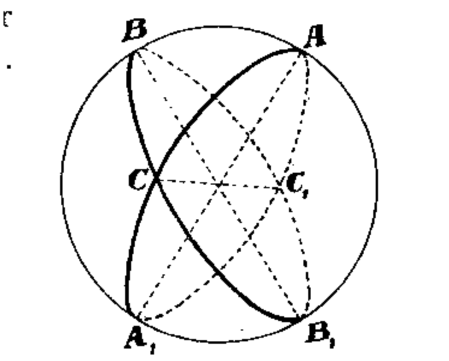


Fig. 66.

The surface of a sphere with radius R is $T = 4\pi R^2$. Figure (13) shows a spherical triangle ABC.

We have also drawn the neighbour triangle to the side a, which is A_1BC, and the neighbour triangle to the side b: AB_1C, and to the side c: ABC_1.
The points A and A_1, B and B_1, C and C_1 lie diametrical opposite, and any triangle and its neighbouring triangle form a double edge, (a half moon) with an angle that is equal to the angle at the top.
For example, the two triangles ABC and A_1BC form a double edge with the angle $A = A_1$.

The area of a double edge separated by an angle 1^0 must be 1/360 times the area of the sphere. Consequently the area of a double edge separated by an angle v is

(6.1) $$T_v = \frac{v}{360}T = \frac{v}{360}4\pi R^2$$

From this follows, writing: $T(ABC)$ for the area of ΔABC.

(6.2) $$\begin{aligned} T(ABC) + T(A_1BC) &= \frac{A}{360}T \\ T(ABC) + T(AB_1C) &= \frac{B}{360}T \\ T(ABC) + T(ABC_1) &= \frac{C}{360}T \end{aligned}$$

And by adding the three equations:

(6.3) $$2T(ABC) + \left(T(ABC) + T(A_1BC) + T(AB_1C) + T(ABC_1)\right) = \frac{A+B+C}{360}T$$

C_1AB lies symmetrically with CA_1B_1, with respect to centre of the sphere, so the two triangles have the same area.
If you look at figure (13) you may convince yourself that the four triangles in the parenthesis in (6.3) together form a half sphere, and the terms therefore are equal to $\frac{1}{2}T$. Inserting this in (6.3) gives:

$$2T(ABC) + \tfrac{1}{2}T = \frac{A+B+C}{360}T \quad \Leftrightarrow \quad 2T(ABC) + \frac{180}{360}T = \frac{A+B+C}{360}T$$

$$2T(ABC) = \frac{A+B+C-180}{360}T \quad \Leftrightarrow$$

(6.4)

$$T(ABC) = \frac{E}{720}T \quad \Leftrightarrow \quad T(ABC) = \frac{E}{180}\pi R^2$$

This is the general expression for the area of a spherical triangle, where $T = 4\pi R^2$

(6.5) $$E = A + B + C - 180^0$$

is called the *spherical excess*.

It is a remarkable fact that the area of a spherical triangle neither depends on the size of the sides nor the size of the angles, but only on the spherical excess.

7. Examples and exercises to the general spherical triangle.

Example 7.1
Given the three sides: $a = 80^0$, $b = 110^0$, $c = 65^0$.

It is always a good idea to draw a plane test triangle, which does not need to be very accurate.

The three angles *A, B, C* can be determined using the cosine relation.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad \Leftrightarrow \quad \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$\cos A = \frac{\cos 80 - \cos 110 \cos 65}{\sin 110 \sin 65} = 0.7362 \quad \Rightarrow \quad A = 21^0.93$$

And in the same manner:

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} = -0.4654 \quad \Rightarrow \quad B = 117^0.74$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} = 0.5400 \quad \Rightarrow \quad C = 57^0.51$$

Example 7.2

Given the three angles: $A = 80^0$, $B = 110^0$, $C = 65^0$.

One might be inclined to solve the three cosine relations to obtain *a, b, c*, but it will lead nowhere.

Instead one should use the polar triangle A_1, B_1, C_1, as we have shown previously the *angles* and *sides* are complementary (180 – v) to the *sides* and *angles* in the original triangle.

$$a_1 = 180 - A,\ b_1 = 180 - B,\ c_1 = 180 - C$$

and subsequently determine

$$A_1 = 180 - a,\ B_1 = 180 - b,\ C_1 = 180 - c.$$

With the given values for the angles, which are the same as the sides in example 7.1, we only need to write:

$$a = 180 - 21.93 = 158^0.07,\ b = 180 - 117.74 = 62^0.26,\ c = 180 - 157.51 = 122^0.49\ .$$

Example 7.3
To avoid that this looks like pure mathematical magic, we repeat example 7.2, but with fresh values for the angles.

$$A = 57^0,\ B = 75^0,\ C = 100^0.$$

We then proceed by writing down the cosine relations for the polar triangle, where $a_1 = 180 - A$, $b_1 = 180 - B$, $c_1 = 180 - C$, and then determine A_1, B_1, C_1.

Finally we determine $a = 180 - A_1$, $b = 180 - B_1$, $c = 180 - C_1$.

$$\cos a_1 = \cos b_1 \cos c_1 + \sin b_1 \sin c_1 \cos A_1 \quad \Leftrightarrow$$

$$\cos(180 - A) = \cos(180 - B)\cos(180 - C) + \sin(180 - B)\sin(180 - C)\cos(180 - a)$$

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a$$

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} = 0.5253 \quad \Rightarrow \quad a = 58^0.31$$

And by permuting the letters:

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C} = 0.1989 \quad \Rightarrow \quad b = 78^0.53$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} = -0.0403 \quad \Rightarrow \quad c = 92^0.31$$

Example 7.4
Given an angle and the two adjacent sides:

$$a = 108^0,\ b = 143^0,\ C = 159^0.$$

The side c can directly be determined by the cosine relation:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C = -0.048 \quad \Rightarrow \quad c = 92^0.7$$

B and C can then in principle be found by the sine relations: $\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}$

$$\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c} \quad \Rightarrow \quad \sin B=\frac{\sin b\sin C}{\sin c}=0.2252 \quad \Rightarrow \quad B=13^0.01 \ \vee \ B=166^0.99$$

$$\frac{\sin A}{\sin a}=\frac{\sin C}{\sin c} \quad \Rightarrow \quad \sin A=\frac{\sin a\sin C}{\sin c}=0.8161 \quad \Rightarrow \quad A=20^0.84 \ \vee A=159^0.16$$

The problem is of course that we get two solutions for each angle, and that we really have no way to decide which angle belongs to the given triangle, and which one to the neighbour triangle to the side c.
In plane trigonometry the same triangle has two solutions, but in spherical geometry there is only one solution, the "other solution" belongs to the neighbouring triangle.
In some cases it is clear which angle to choose, if you draw a plane test triangle.
To circumvent this problem, without resorting to a plane "test triangle" we turn instead to the cosine relation to determine the angles A and B.

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}=\frac{\cos 108-\cos 143\cos 106{,}71}{\sin 143\sin 106.71}=-0.9345 \quad \Rightarrow \quad A=159^0.16$$

$$\cos B=\frac{\cos b-\cos a\cos c}{\sin a\sin c}=-0.9774 \quad \Rightarrow \quad B=166^0.98$$

Example 7.5

In the preceding examples, we have been able to find unknown sides and angles almost in the same manner as in the plane geometry.
But in spherical geometry problems arise, if we consider the two cases:

1. Given an angle, an adjacent and the opposite side, e.g. *A, a, b*.
2. Given a side, an adjacent and an opposite angle, e.g. *A, B, a*.

The two cases are in fact the same, since B can be calculated from the sine relations in the first case, and b can be calculated from the sine relations in the latter.

But in both cases we are left with *A, B, a, b*, but are missing c and C.
The problem arises of course from the fact, that we can not find $C = 180 – (A+B)$, as we do in the plane geometry.

So we are left with the cosine relation. It is correct that we may write two cosine relations, which only have the side c as unknown, but the problem is that the unknown side c appears in the equation both as cos c and sin c.
Attempts to draw the height from C, and the using the formulas for the right-angle spherical triangle lead nowhere. Alternatively we may write the cosine relation for cos a and cos b.

(7.5.1) $\cos a=\cos b\cos c+\sin b\sin c\cos A$ and $\cos b=\cos a\cos c+\sin a\sin c\cos B$

Then we solve both equations for sin *c*, and put the results equal to each other.

$$\sin c = \frac{\cos a - \cos b \cos c}{\sin b \cos A} \quad \text{and} \quad \sin c = \frac{\cos b - \cos a \cos c}{\sin a \cos B}$$

Which gives an equation with the unknown cos *c*, from which *c* can be determined.

$$\frac{\cos a - \cos b \cos c}{\sin b \cos A} = \frac{\cos b - \cos a \cos c}{\sin a \cos B}$$

Once c is determined, the angle *C* may then be found from the cosine relation for *c*.

There is however another possibility, since both of the two cosine relations is a so called homogeneous equation of first degree in cosine and sine.

7.5 Solving the homogeneous equation in cos *x* and sin *x*

The general homogeneous equation in cos *x* and sin *x* may be written:

(7.5.3) $$a \cos x + b \sin x = c$$

To solve the equation we divide by $\sqrt{a^2 + b^2}$.

$$\frac{a}{\sqrt{a^2 + b^2}} \cos x + \frac{b}{\sqrt{a^2 + b^2}} \sin x = \frac{c}{\sqrt{a^2 + b^2}}$$

Introducing the angle *y* by:

$$\cos y = \frac{a}{\sqrt{a^2 + b^2}} \quad \textit{and} \quad \sin y = \frac{b}{\sqrt{a^2 + b^2}},$$

It then follows

(7.5.4) $$\tan y = \frac{b}{a}$$

And the equation becomes hereafter

$$\cos y \cos x + \sin y \sin x = \frac{c}{\sqrt{a^2 + b^2}}$$

And it is rewritten with the help of the addition formulas for cos(*x* - *y*)

$$\cos(x - y) = \frac{c}{\sqrt{a^2 + b^2}}$$

The equation has solutions if:

$$\left| \frac{c}{\sqrt{a^2 + b^2}} \right| \leq 1.$$

And the solution may be written

$$x = y + \cos^{-1} \frac{c}{\sqrt{a^2 + b^2}}$$

8. The plane geometry as a limiting case of the spherical geometry

The radian number α for an arc *a* on a great circle having radius *R* is given by:

$$\alpha = \frac{a}{R}$$

If the *sides* (the arcs) in a spherical triangle are very small compared to the circumference, then the spherical triangle almost appears as plane. (We conceive the earth as flat). And therefore we should expect that the trigonometric formulas for the spherical triangle in this limit are the same as for the plane triangle.

That this indeed the case, one can realize if we make a Taylor expansion to the first significant order of the trigonometric functions. Resulting in:

$$\sin\alpha = \alpha \qquad \text{and} \qquad \cos\alpha = 1 - \tfrac{1}{2}\alpha^2$$

The right-angle spherical triangle

We invent the notation α, β, δ, for the radian number of the sides *a, b, c* in the spherical triangle, and at the same time assume that:

$$\alpha \ll 1, \quad \beta \ll 1, \quad \gamma \ll 1.$$

The lengths of the sides then become:

$$a = \alpha R, \quad b = \beta R, \quad c = \gamma R.$$

The formula (4.5) for the right-angle spherical triangle then becomes:

$$\sin A = \left(\frac{\sin a}{\sin c} \right) = \frac{\sin\alpha}{\sin\gamma} \quad \Rightarrow \quad \sin A \approx \frac{\alpha}{\gamma} = \frac{\alpha R}{\gamma R}$$

Which, is seen to be identical to the formula for the plane right-angle triangle.

$$\sin A = \frac{a}{c}$$

By the same token, when the formula (4.4) for the right-angle spherical triangle

$$\cos c = \cos a \cos b$$

is written with radians, followed by a Taylor expansion to the second order, and keeping only terms up to the second order.

$$\cos\gamma = \cos\alpha\cos\beta$$

$$1-\tfrac{1}{2}\gamma^2 = (1-\tfrac{1}{2}\alpha^2)(1-\tfrac{1}{2}\beta^2) \quad \Leftrightarrow$$
$$\gamma^2 = \alpha^2 + \beta^2 \quad \Leftrightarrow$$
$$R^2\gamma^2 = R^2\alpha^2 + R^2\beta^2 \quad \Leftrightarrow$$
$$c^2 = a^2 + b^2$$

We thus find that the formula $\cos c = \cos a \cos b$ for the right-angle spherical triangle is equivalent to Pythagoras' theorem for the right-angle plane triangle.

The general spherical triangle:
We shall first look at the sine relations for the spherical triangle.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

When they are written with radians

$$\frac{\sin A}{\sin\alpha} = \frac{\sin B}{\sin\beta} = \frac{\sin C}{\sin\gamma}.$$

We expand the sine in the denominator followed by multiply with *R*.

$$\frac{\sin A}{\alpha R} = \frac{\sin B}{\beta R} = \frac{\sin C}{\gamma R} \quad \Leftrightarrow \quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Which are seen to be the sine relations for the plane right-angle triangle.

Next we turn to the cosine relations for the spherical triangle.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

And we write it using radians, followed by a Taylor expansion to the second order.

$$\cos\gamma = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos C$$

$$1-\tfrac{1}{2}\gamma^2 = (1-\tfrac{1}{2}\alpha^2)(1-\tfrac{1}{2}\beta^2) + \alpha\beta\cos C$$
$$1-\tfrac{1}{2}\gamma^2 = 1-\tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta^2 + \alpha\beta\cos C \quad \Leftrightarrow$$
$$\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos C \quad \Leftrightarrow$$
$$R^2\gamma^2 = R^2\alpha^2 + R^2\beta^2 - 2\alpha R\beta R\cos C \quad \Leftrightarrow$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Which, we recognize as the cosine relation for the plane triangle.

9. Exercises

1. Oporto in Portugal and New York City in USA lies almost on the same latitude, as NYC = (40^0 45' *n* ; 74^0 0' *w*) and Oporto = (40^0 45' n ; 8^0 40' w).
(*n:* Northern latitude, *w*: western longitude, and 45' means 45 arc minutes)
If one should sail or fly from Oporto to New York, it is most likely to travel directly west.

a) Explain why straight west is not the shortest route, and calculate the distance you may spare, by taking the shortest path, and present the result in nautical miles.
To your information R_{earth} = 6370 km and 1 nautic mile (*sm*) = 1854 m

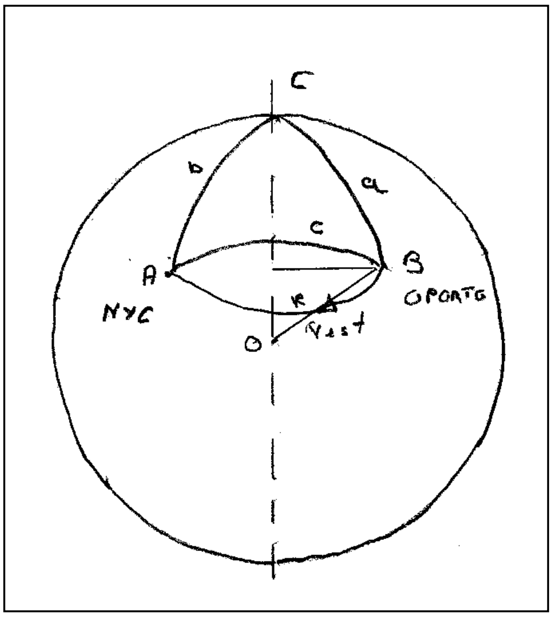
2. Copenhagen lies at (55^0 42' *n*; 12^0 35' *e*) los Los Angeles lies at (34^0 0' *n*; 118^0 10' *w*).
Calculate the spherical distance between CPH and LA.

On flight route goes via Søndre Strømfjord in Greenland (66^0 0' n; 54^0 0 w).
Give an explanation why. (The plane does not land in Greenland to pick up passengers)

Calculate the spherical distances Copenhagen - Søndre Strømfjord – Los Angeles, and compare it to the spherical distance Copenhagen - Los Angeles.

3. The Bermuda triangle is a territory limited by Miami: (25^0 49' n ; 80^0 6' w) - Puerto-Rico: (20^0 0'; 63^0 0') and Bermuda: (32^0 45' ; 65^0 0').
The intension of this exercise is not to make a statement concerning the mysteries in the Bermuda triangle, but only to determine the angles and sides of the Bermuda triangle, and finally give an indication of its area of the Bermuda triangle in km^2.

Solution to exercise 1.



Each time I have given spherical geometry as a final assignment in the Danish 10 – 12 year high school, I have included exercise 1. Actually no student has ever been able to give the correct answer (without help).
So I have chosen to reveal the solution here.
Although it is in accordance with spherical geometry, the conclusion is apt to cause scepticism among non mathematicians.

Since longitude and latitude are given in degrees and arc minutes, we initially convert it to decimal degrees.

This is done by diving by 60 and multiplying by 100, e.g. $40^0.45' = 40^0.75$ and $8^0.40' = 8^0.67$.

The arcs $a = b = 90^0 - 40^0.75 = 49^0.25$, and
$C = 74^0 - 8^0.67 = 64^0.33$.

To determine the length of *c*, we make use of the cosine relation

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$\cos c = \cos 49^0.25 \cos 49^0.25 + \sin 49^0.25 \sin 49^0.25 \cos 64^0.33 = 0.6775$$
$$c = 47^0.57 = \frac{47{,}57}{180}\pi \;\; rad = \;\; 0.8302 \;\; rad$$

The length is obtained by multiplying by the radius R = 6370 *km* of the earth.

$$d_c = cR = 5288\ km = \frac{5288}{1854} sm = 2853 \;\; sm$$

Which is the distance one should travel between Oporto and New York along a great circle.

If you alternatively travel direct west, then you travel by a minor circle with radius $r = R \sin a$, which is $r = 6370 \sin 49^0.25\ km = 4825\ km$

The arc that you traverse is the arc between the two latitudes equal to $C = 64^0.33 = \;1.1228 \;\; rad$
To determine the length d_r you should multiply the arc C with the radius r.

$$d_r = 4825 \cdot 1.1228\ km = 5417\ km = 2922\ sm.$$

The difference in nautical miles (sm) is therefore 2922 – 2853 = 69 sm = 128 *km*.

The example confirms in a practical manner that the shortest path between to points on a sphere is along a great circle.