1. Solve $\log_2 x = \log_x 4$

The equation is simple but a bit tricky.
We put

$$\log_2 x = k,$$

and we then have

$$\log_2 x = k \Leftrightarrow x = 2^k \quad \text{and similarly}$$

$$\log_x 4 = k \Leftrightarrow 4 = x^k$$

At the same time:

$$4 = x^{\log_2 x} \Rightarrow \log_2 4 = \log_2 x \cdot \log_2 x \Rightarrow (\log_2 x)^2 = 2,$$

$$\text{Since: } y = \log_2 4 \Leftrightarrow 4 = 2^y \Rightarrow y = 2$$

Then we have:

$$\log_2 x = \sqrt{2} \Leftrightarrow x = 2^{\sqrt{2}}$$

1a: $\log_{12} 24 = x$. **Find** $\log_2 3$

We have (definition of logarithm) $y = \log_c x \Leftrightarrow x = c^y$
and specially : $y = \log_{12} x \Leftrightarrow x = 12^y$.
Taking the $\log_2$ of both sides og the latter equation we find

$$\log_2 x = \log_2 12^y = y \log_2 12 = \log_{12} x \log_2 12$$

So: $$\log_2 24 = \log_{12} 24 \log_2 12 = x \log_2 12$$

On the other hand: $\log_2 24 = \log_2 3 \cdot 8 = \log_2 3 + 3$

So we have: $\log_2 3 = \log_2 24 - 3 = x \log_2 12 - 3$

2. Solve $2^{(x-1)^2} = x + 2$

Generally a transcendent equation has no analytic solution, so you may often have to resort to a qualified guess (if possible). At a glance, we can see that $x = 0$ is a solution. But there may be other solutions, since otherwise the equation was ridiculously easy. We begin by taking the $\log_2$ on both sides.

$$(x-1)^2 = \log_2(x+2)$$

We shall them consider the function

$$f(x) = (x-1)^2 - \log_2(x+2)$$

We notice that:
$f(0) = 1 - \log_2(2) = 1 - 1 = 0$; $\quad f(1) = 0 - \log_2(3) < 0$

$f(2) = 1 - \log_2(4) = -1$
$f(2.5) = 1.5^2 - \log_2(4.5) = 0.0801$
$f(3) = 4 - \log_2(5) = 2.3906$

$\log_2(x) = \frac{\ln(x)}{\ln 2}$ since $y = \log_2 x \Leftrightarrow x = 2^y$ so $\ln x = y \ln 2 = (\log_2 x)\ln 2 \Leftrightarrow \log_2 x = \frac{\ln x}{\ln 2}$

$(\log_2(x+2))' = \frac{1}{(x+2)\ln 2}$

We then make the series expansion from $x = 2$:

$$f(2+\delta) = f(2) + f'(2)\delta = 0$$

$$f'(x) = 2(x-1) - \frac{1}{(x+2)\ln 2} \qquad f'(2) = 2 - \frac{1}{4\ln 2}$$

$$f(2+\delta) = f(2) + f'(2)\delta = 0 \qquad \Rightarrow \qquad -1 + (2 - \frac{1}{4\ln 2})\delta = 0$$

$$\delta = -\frac{1-\log_2(4)}{2 - \frac{1}{4\ln 2}} \qquad \delta = \frac{1}{2 - \frac{1}{4\ln 2}} = 0.6100$$

$f(2+\delta) = (1.6100)^2 - \ln(3.6100)/\ln 2 = 0.3873$, whereas $f(2.47) = 0.0006$

We then make the series expansion from $x = 2.5$:

$$f(2.5+\delta) = f(2.5) + f'(2.5)\delta = 0$$

$$f'(x) = 2(x-1) - \frac{1}{(x+2)\ln 2} \qquad f'(2.5) = 3 - \frac{1}{4.5\ln 2}$$

$$f(2+\delta) = f(2.5) + f'(2.5)\delta = 0 \qquad \Rightarrow \qquad 0.0801 + (3 - \frac{1}{4.5\ln 2})\delta = 0$$

$$\delta = -\frac{0.0801}{3 - \frac{1}{4.5\ln 2}} \qquad \delta = -\frac{1}{2 - \frac{1}{4\ln 2}} = -0.0299$$

$f(2+\delta) = f(2{,}5 - 0.0299) = 0 \quad x = 2.47 \quad f(x) = 0.0006$

3. Determine *f from* $f(\frac{2x-1}{x-3}) = x^2$

It is actually a trivial exercise.

$y = \frac{2x-1}{x-3} \Leftrightarrow x = \frac{3y-1}{y-2} \quad \Rightarrow \quad f(y) = \left(\frac{3y-1}{y-2}\right)^2$, so we have

$$f(x) = \left(\frac{3x-1}{x-2}\right)^2$$

4. I is given that: $3^a = 5^b$ **: Find an expression for :** $\dfrac{ab}{a+b}$

It is relatively easy to find an expression for $\dfrac{ab}{a+b}$, but it contains either a or *b*.

$$3^a 3^b = 5^b 3^b \iff 3^{a+b} = 15^b \iff (a+b)\log 3 = b\log(15) \iff (a+b) = \frac{b\log(15)}{\log 3}$$

$$3^a = 5^b \iff (3^a)^b = (5^b)^b \iff 3^{ab} = 5^{bb} \iff ab\log 3 = b^2\log 5 \iff ab = \frac{b^2\log 5}{\log 3}$$

$$\frac{ab}{a+b} = \frac{\dfrac{b^2\log 5}{\log 3}}{\dfrac{b\log(15)}{\log 3}} = \frac{b\log 5}{\log 15} = \frac{a\log 3}{\log 15}$$

It is not possible to avoid either *a* or *b* in the result.

5. Solve the equation: $5^x - 3^x = 11$**.**

This is a transcendental equation, and in General a transcendent equation has no analytic solution, so you may often have to resort to a qualified guess (if possible). However this equation has no obvious solution, so we have to do something.

However the equation $5^x - 3^x = 4^x$ has evidently the solution $x = 2$, since $5^2 - 3^2 = 4^2$. So we have reason to believe that the solution does not deviate much from 2.

We therefore look at the function: $f(x) = 5^x - 3^x - 11$, and attempt to solve $f(x) = 0$

Assuming that the solution can be written as $x = 2 + \delta$, we find from the Taylor expansion to the first order term.

$$f(2+\delta) = f(2) + f'(2)\delta$$

$f(2) = 25 - 9 - 11 = 5$

$f'(x) = 5^x \ln 5 - 3^x \ln 3 \quad f'(2) = 5^2 \ln 5 - 3^2 \ln 3$

$$f(x) = 0 \iff f(2+\delta) = f(2) + f'(2)\delta = 0 \iff \delta = -\frac{5}{5^2\ln 5 - 3^2\ln 3} = -0.1648$$

We then find: $x = 2 + \delta = 1.8352$ gives $f(x) = 0.66$

6. Given $3^x = 7^y = 441.$ **Determine** $\dfrac{1}{x} + \dfrac{1}{y}$

First we notice that $441 = 21^2$ so we have two equations $3^x = 21^2$ *and* $7^y = 21^2$

The first equation we raise to the power *y*, and the second equation to the power *x*.

We then have two equations:

$(3^x)^y = (21^2)^y$ *and* $(7^y)^x = (21^2)^x$

$3^{xy} = 21^{2y}$ *and* $7^{yx} = 21^{2x}$

And then we multiply the two equations: $(3\cdot 7)^{xy} = 21^{2x+2y} \iff 21^{xy} = 21^{2x+2y}$

Since the roots are equal then must be the exponents:

$$xy = 2(x+y) \iff \frac{xy}{x+y} = 2 \iff$$

$$\frac{x+y}{xy} = \frac{1}{2} \iff \frac{1}{x} + \frac{1}{y} = \frac{1}{2}$$

7. Solve $x^{-x^x} = 2^{\sqrt{2}}$

Since $x^{-x^x} = \frac{1}{x^{x^x}} = \frac{1}{2^{\sqrt{2}}} \Rightarrow x^{x^x} = 2^{-\sqrt{2}}$,

Since there is no general analytic solution to a transcendent equation, we shall try to transform the right side, so it complies with the left side.

We notice that $\sqrt{2} = 2^{\frac{1}{2}}$ the right side so we have $2^{-\sqrt{2}} = \left(\frac{1}{2}\right)^{2^{\frac{1}{2}}} = \left(\frac{1}{2}\right)^{(\frac{1}{2})^{-(\frac{)1}{2})}}$

Showing that 1/2 would be a solution,
if it was not for the minus sign.

The next rewriting is rather tricky. First we notice that $\sqrt{2} = \frac{2}{\sqrt{2}}$

$$2^{-\sqrt{2}} = \left(\frac{1}{2}\right)^{\frac{2}{\sqrt{2}}} = \left(\frac{1}{2}\right)^{2\left(\left(\frac{1}{2}\right)^{2}\right)^{\frac{1}{\sqrt{2}}}} = \left(\frac{1}{4}\right)^{(\frac{1}{2})^{\frac{1}{2}}}$$

It is noticeable that $\left(\frac{1}{2}\right)^{\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{4}}$ so we have: $2^{-\sqrt{2}} = \left(\frac{1}{4}\right)^{(\frac{1}{4})^{\frac{1}{4}}}$, So we find surprisingly that $x = \frac{1}{4}$

7a. solve $x^{x^5} = 100$

$x^{x^5} = 100$ since 100 = 10·10. it is rather obvious to guess $x = \sqrt[5]{10}$

And indeed: $x^5 = (\sqrt[5]{10})^5 = 10$ and $((\sqrt[5]{10})^{10} = 10^2 = 100)$

8. Solve: $\ln(x) + \ln(x+1) = \ln(x+2)$

(The solution is almost trivial)

Applying the functional rules for the logarithm, we get immediately.

$$\ln(x) + \ln(x+1) = \ln(x+2) \iff$$

$$\ln\left(\frac{x(x+1)}{2x+1}\right) = 0 \iff \frac{x(x+1)}{2x+1} = 1 \iff$$

$$x^2 - x - 1 = 0 \quad d = 1+4 = 5 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2} \Rightarrow x = \frac{1+\sqrt{5}}{2}$$

8a. Solve: $\log_4 10 - \log_4 x = 2$

$\log_4 10 - \log_4 x = 2 \quad \Leftrightarrow \quad \log_4 \frac{10}{x} = 2 \quad \Leftrightarrow \quad \frac{10}{x} = 4^2 \quad \Leftrightarrow \quad x = \frac{5}{8}$

According to the definition of the logarithm it is the inverse function to the exponential function

9. Solve the equation $x^x = 2^{2048}$**.**

It is almost obvious that x must be a power of 2, so we put $x = 2^n$.

$x^x = (2^n)^{2^n} = 2^{n2^n} = 2^{2048}$, so we must determine n such that $n2^n = 2048 \ (= 2 \cdot 2^{10})$, from which it is obvious that $n = 2^3 \quad as \quad 2^3 2^{2^3} = 2^{11}$.

So we find that: $x^x = (2^3)^{2^3} = 8^8$

10. Simplify $\sqrt{53 - 10\sqrt{6}}$

We notice that $10\sqrt{6} = 5 \cdot 2\sqrt{2}\sqrt{3}$

and we write: $(5\sqrt{2} - \sqrt{3})^2 = (5\sqrt{2})^2 + \sqrt{3}^2 - 2 \cdot 5\sqrt{2}\sqrt{3} = 53 - 10\sqrt{6}$

So

$$\left(5\sqrt{2} - \sqrt{3}\right) = \sqrt{53 - 10\sqrt{6}}$$

11. simplify: $(220 - 30\sqrt{35})$

We shall seek to write $(220 - 30\sqrt{35})$ as the binomial square. We notice that $\sqrt{35} = \sqrt{5}\sqrt{7}$.
After some fruitless attempts, we try with:

$$(220 - 30\sqrt{35}) = (a\sqrt{7} - b\sqrt{5})^2 = 7a^2 + 5b^2 - 2ab\sqrt{35}$$

$$2ab = 30 \ \Rightarrow \ ab = 15 \ \Rightarrow \ a = 3 \ and \ b = 5 \ \ or \ \ a = 5 \ \ or \ \ b = 3.$$

It shows out that the first is the right solution, since.

$$(5\sqrt{7} - 3\sqrt{5})^2 = 175 + 45 - 2 \cdot 5 \cdot 7\sqrt{35} = 220 - 30\sqrt{35}$$

12. Solve the equation: $\sqrt{2 + \sqrt{x + 2}} = x$

It is seen that $x = 2$ is a solution.

We take the square on both sides: $2 + \sqrt{x+2} = x^2$

And again take the square on both sides of: $\sqrt{x+2} = x^2 - 2$

$x + 2 = x^4 + 4 - 4x^2 \quad \Leftrightarrow \quad x + 2 = x^2(x^2 - 2^2) + 4$

$x - 2 = x^2(x-2)(x+2) \quad \Leftrightarrow \quad x^2(x+2) = 1 \quad \Leftrightarrow \quad x = 2 \ \vee \ x^3 + 2x^2 - 1 = 0$

We notice that $x = -1$ is a root in the last equation, so we divide by $x+1$.

$x+1 | x^3 + 2x^2 - 1 | x^2 + x - 1$
$x^3 + x^2$
$x^2 - 1$
$x^2 + x$
$-x - 1$

$x^2 + x - 1 = 0. \quad d = 5 \quad x = \frac{-1+\sqrt{5}}{2}$,
where we have discarded the negative root, since x must be positive.

13. Solve $\sqrt{1+\sqrt{x}} = x - 1$

We start by squaring both sides:

$$\left(\sqrt{1+\sqrt{x}}\right)^2 = (x-1)^2 \quad \Leftrightarrow \quad 1+\sqrt{x} = (x-1)^2$$

And we square another time:

$$\sqrt{x}^2 = ((x-1)^2 - 1)^2 \quad \Leftrightarrow \quad x = x^2(x-2)^2 \quad \Leftrightarrow$$
$$x^4 - 4x^3 + 4x^2 - x = 0 \quad \Leftrightarrow \quad x(x^3 - 4x^2 + 4x - 1) = 0$$

We notice that $x = 1$ is a solution. So we divide with x -1.

$x-1 | x^3 - 4x^2 + 4x - 1 | x^2 \; - 3x + 1$
$x^3 - x^2$
$-3x^2 + 4x$
$-3x^2 + 3x$
$x - 1$

$x^2 \; - 3x + 1 = 0$
$d = 9 - 4 = 5$
$x = \frac{3 \pm \sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$

14. Determine $f(x)$ **from** $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$

There is really nothing to say about this problem, other than that every function $f(x) = x^{\alpha}$ fulfils the functional equation, since:

$$f\left(\frac{x}{y}\right) = \left(\frac{x}{y}\right)^{\alpha} = \frac{x^{\alpha}}{y^{\alpha}} \; \frac{f(x)}{f(y)}$$

15. Solve: $x^{x^5} = 5$

Since a transcendent equation cannot be solved analytically, we shall resort to guesswork. In this case, however, it is rather obvious. We try with $x = \sqrt[5]{5}$

$$\sqrt[5]{5}^{\sqrt[5]{5}^5} = \sqrt[5]{5}^5 = 5$$

16. Solve the equation $\sqrt[3]{x+28} + \sqrt[3]{x-28} = 2$

To solve this equation algebraically maybe possible, but at a glance we see that:

$4-2=2 \Leftrightarrow \sqrt[3]{64} - \sqrt[3]{8} = 28$ So our guess is: $x+28=64 \quad \Leftrightarrow \quad x=36 \quad and \quad x-28=8$

The solution is therefore: $x = 36$.

16a. Solve $\sqrt{x-2} + 3 = \sqrt{4 \cdot x + 1}$

The equation may probably be solved by squaring two times, but it is much easier with an eloquent guess. The number 3, suggest that the first square root is 2, that is $x = 6$. And indeed the right side gives 5. Since:

$$\sqrt{6-2} + 3 = \sqrt{4 \cdot 6 + 1}$$

17. Solve $\sqrt[3]{x+3} - \sqrt{x-1} = -6$

We shall not try the algebraic way, lifting both sides of the equation to a higher power, since it is almost always futile. Instead we shall try to set $x + 3 = y^3$, where y is an integral number. The possibilities are: $x+3 = 8, 27, 64, 125$. $x+3=125$ is a promising candidate, since $x-1 = 122-1 = 121$. We find $\sqrt[3]{125} - \sqrt{121} = 5 - 11 = -6$

18. Solve $\sqrt[3]{x+3} = \sqrt{x-1}$

Well, an algebraic treatment is not palatable, since it requires raising to the 6th power. But it is actually very easy to guess $x+3=8 \quad \Leftrightarrow \quad x=5$.

Then the equation is satisfied since: $\sqrt[3]{8} = \sqrt{4}$.

18a. Solve: $\sqrt{x} + \sqrt{y} = \sqrt{250}$

Strange exercise, since all integer solution seem to give $\sqrt{256}$, e.g.

$x = 9^2, x = 7^2$ gives $\sqrt{x} + \sqrt{y} = 16 = \sqrt{256}$.

The same result is obtained with: $(\sqrt{x}, \sqrt{y}) = (10,6),(8,8),(11,5),(14,2)$

However if we proceed, we can isolate y

$\sqrt{y} = \sqrt{250} - \sqrt{x} \quad \Rightarrow y = 250 + x - 2\sqrt{250x} = 250 + x - 2\sqrt{25 \cdot 10x}$

$y = 250 + x - \sqrt{250x} = 250 + x - 10\sqrt{10x}$

To facilitate the calculations we put $x = 10k^2$, so we now have the equation:

$y = 250 + x - 10\sqrt{10x} = 250 + x - 100k$

Since $x \le 250 \quad \Leftrightarrow \quad 10k^2 \le 250 \quad \Leftrightarrow \quad 0 \le k \le 5$.
So let us take it step by step:
$k = 0: \ x = 0 \quad and \quad y = 250$

$k = 1: x = 10, \quad y = 160$

$k = 2: \ x = 40 \quad y = 90$

$k = 3 \quad x = 90 \quad y = 40$

$k = 4: \quad x = 160 \quad y = 10$

$k = 5: \quad x = 250 \quad y = 0$

So these are the solutions, although integers not quadratic numbers. (Strange)

18b. Solve $x^3 - (\sqrt{3+1})x^2 + 3 = 0$

We may not really try to solve this equation by algebra. But a qualified guess is. $x = \sqrt{3}$, and we find:

$$\sqrt{3}^3 - (\sqrt{3}+1)\sqrt{3}^2 + 3 = 3\sqrt{3} - 3\sqrt{3} - 3 + 3 = 0$$

19. Solve the equations: $\sqrt{x} + y = 7$ **and** $x + \sqrt{y} = 11$

A first attempt to solve the equations analytically leads to a 4th order algebraic equation, which cannot be solved but it is easy to see that $(x,y) = (9,4)$, is a solution. To show that this is the only solution in the relevant intervals, we shall do some calculations.
Initially we multiply the first equation by 11 and the second by 7, both equations then become equal to 77, and we may put them equal to each other. After some rearrangement we may write.

$11\sqrt{x} + 11y = 7x + 7\sqrt{y} \quad \Leftrightarrow \quad 7x - 11\sqrt{x} = 11y - 7\sqrt{y} = c$

Since the left side depends only on x and the right side depends only on y, they must be equal to the same constant c.

$7x - 11\sqrt{x} - c = 0$ and $11y - 7\sqrt{y} - c = 0$

We know however that $x = 9$ is a solution to the first equation, so we get 63 – 11 3 –c = 30.
We put $x = t^2$, and find the equation:

$$7t^2 - 11t - 30 = 0 \qquad d = 121 + 4\cdot 7\cdot 30 = 961 = 31^2 \quad t = \sqrt{x} = \frac{11 \pm 31}{14} = \begin{cases} 3 \\ -\frac{10}{7} \end{cases}$$

Since the square root is never negative we have: $\sqrt{x} = 3, \ or \quad x = 9$

y is then determined by $y = \ 7 - \sqrt{x} \quad \Rightarrow \ y = 4$

20. Solve $y'' = (y')^2$

We put $z = y'$ and we then have:

$$z' = z^2 \quad \Leftrightarrow \quad \frac{dz}{dx} = z^2 \quad \Leftrightarrow \quad \int \frac{dz}{z^2} = \int dx \quad \Leftrightarrow \quad -\frac{1}{z} = x + c$$

$$-\frac{1}{y'} = x + c \quad \Leftrightarrow \quad y' = -\frac{1}{x+c} \quad \Leftrightarrow \quad y = \ln|x + c|$$

21. Solve the differential equation: $y' = \sqrt{x+y}$.

We shall first look at the well known equation: $y' = \sqrt{y}$, which is solved by separation:

$$\frac{dy}{dx} = \sqrt{y} \quad dy = \sqrt{y}dx \quad \Leftrightarrow \quad \int \frac{dy}{\sqrt{y}} = x + c \quad \Leftrightarrow \quad 2\sqrt{y} = x + c \quad \Leftrightarrow \quad y = \tfrac{1}{4}(x+c)^2$$

However, we cannot apply this method directly, since we cannot separate the variables.

We start with the substitution: $z = x + y; \quad dz = 1 + dy, so \;\; dy = dz - 1$, so we have:

$$z' - 1 = \sqrt{z} \quad \Leftrightarrow \quad z' = \sqrt{z} + 1 \quad \Leftrightarrow \quad \frac{dz}{dx} = \sqrt{z} + 1 \quad \Leftrightarrow \quad \int \frac{dz}{\sqrt{z} + 1} = x + c$$

The integral cannot be immediately integrated, so we make a new substitution:

$$u = \sqrt{z} + 1 \quad \Rightarrow du = \frac{1}{2\sqrt{z}} dz \quad \Rightarrow \quad dz = 2\sqrt{z}\, du = 2(u-1)du$$, so we get:

$$2\int \frac{(u-1)}{u} du = 2(u - \ln(u))$$

and then we substitute back.

$$2(\sqrt{z} + 1 - \ln(\sqrt{z} + 1)) = 2(\sqrt{x+y} + 1 - \ln(\sqrt{x+y} + 1))$$

So the solution to the differential equation is the functional equation:

$$2(\sqrt{x+y} + 1 - \ln(\sqrt{x+y} + 1)) = x + c$$

However, y cannot be isolated in this equation.

22. Which is the largest $(n!)^2 \, or \;\; n^n$

A trial with n = 1,2,3,4, seem to reveal the answer.

$1!^2 = 1$, $(2!)^2 = 4$, $(3!)^2 = 36$, $(4!)^2 = 576$, $(5!)^2 = 14.400$, whereas
$1^1 = 1$, $2^2 = 4$, $3^3 = 27$, $4^4 = 256$.

To make a formal proof we write the one part of $(n!)^2$ backwards.

$$(n!)^2 = 1 \cdot n \cdot 2 \cdot (n-1) \cdot 3(n-2)........(n-1) \cdot 2 \cdot n \cdot 1$$

The p'th double factor we shall compare to the factor n in n^n.

The p'th double factor is $p(n-p+1)$, so we will solve the inequality.

$$\frac{p(n-p+1)}{n} > 1 \quad or \quad p(n-p+1) > n \quad \Leftrightarrow$$
$$-p^2 + p(n+1) - n > 0 \;\Leftrightarrow\; p^2 - p(n+1) + n < 0$$
$$d = (n+1)^2 - 4n = (n-1)^2$$
$$p = \frac{n+1 \pm (n-1)}{2} \quad \Leftrightarrow \quad p = n \quad or \quad p = 2$$

A polynomial of second degree is negative between the roots, as it is well known, so the solution is:

$$2 < p < n$$

The equality is valid in the end points: $(2!)^2 = 4$ and $2^2 = 4$, $n \cdot 1 = n$

23. Solve: $7x^3 + 12x^2 + 6x + 1 = 0$

A problem that actually is too elementary to be mentioned here.

One sees immediately that $x = -1$, is a solution, division with $x+1$ gives:

$$7x^3 + 12x^2 + 6x + 1 = (x+1)(7x^2 + 5x + 1)$$
$$7x^2 + 5x + 1 = 0; \quad d = 25 - 28 < 0 \quad \text{The only solution is } x = -1$$

24. Determine. $a^{2022} + \frac{1}{a^{2022}} \quad when \quad a^2 + a + 1 = 0$

$a^2 + a + 1 = 0$ has determinant $d = 1 - 4 < 0$, so a cannot be a real number, (which ought to be stated in the problem).

We shall try to solve the problem, without using reference to complex numbers.

$a^4 = (-a-1)^2 = a^2 + 1 + 2a = a^2 + 1 + a + a = a$.

From this follows: $(a^{16}) = (a^4)^4 = a^4 = a$, and consequently $a^{p4} = (a^4)^p = a^p$, so we have:

$a = a^4 = a^{16} = a^{64} = (a^{16})^4 = a^{256} = a^{1024}$

We then decompose $2022 = 2\cdot 1024 - 26 = 2\cdot 1024 - 16 - 8 - 2$ and we get

$$a^{2022} = a^{2\cdot 1024-26} = a^{2\cdot 1024-16-8-2}\ \frac{a^2}{a\cdot a^2 a^2} = \frac{a^2}{a^2} = 1\text{, so}$$

$$a^{2022} + \frac{1}{a^{2022}} = 2$$

We can make it easier: From $a^4 = a$ follows: $a^3 = 1$, so $(a^3)^n = a^{3\cdot n} = 1$,

$2202 = 3\cdot 734$, so $a^{2022} + \dfrac{1}{a^{2022}} = (a^3)^{734} + \dfrac{1}{(a^3)^{734}} = 1 + 1 = 2$

25. Determine *f* (12) from $f(x+\sqrt{x}) = x - \sqrt{x}$

$f(x+\sqrt{x}) = x - \sqrt{x},\quad find\ \ f(12).\ \ x + \sqrt{x} = 12\ \ gives\ \ \sqrt{x} = 3$, so $f(12) = 9 - 3 = 6$

25a. Determine *f* from: $f(x+\frac{1}{x}) = x - \frac{1}{x}$

$$y = x + \frac{1}{x} \iff x^2 - yx + 1 = 0\ \ ;\ \ d = y^2 - 4;\ \ x = \frac{y \pm \sqrt{y^2-4}}{2}$$

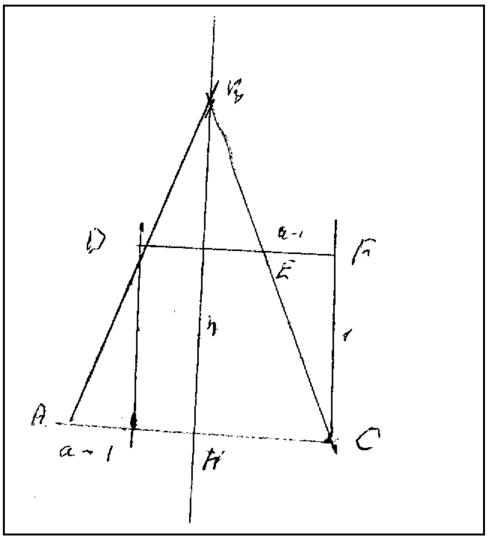
$$Also\quad y = \frac{x^2+1}{x} = x - \frac{1}{x} + \frac{2}{x} \Rightarrow x - \frac{1}{x} = y + \frac{2}{x} = y + \frac{2}{\frac{y\pm\sqrt{y^2-4}}{2}} =$$

We put

$$y + \frac{4}{y \pm \sqrt{y^2-4}}\quad so\quad f(y) = y + \frac{4}{y \pm \sqrt{y^2-4}}$$

$$f(x) = x + \frac{4}{x \pm \sqrt{x^2-4}}$$

26. A geometrical problem



Find the area of the triangle.
The side in the triangle is *a*. The side en the square is 1.
From the triangular triangles:

$$\Delta ADG \propto \Delta ABH \quad and \quad \Delta CFE \propto \Delta CBH$$

$$\frac{AG}{GD} = \frac{AH}{BH}\quad (and\ \ \frac{EF}{FC} = \frac{AH}{BH})$$

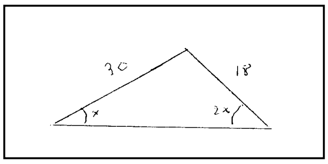
$$\frac{a-1}{1} = \frac{½a}{h}\quad (and\ \ \frac{a-1}{1} = \frac{½a}{h}) \Rightarrow a - 1 = \frac{a}{2h}$$

$$h = \frac{\sqrt{3}}{2} a \textit{ follows from } a^2 = h^2 + \left(\frac{a}{2}\right)^2$$

$$a - 1 = \frac{a}{\sqrt{3a}} \Rightarrow a - 1 = \frac{1}{\sqrt{3}} \Rightarrow a = 1 + \frac{1}{\sqrt{3}}$$

$$T = \tfrac{1}{2} ha = \frac{\sqrt{3}}{2} a^2 = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}}{6}(\sqrt{3}+1)^2$$

27. Geometrical problem. Area of a triangle.



In a triangle two sides are 18 and 30. The angle with the baseline c are x and $2x$.
We use the formula for the area. $T = \tfrac{1}{2} ab \sin C = \tfrac{1}{2} ac \sin B = \tfrac{1}{2} bc \sin A$
So we have: $T = \tfrac{1}{2} \cdot 18 \cdot c \sin 2x = \tfrac{1}{2} \cdot 30 \cdot c \sin x \Rightarrow$
$18 \sin 2x = 30 \sin x \Rightarrow 18 \cdot 2 \sin x \cos x = 30 \sin x \Rightarrow$
$\cos x = \dfrac{5}{6} \Rightarrow \sin x = \sqrt{1 - (\tfrac{5}{6})^2} = \sqrt{\tfrac{11}{36}}$

$T = \tfrac{1}{2} 18 \cdot 30 \sin(180 - 3x) = 270 \sin 3x$
$\sin 3x = \sin(2x + x) = 2 \sin 2x \cos x = 4 \sin x \cos^2 x$
So we get for the area:

$$T = 270 \cdot 4 \cdot \tfrac{\sqrt{11}}{6} \cdot \tfrac{25}{36} = 5 \cdot \tfrac{\sqrt{11}}{2} \cdot 25 = 125 \tfrac{\sqrt{11}}{2}$$

28. Solve: $x^{\sqrt{x}} = \tfrac{1}{2}$

It is immediately seen that $x = \tfrac{1}{4}$ is a solution since $(\dfrac{1}{4})^{\sqrt{\frac{1}{4}}} = (\dfrac{1}{4})^{\frac{1}{2}} = \sqrt{\tfrac{1}{4}} = \tfrac{1}{2}$

29. Determine *f*, when $f(x + \dfrac{1}{x} + 4) = x^2 + \dfrac{1}{x^2} + 16$

We put $y = x + \dfrac{1}{x} + 4$ and calculate $y^2 = (x + \dfrac{1}{x} + 4)^2 = x^2 + \dfrac{1}{x^2} + 16 + 2 + 8x + \dfrac{8}{x}$

$y^2 = x^2 + \dfrac{1}{x^2} + 8(x + \dfrac{1}{x} + 4) - 14 \Rightarrow$ so we have:

$x^2 + \dfrac{1}{x^2} = y^2 - 8y + 14 \Rightarrow x^2 + \dfrac{1}{x^2} + 16 = y^2 - 8y + 30$

Since this expression is equal to the right side of the equation: $f(x + \dfrac{1}{x} + 4) = x^2 + \dfrac{1}{x^2} + 16$, we have: $f(y) = y^2 - 8y + 30$ or

$$f(x) = x^2 - 8x + 30$$

29a. Determine *f* where: $2f(x+y)+6y^3 = f(x+2y)+x^3$**.**

We notice that we find an expression for $f(x)$, if we put $y = 0$, we get:

$$2f(x) = f(x)+x^3 \quad \Leftrightarrow \quad f(x) = x^3$$

We shall then show that this correspond to the defining equation.

$$2(x+y)^3 + 6y^3 = (x+2y)^3 + x^3$$
$$2x^3 + 2y^3 + 6x^2y + 6xy^2 + 6y^3 = x^3 + 8y^3 + 6x^2y + 6xy^2 + x^3$$

And so it does.

30. Determine *f* from the expression: $\int \frac{f(x)}{x}dx = xf(x)+c$**.**

We do this by differentiating the right side to get the integrand.

$$(xf(x)+c)' = \frac{f(x)}{x} \quad \Rightarrow \quad xf'(x)+f(x) = \frac{f(x)}{x} \quad or \quad xy'+y-\frac{y}{x} \quad \Leftrightarrow$$
$$y'+y(\frac{1}{x}-\frac{1}{x^2}) = 0.$$
$$Let \;\; G(x) = \int (\frac{1}{x}-\frac{1}{x^2})dx \quad \Rightarrow \quad G(x) = \ln x + \frac{1}{x} + k \quad such \;\; that \quad G'(x) = (\frac{1}{x}-\frac{1}{x^2})$$
$$The \;\; equation \;\; can \;\; then \; be \; written: \quad y'+yG'(x) = 0.$$

We then multiply the equation with: $\exp(G(x))$ so we have:

$$y'e^{G(x)} + ye^{G(x)}G'(x) = 0.$$

Which may also be written:

$$(ye^{G(x)})' = 0 \quad \Leftrightarrow \quad ye^{G(x)} = c \quad \Leftrightarrow$$

$$y = \frac{c}{e^{G(x)}} = ce^{-G(x)} = ce^{-\ln x - \frac{1}{x}} \quad \Leftrightarrow \quad y = \frac{c}{x}e^{-\frac{1}{x}}$$

31. (A classical mathematical joke)

Alice is eight years older than Bob. Within one year Alice will be twice as old as Bob is now.
A = age of Alice, B = Age of Bob
What are their ages? The statements above can be expressed in the two equations:
A = B + 8, and A+1 =2B. And there are easily solved to give: A= 17 and B= 9.

31a. Solve the system of equations $x^2+xy=5 \quad and \quad y^2+xy=11$

We add the two equations to give $x^2+y^2+2xy=16 \iff (x+y)^2=16 \iff x+y=\pm 4 \Rightarrow$

$y=-x\pm 4$ we first look at: $y=4-x$, which we insert in the first equation.

$$y=4-x \quad and \quad x^2+xy=5 \quad \Rightarrow \quad x^2+x(4-x)=5 \quad \iff \quad x=\frac{5}{4} \quad and \quad y=\frac{11}{4}$$

$$y=-4-x \quad and \quad x^2+xy=5 \quad \Rightarrow \quad x^2+x(-4-x)=5 \quad \iff \quad x=-\frac{5}{4} \quad and \quad y=-\frac{11}{4}$$

32. Solve the equation: $x^4+4x-1=0$

We do some rewriting

$$x^4+4x-1=(x^2+1)^2-2x^2+4x-2=(x^2+1)^2-2(x^2+1-2x) \quad \iff$$

$$(x^2+1)^2-2(x-1)^2=0 \quad \iff$$

$$\sin ce \quad a^2-b^2=(a-b)(a+b) \quad we\ have$$

$$((x^2+1)-\sqrt{2}(x-1))((x^2+1)+\sqrt{2}(x-1))=0 \quad \iff$$

$$(x^2+1)-\sqrt{2}(x-1)=0 \quad \vee \quad (x^2+1)+\sqrt{2}(x-1)=0 \quad \iff$$

$$x^2-\sqrt{2}x+\sqrt{2}+1)=0 \quad \vee \quad x^2+\sqrt{2}x+1-\sqrt{2})=0$$

$$d=2-4(\sqrt{2}+1)<0 \qquad d=2-4(1-\sqrt{2})=4\sqrt{2}-2 \quad \iff$$

$$x=\frac{-\sqrt{2}\pm\sqrt{4\sqrt{2}-2}}{2}$$

33. Determine the function *f*, from the equation $f(x)+f\left(\frac{1}{1-x}\right)=x$

We shall rewrite the equation substituting *x* by *y*.

$$f(y)+f\left(\frac{1}{1-y}\right)=y$$

We put: $y=\frac{1}{1-x}$ and $\frac{1}{1-y}=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x}$

and we find:

$$f(\frac{1}{1-x})+f\left(\frac{x-1}{x}\right)=\frac{1}{1-x} \quad \iff$$

$$f(\frac{1}{1-y})+f\left(\frac{y-1}{y}\right)=\frac{1}{1-y}$$

We put:
$$y=\frac{1}{1-x} \Rightarrow \frac{1}{1-y}=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x} \quad and$$
$$\frac{y-1}{y}=\frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}=\frac{\frac{x}{1-x}}{\frac{1}{1-x}}=x$$

$$f(\frac{x-1}{x})+f(x)=\frac{x-1}{x}$$

We then have 3 equations:

I: $f(x)+f\left(\frac{1}{1-x}\right)=x$ II: $f(\frac{1}{1-x})+f\left(\frac{x-1}{x}\right)=\frac{1}{1-x}$ III: $f(\frac{x-1}{x})+f(x)=\frac{x-1}{x}$

If calculate: I + III –II we find: $2f(x)=x+\frac{x-1}{x}-\frac{1}{1-x}=\frac{x^2(1-x)+(x-1)(1-x)-x(1-x)}{x(1-x)}$

34. Solve the equation: $4^x+6^x=9^x$

We divide by: 4^x : $1+\frac{6}{4^x}^x=\frac{9}{4^x}^x \Rightarrow 1+\left(\frac{3}{2}\right)^x=\left(\left(\frac{3}{2}\right)^x\right)^2$

If we put $y=\left(\frac{3}{2}\right)^x$ we find the quadratic equation:

$$y^2-y-1=0; \quad d=1+4=5$$
$$y=\frac{1\pm\sqrt{5}}{2}=\frac{1+\sqrt{5}}{2}$$

So: $\left(\frac{3}{2}\right)^x=\frac{1+\sqrt{5}}{2} \Leftrightarrow x=\frac{\ln\left(\frac{1+\sqrt{5}}{2}\right)}{\ln\frac{3}{2}}$

35. Two trivial equations: $5^x-3^x=16$ **and** $n!=n^3-n$

$5^x-3^x=16 \Leftrightarrow 3^x+4^2=5^x \Leftrightarrow x=2$

$n!=n^3-n \Leftrightarrow n=5$

36. Solve: $x+\sqrt{x+\sqrt{x+\sqrt{x+.....}}}=20$

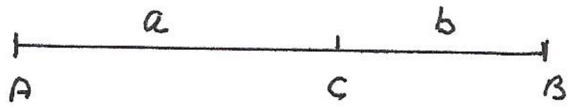
$\Leftrightarrow x+\sqrt{x}=20 \Leftrightarrow \sqrt{x}^2+\sqrt{x}-20=0$

$d=1+4\cdot 20=81; \quad \sqrt{x}=\frac{-1\pm 9}{2}=4 \Leftrightarrow x=16$

37. The golden ratio

The golden ratio is the designation for dividing a line segment having the length $a + b$ into two pieces a and b, which satisfy the relation:

$$\frac{a}{b} = \frac{a+b}{a}$$



The equation can be rewritten as: $\frac{a}{b} = 1 + \frac{b}{a}$.

And if we put $\frac{a}{b} = x$, then the golden ratio will satisfy the equation:

$$x = 1 + \frac{1}{x} \iff x^2 - x - 1 = 0 \iff x = \frac{1 \pm \sqrt{5}}{2}.$$

The ratio between the two pieces is : $\frac{a}{b} = x = \frac{1+\sqrt{5}}{2} \approx 1{,}618$.

38. Problem: Determine $\left(\frac{1+\sqrt{5}}{2}\right)^{12}$.

Any attempt on a direct calculation is doomed to fail,

but if we put $x = \frac{1+\sqrt{5}}{2}$, then x satisfy the equation:

$x^2 - x - 1 = 0 \iff x^2 = x + 1$

$x^4 = (x+1)^2 = x^2 + 2x + 1 = x + 1 + 2x + 1 = 3x + 2$

$x^8 = (3x+2)^2 = 9x^2 + 4 + 12x = 9(x+1) + 12x + 4 = 21x + 13$

$x^{12} = x^4 x^8 = (3x+2)(21x+13) = 63x^2 + 39x + 42x + 26 = 63x + 63 + 81x + 26$

$x^8 = (3x+2)^2 = 9x^2 + 4 + 12x = 9(x+1) + 12x + 4 = 21x + 13$

$x^{12} = 144x + 89 = 144\left(\frac{1+\sqrt{5}}{2}\right) + 89$

39. Determine *f* from : $f(x+\sqrt{x}) = x - \sqrt{x}$

$$(x+\sqrt{x})(x-\sqrt{x}) = x^2 - x \quad \textit{and we put} \quad y = x + \sqrt{x}$$

We notice that

$$f(x+\sqrt{x}) = f(y) = x - \sqrt{x} = \frac{(x-\sqrt{x})(x+\sqrt{x})}{x+\sqrt{x}} = \frac{x^2 - x}{y}$$

we need to express x by y, which is not at simple as it sounds.

$$y = x + \sqrt{x} \iff \sqrt{x}^2 + \sqrt{x} - y = 0; \quad d = 1 + 4y; \Rightarrow \sqrt{x} = \frac{-1+\sqrt{1+4y}}{2}$$

$y = x + \sqrt{x} \quad \Leftrightarrow \quad y - x = \sqrt{x} \quad \Leftrightarrow \quad y^2 + x^2 - 2xy = x \quad \Rightarrow x^2 - x = 2xy - y^2$

$f(y) == \dfrac{x^2 - x}{y} = \dfrac{2xy - y}{y} = 2x - 1 = \frac{1}{2}(2\sqrt{x})^2 - 1 = \frac{1}{2}(1 + (1 + 4y) - 2\sqrt{1 + 4y}) - 1$

$f(y)$ =1+2y - $\sqrt{1+4y}$) -1 = 2y- $\sqrt{1+4y}$) .

40. Solve: $\sqrt{x+5} + \sqrt{20-x} = 7$

We put $y^2 = x + 5$

$\sqrt{y^2} + \sqrt{25 - y^2} = 7 \quad \Rightarrow \quad \sqrt{25 - y^2} = 7 - y \quad \Rightarrow 25 - y^2 = (7 - y)^2$

$y^2 - 7y + 12 = 0; \quad d = 49 - 48 = 1 \quad y = \dfrac{7 \pm 1}{2} = \left(\begin{array}{l} 4 \\ 3 \end{array} \right. \quad y^2 = \left(\begin{array}{l} 16 \\ 9 \end{array} \right. \quad \Leftrightarrow \quad x = y^2 - 5 = \left(\begin{array}{l} 11 \\ 4 \end{array} \right.$

41. Solve for *f*: $f(x+y) = xf(y) = yf(x)$

$xf(y) = yf(x) \quad \Leftrightarrow \quad \dfrac{f(x)}{x} = \dfrac{f(y)}{y} = const = k$

The ration must be constant, since the first term depends on x only, and that the second term depends only on y, so $f(x) = kx \quad and \quad f(y) = ky$

$f(x+y) = xf(y) = yf(x) = xky = ykx = kxy$, so

$$f(x+y) = kxy$$

42. Solve: $4^{\frac{1}{x}} - 6^{\frac{1}{x}} = 9^{\frac{1}{x}}$

We divide the equation by $4^{\frac{1}{x}}$: $1 - (\frac{6}{4})^{\frac{1}{x}} = (\frac{9}{4})^{\frac{1}{x}} \quad \Leftrightarrow$

$1 - (\frac{3}{2})^{\frac{1}{x}} = ((\frac{3}{2})^2)^{\frac{1}{x}} \quad \Leftrightarrow 1 - (\frac{3}{2})^{\frac{1}{x}} = ((\frac{3}{2})^{\frac{1}{x}})^2$ We put $y = (\frac{3}{2})^{\frac{1}{x}}$ and find:

$1 - y = y^2 \quad \Leftrightarrow \quad y^2 + y - 1 = 0 \quad \Leftrightarrow \quad y = \dfrac{-1 \pm \sqrt{5}}{2} = \dfrac{-1 + \sqrt{5}}{2}$

$(\frac{3}{2})^{\frac{1}{x}} \dfrac{-1+\sqrt{5}}{2} \quad \Leftrightarrow \quad \dfrac{1}{x}\log(\frac{3}{2}) = \log(\dfrac{-1+\sqrt{5}}{2}) \quad \Leftrightarrow \quad x = \dfrac{\log(\frac{3}{2})}{\log(\frac{-1+\sqrt{5}}{2})}$

43. Determine *f* from: $f\left(\frac{x-3}{x+1}\right)+f\left(\frac{x+3}{1-x}\right)=x \quad f(y)+f(z)=x$

We put $y=g(x)=\frac{x-3}{x+1} \Leftrightarrow y(x+1)=x-3 \Leftrightarrow x=g^{-1}(y)=z(y)=\frac{y+3}{1-y}$

We put $x=\alpha x+\beta x; \quad where \quad \alpha+\beta=1$ and set $f(y)=f(\frac{x-3}{1+x})=\alpha x \quad and \quad f(z)=f(\frac{x+3}{1-x})=\beta x$

$f(y)=\alpha x=\alpha g^{-1}(y)$ (and $f(g^{-1}(x))=\beta x$)

$f(y)=\alpha\frac{y+3}{1-y} \quad \Leftrightarrow \quad f(x)=\alpha\frac{x+3}{1-x}$.

Since we have found one expression for $f(x)$, we do not need the other expression.

$$f\left(\frac{x-3}{x+1}\right)+f\left(\frac{x+3}{1-x}\right)=x \quad \Leftrightarrow \quad f(g(x))+f(g^{-1}(x))=x,\ where\ \ y=g(x)=\frac{x-3}{x+1}$$

44. Given: $x+\frac{1}{x}=5$ **compute** $x^5+\frac{1}{x^5}$

$(x+\frac{1}{x})^2=x^2+\frac{1}{x^2}+2=25 \quad \Leftrightarrow \quad x^2+\frac{1}{x^2}=23$

$(x^2+\frac{1}{x^2})(x+\frac{1}{x})=x^3+x+\frac{1}{x}+\frac{1}{x^3}=23\cdot 5 \quad \Rightarrow \quad x^3+\frac{1}{x^3}=23\cdot 5-(x+\frac{1}{x})=22\cdot 5$

$(x^2+\frac{1}{x^2})^2=23^2 \quad \Leftrightarrow \quad x^4+\frac{1}{x^4}+2=23^2 \quad \Leftrightarrow \quad x^4+\frac{1}{x^4}=23^2-2$

$(x^4+\frac{1}{x^4})(x+\frac{1}{x})=x^5+x^3+\frac{1}{x^3}+\frac{1}{x^5}=x^5+\frac{1}{x^5}+x^3+\frac{1}{x^3}=(23^2-2)5$

$x^5+\frac{1}{x^5}=(23^2-2)5-(x^3+\frac{1}{x^3})=(23^2-2)5-22\cdot 5=23^2-120$

45. solve $6^x+6^y=42 \quad and \quad x+y=3$

Actually an immediate guess is $x = 1$ and $y = 2$, which sees to fit, but let us control that.

$$x+y=3 \quad \Rightarrow \quad 6^{x+y}=6^3 \quad \Leftrightarrow \quad 6^x6^y=6^3 \quad then \quad 6^y=\frac{6^3}{6^x}$$

Which we insert in the first equation:

$6^x+\frac{6^3}{6^x}=42$ We put $z=6^x$ and we then have

$z^2-42z+6^3=0$

$d=42^2-4\cdot 6^3=900=30^2 \quad z=\frac{42\pm 30}{2}=\begin{pmatrix}36\\6\end{pmatrix} \quad \Rightarrow \quad 6^x=36 \quad or \quad 6^x=6 \quad \Leftrightarrow$

$$x=2 \quad or \quad x=1$$

46. Given $x+\sqrt{x}=5.\ Find\quad x+\frac{5}{\sqrt{x}}$

We divide

$x+\sqrt{x}=5\ with\ \sqrt{x}$ to give $\sqrt{x}+1=\frac{5}{\sqrt{x}}$, This inserted in $x+\frac{5}{\sqrt{x}}=c$

To give: $x+\sqrt{x}+1=c\ \Rightarrow\ 5+1=c\ \Leftrightarrow\ c=6$

47. Simplify $\sqrt[3]{2+\sqrt{5}}$ **.**

That is find a number, which raise to the third power is $2+\sqrt{5}$.

After some fruitless efforts, we multiply and divide with 8.

$$\frac{8(2+\sqrt{5})}{8}=\frac{16+8\sqrt{5}}{2^3}$$

Now the idea is to write $16+8\sqrt{5}$ as $(a+b\sqrt{5})^3$, where *a* and *b* are integral numbers.

$(a+b\sqrt{5})^3=a^3+3b^3 5\sqrt{5}+3a^2b\sqrt{5}+3ab^2 5$

If we collect the terms with the square root $\sqrt{5}$, we have:

$3b^3 5\sqrt{5}+3a^2b\sqrt{5}$ .so $3b^3 5+3a^2b=8$,

which is only possible if $a = b = 1$

Indeed: $(1+\sqrt{5})^3=1+5\sqrt{5}+3\cdot 1^2\sqrt{5}+3\cdot 1\cdot 5=16+8\sqrt{5}$

48. Solve $x^{x^5}=5$

As mentioned several times, a transcendent equation has in general no analytic solution, so we shall try with a qualified guess. The guess must include the number 5.

It is rather obvious to try with $x=\sqrt[5]{5}$, and indeed:

$$\sqrt[5]{5}^{\sqrt[5]{5}^5}=\sqrt[5]{5}^5=5$$

49. solve $9^x+15^x=25^x$

$$9^x+15^x=25^x\ \Leftrightarrow\ 3^x3^x+3^x5^x=5^x5^x$$

The equation can be simplified if we divide the equation with 3^x5^x .

$$3^x3^x + 3^x5^x = 5^x5^x \quad \Leftrightarrow \quad \frac{3^x}{5^x} + 1 = \frac{5^x}{3^x}$$

If we put $y = \frac{3^x}{5^x}$ the equation can be written. $y + 1 = \frac{1}{y} \quad \Leftrightarrow \quad y^2 + y - 1 = 0$

$d = 1 + 4 \qquad y = \frac{-1 \pm \sqrt{5}}{2} = \frac{-1 + \sqrt{5}}{2}$

$$\frac{3^x}{5^x} = \left(\frac{3}{5}\right)^x = \frac{-1+\sqrt{5}}{2} \quad \Leftrightarrow \quad x = \frac{\ln\left(\frac{-1+\sqrt{5}}{2}\right)}{\ln\left(\frac{3}{5}\right)}$$

50. Solve: $\frac{1}{x} - \frac{1}{y} = \frac{1}{x+y} \quad \textit{find} \quad \frac{y}{x}$

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{x+y} \quad \Leftrightarrow \quad \frac{x-y}{xy} = \frac{1}{x+y} \Leftrightarrow \quad \frac{(x-y)(x+y)}{xy} = 1$$

$$\frac{(x^2-y^2)}{xy} = 1$$

divide the fraction by x^2 on the left hand side.

$$\frac{1-\left(\frac{y}{x}\right)^2}{\frac{y}{x}} = 1 \quad \Leftrightarrow \quad \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1 \quad \Leftrightarrow \quad z^2 + z - 1 = 0 \quad d = 1 + 4 = 5 \quad z = \frac{-1+\sqrt{5}}{2}$$

$$\frac{y}{x} = \frac{-1+\sqrt{5}}{2}$$

51. Solve $\log_{4x} x + \log_{\frac{x}{2}} x = 2$

From the definition of the logarithm with base g we have:

$\log_g x = y \quad \Leftrightarrow \quad x = g^y \Leftrightarrow \log_g x = y \log_g g = y$

If we have two bases a and b, we can make the identities

$a = b^{\log_b a} \quad \log_a(a) = 1 = \log_a b^{\log_b a} = \log_b a \cdot \log_a b \quad \Rightarrow \quad \log_b a \cdot \log_a b = 1$

We may use this identity to the two logs.
The advantage is, that we now have the same base: $a + b = 2$

$\log_{4x} x = \frac{1}{\log_x 4x} = a \quad \Leftrightarrow \quad \log_x 4x = \frac{1}{a}$.

$\log_{\frac{x}{2}} x = \frac{1}{\log_x \frac{x}{2}} = b \quad \Rightarrow \quad \log_x \frac{x}{2} = \frac{1}{b}$

$\log_x 4x = \frac{1}{a} \quad \Leftrightarrow \quad (4x) = x^{\frac{1}{a}} \qquad \log_x \frac{x}{2} = \frac{1}{b} \quad \Leftrightarrow \quad \frac{x}{2} = x^{\frac{1}{b}}$

$(4x)^a = x \qquad \left(\frac{x}{2}\right)^b = x \quad \Rightarrow \quad (4x)^a = \left(\frac{x}{2}\right)^b \quad \Leftrightarrow \quad (4x)^a = \left(\frac{x}{2}\right)^{2-a}$

$(2x^2)^a = \left(\frac{x}{2}\right)^2$

From these equations, we have really no way to determine *a,* We try a guess $a = \frac{1}{2}$, which gives

$\sqrt{2}x = \frac{x^2}{4} \quad \Leftrightarrow \quad x = 4\sqrt{2}$

51a. Solve for *x*: $\ln(\sqrt{1+x^2} - x) = 4$

$\ln(\sqrt{1+x^2} - x) = 4$. We put $\sqrt{1+x^2} - x = y \quad \Leftrightarrow \sqrt{1+x^2} = y + x \quad \Leftrightarrow \quad 1 + x^2 = (x+y)^2 \quad \Leftrightarrow$
$1 + x^2 = x^2 + y^2 + 2xy \quad \Leftrightarrow \quad x = \frac{1-y^2}{2y}$

$\ln(y) = 4 \quad \Leftrightarrow \quad y = e^4 \quad \Rightarrow \quad x = \frac{1-(e^4)^2}{2e^4}$

52. Determine *f,* where $f(xy) = f(x) - f(y)$

For the logarithmic function *log*, we have $\log(xy) = \log(x) + \log(y)$, so *f* must evidently be:

$f(xy) = \log(\frac{x}{y}) = f(x) - f(y)$

53. Solve $y^2 + (y')^2 = 1$

$$y^2 + (y')^2 = 1 \quad \Leftrightarrow \quad y' = \pm\sqrt{1-y^2}$$

It is obvious to guess at $y = \cos(x+\varphi)$, since $y' = -\sin(x+\varphi) = \pm\sqrt{1-\cos^2(x+\varphi)}$

53a. Solve $xy' + 2y = x^3$

$xy' + 2y = x^3 \quad \Leftrightarrow \quad y' + \frac{2y}{x} = x^2$

This is a standard first order linear differential equation, and the standard way to the solution is to multiply the equation (in this case) with: $G = \exp(\int \frac{2}{x} dx) = \exp(\ln x^2) = x^2$

$$xy'+2y=x^3 \quad \Leftrightarrow \quad y'+\frac{2}{x}y=x^2 \quad \Leftrightarrow \quad G(x)y'+G(x)\frac{2}{x}y=x^2G(x) \quad \Leftrightarrow$$

$$(G(x)y)'=x^2G(x) \quad \Leftrightarrow \quad (G(x)y)=\int x^2G(x)dx=\int x^2\exp(\int\frac{2}{x}dx)dx=$$

$$(G(x)y)=\int x^2\exp(2\ln x)dx=\int x^2x^2dx=\tfrac{1}{5}x^5+c$$

$$y=\frac{\frac{1}{5}x^5+c}{x^2} \quad \Leftrightarrow \quad y=\tfrac{1}{5}x^3+\frac{c}{x^2}$$

54. $3^x=7^y=441$ **determine:** $\frac{1}{x}+\frac{1}{y}$

$$3^x=7^y=441 \quad \Leftrightarrow \quad 3^x=7^y=21^2 \quad \Rightarrow$$

$$3=(21)^{\frac{2}{x}} \quad \wedge \quad 7=(21)^{\frac{2}{y}} \quad \Rightarrow \quad 3\cdot 7=21^1=(21)^{\frac{2}{x}}\,(21)^{\frac{2}{y}}=(21)^{\frac{2}{x}+\frac{2}{y}} \quad \Rightarrow \quad \frac{2}{x}+\frac{2}{y}=1 \quad \Rightarrow$$

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$$

55. $5^x=7^y=1225$ **determine:** $\frac{xy}{x+y}$

$$5^x=7^y=1225=35^2 \quad \Leftrightarrow \quad 5=35^{\frac{2}{x}} \quad \wedge \quad 7=35^{\frac{2}{y}}$$

$$5\cdot 7=35=35^{\frac{2}{x}+\frac{2}{y}} \quad \Rightarrow \frac{2}{x}+\frac{2}{y}=1 \quad \Rightarrow \frac{xy}{x+y}=2$$

55a. Solve: $x^{\frac{1}{x}}=\sqrt{2}$

$x^{\frac{1}{x}}=\sqrt{2} \quad \Leftrightarrow \quad x=\sqrt{2}^{\,x}$: The root is seen to be $x = 2$, since $2=\sqrt{2}^{\,2}$

56. Solve: $4^{x+1}+4^{3-x}=257$

$$4^{x+1}+4^{3-x}=257 \quad \Leftrightarrow \quad 4\cdot 4^x+\frac{4^3}{4^x}=257$$

If we put: $y=4^x$ we have

$$4y+\frac{4^3}{y}=257 \quad \Leftrightarrow \quad 4y^2-257y+4^3=0 \quad ; \quad d=257^2-4^5=65025=255^2$$

$$y=4^x=\frac{257\pm 255}{8} \quad \Rightarrow 4^x=\begin{Bmatrix}64\\ \frac{1}{4}\end{Bmatrix} \quad \Leftrightarrow \quad x=\begin{cases}4\\ -1\end{cases}$$

57. Two simple geometric problems

a) In a right angle triangle, the two kathedras are *a* =35 and *b* = 84. Find the height h_c on *c*.

We have $c=\sqrt{a^2+b^2}$ $c=\sqrt{35^2+84^2}=72.95$.

For the area $T=\frac{1}{2}ab=\frac{1}{2}h_c c \quad \Rightarrow \quad h_c=\frac{ab}{c}=40.30$

b) In a right angle triangle, the perimeter $a+b+c=216$. The area *T* of the triangle is 1944.

Determine *a, b, c.*We have the following equations:

$$a+b+c=216 \qquad a^2+b^2=c^2 \qquad T=\frac{1}{2}ab$$

$$(a+b)^2=(216-c)^2 \quad \Leftrightarrow \quad \begin{array}{l} a^2+b^2+2ab=c^2+216^2-2\cdot 216\cdot c \quad \Leftrightarrow \\ 2ab=216^2-2\cdot 216\cdot c \Rightarrow \ c=\dfrac{216^2-4T}{2\cdot 216}=90 \end{array}$$

$a+b=126 \qquad ab=2T=2\cdot 1944$ We put $a+b=p \quad ab=q \quad \Rightarrow$

$$a+\frac{q}{a}=p \quad \Leftrightarrow \quad a^2-pa+q=0 \ \ ; \ \ d=p^2-4q \qquad a=\frac{p\pm\sqrt{d}}{2}=\frac{126\pm\sqrt{8100}}{2}$$

$$a=\frac{126\pm 90}{2}=\begin{cases}108\\ 18\end{cases} \qquad b=216-90-18=108$$

58. Given: $x+\frac{1}{x}=-1$ Determine $x^{2020}+\frac{1}{x^{2020}}$.

$$x+\frac{1}{x}=-1 \quad \Leftrightarrow \quad x^2+x+1=0 \quad \Leftrightarrow \quad x^2=-x-1 \quad \Rightarrow$$

$$x^4=(-x-1)^2 \quad \Leftrightarrow \quad x^4=x^2+1+2x=-x-1+1+2x=x$$

$$x^4=x \quad \Rightarrow \quad x^3=1 \quad \Rightarrow \quad (x^3)^n=x^{3n}=1$$

2020 =673·3+2 $\quad \Rightarrow \quad x^{2020}=x^{3\cdot 673}x^2=x^2$

We have thus: $x^{2020}+\frac{1}{x^{2020}}=x^2+\frac{1}{x^2}=-1$, since

$$x+\frac{1}{x}=-1 \quad \Rightarrow \quad (x+\frac{1}{x})^2=1 \quad \Leftrightarrow \quad x^2+\frac{1}{x^2}+2=1 \quad \Leftrightarrow \quad x^2+\frac{1}{x^2}=-1$$

59. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

We notice that: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$, since $\frac{1}{n} - \frac{1}{(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$, so we have:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1} - \sum_{n=1}^{\infty} \frac{1}{(n+1)} = 1$$

60. Solve: $\frac{dy}{dx} = \frac{x-y}{x+y}$

If you want to solve this equation by separation of variables, you shall need a trick, which is not obvious. We put $y = z(x)x = zx$. Then we find.

$$\frac{dy}{dx} = xz' + z = \frac{x - zx}{x + zx} \quad \Leftrightarrow \quad xz' = -z + \frac{x - zx}{x + zx} = \frac{-z(x + zx) + x - zx}{x + zx} = \frac{x - 2zx - z^2 x}{x + zx} =$$

$$\frac{x(1 - 2z - z^2)}{x(1+z)} \quad \Leftrightarrow \quad xz' = \frac{(1 - 2z - z^2)}{(1+z)} \quad \Leftrightarrow \quad \frac{1+z}{1 - 2z - z^2} dz = \frac{1}{x} dx \quad \Leftrightarrow$$

$$\int \frac{1+z}{1 - 2z - z^2} dz = \int \frac{1}{x} dx \quad \Leftrightarrow \quad -\tfrac{1}{2} \ln(|\Leftrightarrow 1 - 2z - z^2 |) = \ln|x| \quad \Leftrightarrow \quad \frac{1}{\sqrt{|1 - 2z - z^2|}} = |x|$$

$$z = \frac{y}{x} \quad \Rightarrow \quad \frac{1}{\sqrt{|1 - 2\frac{y}{x} - (\frac{y}{x})^2|}} = |x| \quad \Leftrightarrow \quad \frac{1}{1 - 2\frac{y}{x} - (\frac{y}{x})^2} = x^2 \quad \Leftrightarrow \quad x^2 - 2yx - y^2 = 1 \quad \Leftrightarrow$$

$$y^2 + 2yx - x^2 + 1 \;\; ; \; d = 4x^2 + 4(x^2 - 1) = 8x^2 - 4. \quad y = \frac{-2x \pm 2\sqrt{2x^2 - 1}}{2} \quad \Leftrightarrow \quad y = -x + \sqrt{2x^2 - 1}$$

61. Solve: $\sqrt[3]{14 + \sqrt{x}} + \sqrt[3]{14 - \sqrt{x}} = 4$

Traditional method of solving by raising to the third power, is hardly palatable.
Instead we shall make a qualified guess. We notice that: $1^3 = 1$, $2^3 = 8$ and $3^3 = 27$.
We then stipulate that the first term is equal to 3, and the second term is equal to 1, then we have:

$$\sqrt[3]{14 + \sqrt{x}} = 3 \quad \Leftrightarrow \quad 14 + \sqrt{x} = 27 \quad \Leftrightarrow \quad \sqrt{x} = 13$$

$$\sqrt[3]{14 - \sqrt{x}} = 1 \quad \Leftrightarrow \quad 14 - \sqrt{x} = 1 \quad \Leftrightarrow \quad \sqrt{x} = 13$$

So $\sqrt{x} = 13 \quad \Leftrightarrow \quad x = 169$ is the solution

62. Solve: $\sqrt{2x} = \sqrt[3]{x}$

$$\sqrt{2x} = \sqrt[3]{x} \quad \Leftrightarrow \quad (2x)^{\frac{1}{2}} = x^{\frac{1}{3}} \quad \Leftrightarrow \left((2x)^{\frac{1}{2}} \right)^6 = \left(x^{\frac{1}{3}} \right)^6 \quad \Leftrightarrow \quad (2x)^3 = x^2 \quad \Leftrightarrow \quad x = \frac{1}{8}$$

63. Solve: $\sqrt{x-1}=\sqrt[3]{x+3}$

If we should make a qualified guess, it should be: $x = 5$, since $\sqrt{5-1}=\sqrt[3]{5+3} \;\; = 2$

$$\sqrt{x-1}=\sqrt[3]{x+3} \quad \Leftrightarrow \quad (x-1)^{\frac{1}{2}}=(x+3)^{\frac{1}{3}} \quad \Leftrightarrow \quad \left((x-1)^{\frac{1}{2}}\right)^6=\left((x+3)^{\frac{1}{3}}\right)^6 \quad \Leftrightarrow$$

$$(x-1)^3=(x+3)^2 \quad \Leftrightarrow \quad x^3-3x^2+3x-1=x^2+9+6x \quad \Leftrightarrow \quad x^3-4x^2-3x-10=0$$

If an integral number is a solution, it is one of the divisors in 10: $\pm 1, \pm 2, \pm 5, \pm 10$
It seems that x = 5 is a root, so we make division with $x – 5$.

$$x-5 \,|\, x^3-4x^2-3x-10 \,|\, x^2+x+2$$
$$x^3-5x^2$$
$$x^2-3x$$
$$x^2-5x$$
$$2x-10$$
$$2x-10$$

$x^2+x+2=0$ No solution

The solution is: $x = 5$

63a. Solve: $\sqrt{9x-2}-\sqrt{4x-3}=\sqrt{x+1}$

$\sqrt{9x-2}-\sqrt{4x-3}=\sqrt{x+1}$. x must chosen, such that $9x – 2$, $4x – 3$, $x +1$ are quadratic numbers.
It is easily seen that $x = 3$ qualify. Since: $\sqrt{9\cdot 3-2}-\sqrt{4\cdot 3-3}=\sqrt{3+1}$

64. Solve: $(x-1)^3=(x+3)^2$

This is actually exactly the same exercise as 63.
$(x-1)^3=(x+3)^2 \Leftrightarrow \quad x^3-3x^2+3x-1=x^2+9+6x \quad \Leftrightarrow \quad x^3-4x^2-3x-10=0$
If an integral number is a solution, it is one of the divisors in 10: $\pm 1, \pm 2, \pm 5, \pm 10$
It seems that $x = 5$ is a root, so we make division with $x – 5$.

$$x-5 \,|\, x^3-4x^2-3x-10 \,|\, x^2+x+2$$
$$x^3-5x^2$$
$$x^2-3x$$
$$x^2-5x$$
$$2x-10$$
$$2x-10$$

$x^2+x+2=0$ No solution

So the only solution is $x = 5$

65. solve: $(x-1)^5 + (x+3)^5 = 242(x+1)$

This is a trick exercise, since we can see that if $x = 0$, then the second term on the left is 3^5 =243, However the first term is -1, so the solution is $x = 0$: $-1 + 243 = 242(0+1)$
and if x exceed 0, the second term will exceed the right hand side. E.g. If $x = 1$, then we have, rhs = 484, and lhs = 1024, x=2: gives: 3126 = 724.
However, -1 is also a solution since. $(-2)^5 + (2)^5 = 242(-1+1)$

66. In a triangle *B*=30⁰, *C*=45⁰, *a*=8. Find the sides *b* and *c*

It follows that: $A = 180 - B - C = 105$. We write the sine relations:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad \frac{8}{\sin 105} = \frac{b}{\sin 30} = \frac{c}{\sin 45} \Rightarrow$$

$$\sin 30 = \tfrac{1}{2}\text{;} \quad \sin 45 = \frac{\sqrt{2}}{2} \quad ;$$

$$\sin 105 = \sin(45+60) = \sin 45 \cos 60 + \cos 45 \sin 60 = \frac{\sqrt{2}}{2}\frac{1}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4}(1+\sqrt{3})$$

$$b = \frac{8\sin 30}{\sin 105} = \frac{8\frac{1}{2}}{\frac{\sqrt{2}}{4}(1+\sqrt{3})} = \frac{16}{\sqrt{2}(1+\sqrt{3})} \quad ; \; c = \frac{8\sin 45}{\sin 105} = \frac{8\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{4}(1+\sqrt{3})} = \frac{16}{(1+\sqrt{3})}$$

67. Solve: $e^{dy} = x^{dx}$ **???**

$$e^{dy} = x^{dx} \Leftrightarrow \quad \ln e^{dy} = \ln x^{dx} \quad \Leftrightarrow \quad dy = dx \ln x \quad \Leftrightarrow \quad \frac{dy}{dx} = \ln x \quad \Leftrightarrow \quad y = \int \ln x dx \quad \Leftrightarrow$$

$$y = x\ln x - x + c$$

68. Solve $\log(\ln x) = \ln(\log x)$

All logarithmic functions are proportional. We shall therefore express log(x) by ln(x).
By the definition of the logarithm, we have:

$$y = \log x \quad \Leftrightarrow \quad x = 10^y \quad \Rightarrow \quad \ln x = y \ln 10 = \log x \ln 10 \quad \Leftrightarrow \quad \log x = \frac{\ln x}{\ln 10}$$

$$\log(\ln x) = \ln(\log x) \quad \Leftrightarrow \quad \frac{\ln(\ln x)}{\ln 10} = \ln(\frac{\ln x}{\ln 10})$$

We put $y = \ln x \;\; \textit{and} \quad \beta = \frac{1}{\ln 10} \quad \textit{and we then have}:$

$$\beta \ln y = \ln(\beta y) \quad \Leftrightarrow \quad \ln y^{\beta} = \ln(\beta y) \quad \Leftrightarrow \quad y^{\beta} = \beta y$$

$$y^{\beta-1} = \beta \quad \Leftrightarrow \quad (\beta-1)\ln y = \ln\beta \quad \Leftrightarrow \ln y = \frac{\ln\beta}{(\beta-1)} \quad \Leftrightarrow$$

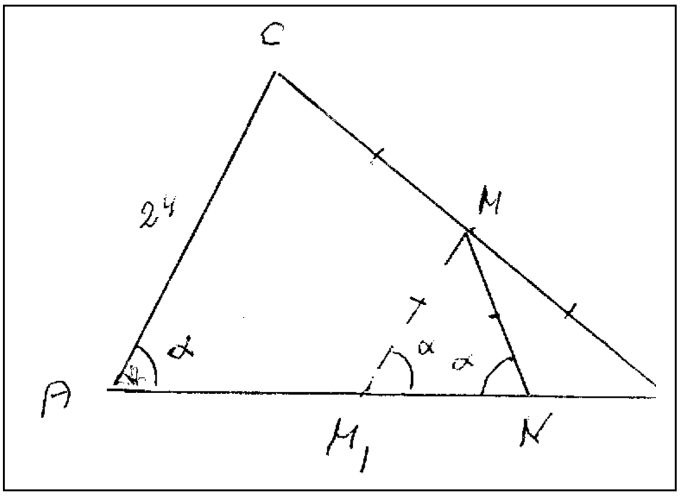
$$\ln(\ln x) = \frac{-\ln(\ln 10)}{(\frac{1}{\ln 10}-1)} \quad \Leftrightarrow \quad x = \exp\left(\exp\left(\frac{\ln(\ln 10)}{(1-\frac{1}{\ln 10})}\right)\right)$$

69. Given that $24^{1-k} = 3$, determine 2^k

$$24^{1-k} = 3 \quad \Leftrightarrow \quad (1-k)\log_2(24) = \log_2 3 \quad \Leftrightarrow \quad (1-k) = \frac{\log_2 3}{3+\log_2 3} \quad \Leftrightarrow \ k = 1 - \frac{\log_2 3}{3+\log_2 3}$$

$$k = \frac{3}{3+\log_2 3} = \frac{3}{\log_2 24}. \quad \Leftrightarrow \quad 2^k = 2^{\frac{3}{\log_2 24}}$$

70. A (simple) geometric problem



The figure shows a triangle *ABC,* where the line *MN* has been drawn, and we have drawn a line MM_1 parallel to *AC*.

We want to find the length of *NM*. *M* is the midpoint of *BC*, so MM_1 is a midpoint transversal. This allows us to write:

$$\frac{MM_1}{24} = \frac{MC}{CB} = \frac{1}{2} \quad \Rightarrow \quad MM_1 = 12$$

On the other hand the angle NM_1M is also α since MM_1 is parallel to *AC* . The two sides MM_1 and *NM* in the triangle MM_1N are equal, so *MN* = 12.

71. Solve for *x*: $\ln(\sqrt{1+x^2}+x) = y$

$$\ln(\sqrt{1+x^2}+x) = y \quad \Leftrightarrow \quad \sqrt{1+x^2}+x = e^y \quad \Leftrightarrow \quad \sqrt{1+x^2} = e^y - x \quad \Leftrightarrow$$

$$1+x^2 = (e^y-x)^2 \quad \Leftrightarrow \quad 1+x^2 = e^{2y}+x^2-2xy \quad \Leftrightarrow \quad 0 = e^{2y}-2xy-1 \quad \Leftrightarrow \quad x = \frac{e^{2y}-1}{2y}$$

72. Factorize: $x^{10}+x^5+1$

$x^{10}+x^5+1$ It is obvious that we try with: (x^5+1) which gives $x^{10}+2x^5+1$, so we must subtract x^5 and then we have $(x^5+1)^2 - x^5 = x^{10}+x^5+1$. We then write x^5 as $(x^{\frac{5}{2}})^2$.

$$x^{10}+x^5+1 = (x^5+1)^2 - (x^{\frac{5}{2}})^2 = (x^5+1-x^{\frac{5}{2}})(x^5+1+x^{\frac{5}{2}})$$

73. Determine *f* from $f(x+\sqrt{x^2+1})=\dfrac{x}{x+1}$

$f(x+\sqrt{x^2+1})=\dfrac{x}{x+1}$ We put $y=x+\sqrt{x^2+1}$ and solve for x.

$(y-x)^2=(\sqrt{x^2+1})^2 \quad \Leftrightarrow \quad y^2+x^2-2xy=x^2+1 \quad \Leftrightarrow \quad x=\dfrac{y^2-1}{2y}$

$$f(y)=\frac{\frac{y^2-1}{2y}}{\frac{y^2-1}{2y}+1}=\frac{\frac{y^2-1}{2y}}{\frac{y^2+2y-1}{2y}}=\frac{y^2-1}{y^2+2y-1}$$

$$f(x)=\frac{\frac{x^2-1}{2x}}{\frac{x^2-1}{2x}+1}=\frac{\frac{x^2-1}{2x}}{\frac{x^2+2x-1}{2x}}=\frac{x^2-1}{x^2+2x-1}$$

73. A strange equation, with no solution $x+x^{1-x}=2^8$

$x+x^{1-x}=2^8 \quad \Leftrightarrow \quad x+\dfrac{1}{x^{x-1}}=2^8 \quad \Leftrightarrow \quad xx^{x-1}+1=2^8x^{x-1} \quad \Leftrightarrow \quad x^x+1=\dfrac{2^8x^x}{x}$

But this equation has no solution, because the 1 on the left hand side.

If we put $\dfrac{2^8}{x}=1$ we get 1=0. So we must have $\dfrac{2^8}{x}>1 \quad \Leftrightarrow \quad x<2^8$ On the other hand, x must be a power of 2. But if we put $x=2^7$ we get: $x^x+1=\dfrac{2^8x^x}{2^7}=2x^x$. So there is no integer solution.

(A numerical solution suggest, tht there might be a solution at 7.1716

74. Determine *f* from the equation: $3f(-x)+f(\frac{1}{x})+f(x)=x$

We proceed as follows. First the equation:

1) $3f(-x)+f(\frac{1}{x})+f(x)=x$ — We replace x by $-x$ to get
2) $3f(x)+f(-\frac{1}{x})+f(-x)=-x$ — Then we replace x by $\frac{1}{x}$ in the original equation.
3) $3f(-\frac{1}{x})+f(x)+f(\frac{1}{x})=\frac{1}{x}$ — Finally we replace x by $-\frac{1}{x}$.
4) $3f(\frac{1}{x})+f(-x)+f(-\frac{1}{x})=-\frac{1}{x}$

However to simplify the calculations, we shall make the following assignments

$a=f(x);\ b=f(-x);\ c=f(\frac{1}{x});\ d=f(-\frac{1}{x})$

And we get the following set of equations:

1) $3b + c + a = x$

2) $3a + d + b = -x$

3) $3d + a + c = \frac{1}{x}$

4) $3c + b + d = -\frac{1}{x}$

In principle we have 4 equations that can be solved, but we are in a situation, where we cannot move variables from one side to another, so we use some tricks.
First we add all the equations:

$$5a + 5b + 5c + 5d = x - x + \frac{1}{x} - \frac{1}{x} = 0 \quad so$$
$$a + b + c + d = 0$$

Then we add (1) + (2):
$4a + 4b + c + d = x - x = 0$
which we write $3(a+b) + a + b + c + d = 3(a+b) + 0 = 0 \Rightarrow a + b = 0$ and then $c + d = 0$
$a = -b \quad c = -d$.
We have from (1): $3b + c + a = x \Rightarrow 2b + c = x$
And from (4)
$3c + b + d = -\frac{1}{x} \Rightarrow 2c + b = -\frac{1}{x}$
these equations are solved:

$$2b + c = x \qquad 4b + 2c = 2x$$
$$2c + b = -\frac{1}{x} \qquad 2c + b = -\frac{1}{x}$$
$$3b = 2x + \frac{1}{x} \Rightarrow b = \frac{2x + \frac{1}{x}}{3} = \frac{2x^2 + 1}{3x}$$
$$f(-x) = \frac{2x^2 + 1}{3x} \Rightarrow f(x) = -\frac{2x^2 + 1}{3x}$$

75. Solve $x \ln x = e$

$$x \ln x = e \iff \ln x^x = e \iff x^x = e^e \iff x = e$$

76. Given: $x\sqrt{x} - 11\sqrt{x} = 10$. Find an expression for $x - \sqrt{x}$

$x\sqrt{x} - 11\sqrt{x} = 10$. We add and subtract x and then add and subtract $10x$

$$x\sqrt{x} - x + x - 11\sqrt{x} = 10 \iff \sqrt{x}(x - \sqrt{x}) + x + 10x - 11\sqrt{x} - 10x = 10 \iff$$
$$\sqrt{x}(x - \sqrt{x}) + 11(x - \sqrt{x}) - 10x = 10$$

We add and subtract $10\sqrt{x}$

$\sqrt{x}(x-\sqrt{x})+11(x-\sqrt{x})-10x+10\sqrt{x}-10\sqrt{x}=10 \quad \Leftrightarrow$

$\sqrt{x}(x-\sqrt{x})+11(x-\sqrt{x})-10(x-\sqrt{x})-10(\sqrt{x}+1)=0 \quad \Leftrightarrow$

$\sqrt{x}(x-\sqrt{x})+(x-\sqrt{x})-10(\sqrt{x}+1)=0 \quad \Leftrightarrow$

$(x-\sqrt{x})(\sqrt{x}+1)-10(\sqrt{x}+1)=0 \quad \Leftrightarrow$

$(\sqrt{x}+1)(x-\sqrt{x}-10)=0 \quad \Leftrightarrow \quad x-\sqrt{x}=10$

77. Determine *f* from $f(x)^2-f(-x)^2=4x$

$f(x)^2-f(-x)^2=4x$. It does lead nowhere to try to make a substitution for *x*, Instead we recall that $(x\pm1)^2=x^2+1\pm2x$ so: $f(x)^2-f(-x)^2=(x\pm1)^2=x^2+1+2x-(x^2+1-2x)=4x$

78. Simplify $\sqrt[3]{7+\sqrt{50}}+\sqrt[3]{7-\sqrt{50}}=a+b$**.**

We shall try to write to write y^3. $\sqrt[3]{7+\sqrt{50}}+\sqrt[3]{7-\sqrt{50}}=y^3$

First we notice that $\sqrt[3]{7+\sqrt{50}}\cdot\sqrt[3]{7-\sqrt{50}}=a\cdot b=\sqrt[3]{(7+\sqrt{50})\cdot(7+\sqrt{50})}=\sqrt[3]{49-50}=\sqrt[3]{-1}=-1$

We then calculate:

79. Simplify: $\sqrt{42022^2-42022-42021}$

$\sqrt{42022^2-42022-42021}=\sqrt{42022^2-42022-42021-1+1}=$

$\sqrt{42022^2-42022-42022+1}=\sqrt{42022^2-2\cdot42022+1}$

We put $x=42022$:

$\sqrt{x^2-2x+1}=\sqrt{(x-1)^2}=x-1,\ so$

$\sqrt{42022^2-42022-42021}=42022-1$

80. Simplify $a=45678^3-45676^3$ **Find an expression for** $\dfrac{a-2}{6}$

We put $x=45676$ So

$$(x+2)^3-x^3=x^3+6x^2+12x+8-x^3=6x^2+12x+8)=$$

$a=45678^3-45676^3$ $6(x^2+2x+1)+2=6(x+2)^2+2=$

$$\sqrt{6}^2(x+2)^2+2$$

$a=6(x+2)^2+2$

$\dfrac{a-2}{6}=(x+2)^2 \quad \Leftrightarrow \quad \pm\sqrt{\dfrac{a-2}{6}}=x+2$

81. Given $x+y=1.$ **and** $x^2+y^2=2$**, determine:** $x^{11}+y^{11}$

$(x+y)^2=1=x^2+y^2+2xy \Leftrightarrow \quad 1=2+2xy \quad \Rightarrow \quad 2xy=-1$

$(x^2+y^2)^2 = 4 = x^4+y^4+2x^2y^2 \Leftrightarrow \quad x^4+y^4+1=4 \quad \Rightarrow \quad x^4+y^4=3$

$(x^4+y^4)^2 = 9 = x^8+y^8+2x^4y^4 \Leftrightarrow 9 = x^8+y^8+1 \quad \Leftrightarrow \quad x^8+y^8=8$

$(x+y)^3 = (x+y)(x+y)^2 = (x^3+3x^2y+3xy^2+y^3) = x^3+2x^2y+2xy^2+x^2y+xy^2+y^3 =$

$x^3+3xy(x+y)+y^3 = x^3+\frac{3}{2}2xy(x+y)+y^3 \Leftrightarrow x^3+\frac{3}{2}(-1)+y^3=1 \quad \Leftrightarrow \quad x^3+y^3=\frac{5}{2}$

$(x^3+y^3)(x^2+y^2) = \frac{5}{2}2 = 5 = x^5+y^5+(x^3y^2+y^3x^2) = x^5+y^5+x^2y^2(x+y) = x^5+y^5+\frac{1}{2} \quad \Leftrightarrow$

$x^5+y^5=\frac{9}{2}$

$(x^8+y^8)(x^3+y^3) = 8\frac{5}{2} = 20 = x^{11}+y^{11}+x^8y^3+x^3y^8 = x^{11}+y^{11}+x^3y^3(x^5+y^5) \quad \Leftrightarrow$

$20 = x^{11}+y^{11}-\frac{1}{2}(\frac{9}{2}) \quad \Leftrightarrow \quad x^{11}+y^{11}=24\tfrac{1}{4}$

82. given $5x+\frac{1}{7x}=15 \qquad 49x^2+\frac{1}{25x^2}=?$

This is actually very simple. First we multiply the equation with $\frac{7}{5}$

$\frac{7}{5}(5x+\frac{1}{7x}=15) \quad \Rightarrow \quad 7x+\frac{1}{5x}=21$

Then we take the square of the last equation

$49x^2+\frac{1}{25x^2}+2(7x\cdot\frac{1}{5x})=441$

$49x^2+\frac{1}{25x^2}=441-\frac{14}{5}$

83. Given $x^2-y^2=9 \; and \quad xy=3$**. Determine *x+y***

$x^2-y^2=9 \; and \quad xy=3 \quad \Rightarrow$

$(x^2-y^2)^2 = 81 = x^4+y^4-2x^2y^2 = x^4+y^4-18 \quad \Rightarrow \; x^4+y^4 = 81+18 = 99$

$(x^2+y^2)^2 = x^4+y^4+2x^2y^2 = x^4+y^4+18 = 99+18 = 117 \quad \Rightarrow \quad x^2+y^2=\sqrt{117}$

$(x+y)^2 = x^2+y^2+2xy = \sqrt{117}+6$

$x+y=\sqrt{\sqrt{117}+6}$

84. Determine $(45678)^3-(45676)^3$

$(45678)^3-(45676)^3$ We put $a=45677$, and then it can be written:

$(a+1)^3-(a-1)^3 = a^3+1+3a^2+3a-(a^3-1-3a^2+3a) = 6a^2+2$

85. Solve: $\sqrt{x+1}+\sqrt{4x-3}=\sqrt{9x-2}$

$\sqrt{x+1}+\sqrt{4x-3}=\sqrt{9x-2}$

First we notice that if x =3 then

$9\cdot 3-2=27-2=25, \quad \Rightarrow \quad \sqrt{9x-2}=5$

In a similar way, we have:

$\sqrt{x+1}=\sqrt{3+1}=\sqrt{4}=2$

$\sqrt{4x-3}=\sqrt{4\cdot 3-3}=\sqrt{9}=3$

And subsequently:

$\sqrt{x+1}+\sqrt{4x-3}=\sqrt{9x-2}$ =2+3=5

So $x=3$ *is the* solution

83. Solve: $4\cdot 4^{x}+4^{-x}=257$

We put $y=4^{x}$ and get:

$$4y+\frac{4^{3}}{y}-257=0 \;\Leftrightarrow$$

$$4y^{2}-257y+4^{3}=0 \qquad d=257^{2}-4\cdot 4\cdot 4^{3}=255^{2}$$

$$y=\frac{257\pm 255}{8} \quad \Rightarrow$$

$$4y^{2}-257y+4^{3}=0 \qquad d=257^{2}-4\cdot 4\cdot 4^{3}=255^{2}$$

$$y=\frac{257\pm 255}{8} \quad \Leftrightarrow \quad y=\frac{2}{8} \;\vee\; y=\frac{512}{8} \;\Leftrightarrow\; y=\frac{1}{4} \;\Leftrightarrow\; \vee \quad y=64$$

$$4^{x}=4^{-1} \;\vee\; 4^{x}=4^{3}$$

$$x=-1 \;\vee\; x=3$$

84 Determine the integral numbers solutions to $x^{2}+y^{2}=1997(x-y)$

$$x^{2}+y^{2}=1997(x-y) \quad \Leftrightarrow$$

$$2(x^{2}+y^{2})=2\cdot 1997(x-y)$$

$$x^{2}+y^{2}+x^{2}+y^{2}-2\cdot 1997(x-y)$$

$$(x+y)^{2}+(x-y-1997)^{2}=1997^{2}$$

Now it is a well known fact, that the following expressions are Pythagoras numbers:

$a=2mn, \quad b=n^{2}-m^{2}, \quad c=n^{2}+m^{2}$, Since then: $a^{2}+b^{2}=c^{2}$

$$(2mn)^{2}+(n^{2}-m^{2})^{2}= \; 4m^{2}n^{2}+n^{4}+m^{4}-2m^{2}n^{2}=(n^{2}+m^{2})^{2}$$

If we compare with our expression, we can see that $n^{2}+m^{2}$ =1997.

Here we have to guess: If $m = n$ we find $n = \sqrt{\frac{1997}{2}} \cong 31.6$, so we make the assumption that m and n, are in the vicinity of 31.
If we plot a circle with radius $\sqrt{1997}$ the only point with integral coordinates is (34,29)
The solution is therefore, n =29 and m =34, since $34^2 + 29^2 = 1156 + 841 = 1997$.

$2mn = x + y$ $n^2 - m^2 = x - y - 1997$, $n^2 + m^2 = 1997$. Solving:

$2mn = x + y$ and $n^2 - m^2 = x - y - 1997$ gives:

$n^2 - m^2 + 2mn + 1997 = 2x$ and $2mn - n^2 + m^{2-}1997 = 2y$

Which gives: $x = 1827$ and $y = 145$

85. *f* is given by the expression: $f(x+\frac{1}{x}) = x^3 + \frac{1}{x^3}$. Determine *f*(4).

We can see that if :

$x + \frac{1}{x} = 4 \Leftrightarrow x^2 - 4x + 1 = 0 \Leftrightarrow x = \frac{4 - \sqrt{16-4}}{2}$ $x = -2 - \sqrt{3} \vee x = -2 + \sqrt{3}$

$f(4) = (-2+\sqrt{3})^3 + \frac{1}{(-2+\sqrt{3})^3} \vee f(4) = (-2-\sqrt{3})^3 + \frac{1}{(-2-\sqrt{3})^3}$

But as we can see: these two numbers are equal, only if $x^3 + \frac{1}{x^3} = 0 \Leftrightarrow \frac{x^6+1}{x^3} = 0$

So the first expression is valid only for $x>0$ and the other only for $x<0$

86. Find all the solutions to $x^4 + 4x - 1 = 0$

$x^4 + 4x - 1 = 0 \Leftrightarrow$

$(x^2-1)^2 - 1 + 2x^2 + 4x - 1 = 0$

$(x^2-1)^2 + 2x^2 + 4x - 2 = 0$

$(x^2-1)^2 + 2(x^2 + 2x - 1)$

$x^2 + 2x - 1 = 0$ $d = 4 + 4 = 8$

$x = -2 \pm \sqrt{2}$

$(x^2-1)^2 + 2(x + 2 + \sqrt{2})(x + 2 - \sqrt{2}) = 0$

We can see that this line of path does not lead to the solution

I have then tried to transform the left side of the expression into a product of two second degree polynomials.

$x^4 + 4x - 1 = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (a+c)x^3 + (ac + d + b)x^2 + (ad + bc)x + bd$

And we then find by comparison.

$a+c=0;\quad ac+d+b=0;\quad ad+bc=4;\quad bd=-1$

$c=-a;\quad -a^2+d+b=0;\quad ad-ba=4;\quad bd=-1$

$-a^2+d+b=0;\quad a(d-b)=4;\quad bd=-1$

$b+d=a^2\quad d-b=\frac{4}{a}\quad 2d=a^2+\frac{4}{a}\quad 2b=a^2-\frac{4}{a}$

$a^4-\dfrac{16}{a^2}+4=0\quad$ *we put* $e=a^2$ *and find*

$e^3+4e-16=0$; $e=2$ *is a root*

$e^3+4e+16=(e-2)(e^2+2e+4)$

$(e^2+2e+4)=0\quad d=4-16<0$

The only solution is for $e = 2$.

$e=2 \;\Rightarrow\; a=\sqrt{2} \;\Rightarrow\; c=-\sqrt{2} \;\Rightarrow$

$2b=a^2-\frac{4}{a} \;\Rightarrow\; b=1-\frac{2}{\sqrt{2}}=1-\sqrt{2}$

$2d=a^2+\frac{4}{a}\Rightarrow\; d=1+\frac{2}{\sqrt{2}}=1+\sqrt{2}$

$x^4+4x-1=0=(x^2+\sqrt{2}x+(1-\sqrt{2}))(x^2-\sqrt{2}x+(1+\sqrt{2}))$

The second factor has no roots since $d=2-4(1+\sqrt{2})=-2-4\sqrt{2}$

The discriminator $d=2-4(1-\sqrt{2})=4\sqrt{2}-2$ for the first factor is, however positive:

$$x=\frac{-\sqrt{2}\pm\sqrt{4\sqrt{2}-2}}{2}$$

is the only solutions to $x^4+4x-1=0$

87. Solve $\sqrt{x-2}+3=\sqrt{4x+1}$

It is obvious to guess a *x,* which make the square roots an integer. So we guess $x = 6$, which sees is a solution: since: $\sqrt{x-2}+3=\sqrt{4x+1}$ becomes:

$\sqrt{6-2}+3=\sqrt{4\cdot 6+1} \;\Leftrightarrow\; \sqrt{4}+3=\sqrt{25} \;\Leftrightarrow\; 5=5$

88. Simplify $\sqrt{53-10\sqrt{6}}$ **.**

Since $\sqrt{6}=\sqrt{3}\sqrt{2}$ We shall try to write $\sqrt{53-10\sqrt{6}}$ as $a\sqrt{2}-b\sqrt{3}$

$\left(a\sqrt{2}-b\sqrt{3}\right)^2=2a^2+3b^2-2ab\sqrt{6}$.

We find $2ab=10$ and $2a^2+3b^2=53$

It easily seen, that this is fulfilled if $a=5 \wedge b=1$, since $2\cdot 25+2\cdot 1=53$.

But it may also be derived from; $b=\dfrac{5}{a} \;\Rightarrow\; 2a^2+3b^2=2a^2+3\dfrac{25}{a^2}=53$

Thus we have: $\left(5\sqrt{2}-\sqrt{3}\right)^2 = 53-10\sqrt{6}$ and $\sqrt{53-10\sqrt{6}} = 5\sqrt{2}-\sqrt{3}$

89. Determine *f* from $f(x+\frac{1}{x}) = x-\frac{1}{x}$

$f(x+\frac{1}{x}) = x-\frac{1}{x}$. Replacing x by $\frac{1}{x}$ gives the same argument to f, but the negative result.

But this is only possible if $x-\frac{1}{x}=0 \iff x=\pm 1$. So we have only two values

$f(2)=0$ *and* $f(-2)=0$

We shall write this as: $f(y) = x-\frac{1}{x}$ and we notice that $yf(y) = x^2-\frac{1}{x^2}$

$y = x+\frac{1}{x} \Rightarrow x^2-xy+1=0$; $d = y^2-4 \Rightarrow x = \frac{y\pm\sqrt{y^2-4}}{2}$

$f(y) = x-\frac{1}{x} = \frac{x^2-1}{x} = \frac{x^2-1}{x} = \frac{xy-1-1}{x} = y-\frac{2}{x} = y-\frac{4}{y\pm\sqrt{y^2-4}}$

So (a bit distressing) $f(x) = x-\frac{4}{x\pm\sqrt{x^2-4}}$

90. Given $x^{(x-1)^2} = 2x+1$ Determine an expression for $x-\frac{1}{x}$

$x^{(x-1)^2} = 2x+1$.We start by rewriting $(x-1)^2 = x^2+1-2x = x^2+2-(2x-1)$, so we have:

$x^{x^2+2-(2x-1)} = 2x+1 \iff x^{x^2+2}x^{-(2x-1)} = 2x+1 \iff x^2x^{x^2} = (2x+1)\,x^{(2x-1)}$

We shall then apply the rule: $ax^a = bx^b \iff a=b$ to get: $x^2 = 2x+1$

$x^2 = 2x+1 \iff x-\frac{1}{x} = 2$

91. In a right angled triangle: *a* + *b*+ *c* =216 and *T* = 1944. Find the sides.

Given $a + b + c = 216$ and $T = 1944$.

From elementary geometry we have: $T = \frac{1}{2}ab$ *and* $c^2 = a^2+b^2$.

$(a+b)^2 = (216-c)^2 \iff a^2+b^2+2ab = 216^2+c^2+2\cdot 216c \iff$

$4\cdot 1944 = 216^2+2\cdot 216c \iff c = 90$

$a+b = 216-c = 126.\quad ab = 2T = 3888 \Rightarrow$

$b = 126-a; \Rightarrow ab = a(126-a) = 3888$

$-a^2+126a-3888 = 0 \quad d = 126^2-4\cdot 3888 = 324 = 18^2$

$a = \frac{-126\pm 18}{-2} = \begin{cases}72\\54\end{cases}$

Since there is complete symmetry between a and b , b will obtain the same values as

92. Given $a^2 - b^2 = 9$ and $ab = 3$. Determine $a + b$

Given: $a^2 - b^2 = 9$ and $ab = 3$. Determine $a + b$. We have

$$a^2 - b^2 = (a-b)(a+b)$$
$$(a+b)^2 = a^2 + b^2 + 2ab$$
$$(a-b)^2 = a^2 + b^2 - 2ab$$

And thus

$(a+b)^2 - (a-b)^2 = a^2 + b^2 + 2ab - (a^2 + b^2 - 2ab) = 4ab = 12$ So

$$(a+b)^2 - (a-b)^2 = 12 \quad and \quad (a-b)(a+b) = 9$$

We put $a + b = x \quad and \quad a - b = y$ and we then have the two equations:

$$x^2 - y^2 = 12 \quad and \quad xy = 9 \quad \Rightarrow \quad x^2 - \left(\frac{9}{x}\right)^2 = 12 \quad \Leftrightarrow \quad x^4 - 12x^2 - 81 = 0$$

$$d = 144 + 4 \cdot 81 = 468; \quad x^2 = \frac{12 \pm \sqrt{468}}{2} \quad \Leftrightarrow \quad x^2 = 6 \pm \sqrt{117}$$

If we discard the negative root, we find: $a + b = \sqrt{6 + \sqrt{117}}$
(One should notice that the result differs from that is done on the YouTube homepage)

93. A very simple quadratic equation $\sqrt{x-2} + 3 = \sqrt{4x+1}$

It is the easiest to guess x, so that $x - 2$ and $4x+1$ become quadratic numbers.
Firstly $x - 2 = 1$ so $x = 3$, and that gives $4x+1 = 19$, and 2 +3 =5, so $x = 3$ is a not solution.
Next $x - 2 = 4$ so $x = 6$, and that gives $4x+1 = 25$, and 2 +3 =5, so $x = 6$ is a solution.
Next $x - 2 = 9$ gives $x = 11$ and $4x+1 = 45$, not a solution.
Next $x - 2 = 16$ gives $x = 18$ and $4x+1 = 73$, not a solution.

$x = 6$ is the only solution

94. A very simple cubic equation $\sqrt[3]{2-x} + \sqrt{x-1} = 1$

It is the easiest to guess x, so that 2 - x becomes a cubic number.
$2 - x = 1$ gives $x = 1$ and $x + 1 = 0$ so that $\sqrt[3]{2-x} + \sqrt{x-1} = 1 \Leftrightarrow 1 - 0 = 1$
$2 - x = -8 \Rightarrow x = 10 \Rightarrow x - 1 = 9$, so that: $\sqrt[3]{2-x} + \sqrt{x-1} = -2 + 3 = 1$
$2 - x = -27 = (-3)^3 \Rightarrow x = 29 \Rightarrow x - 1 = 28$, so that: $\sqrt[3]{2-x} + \sqrt{x-1} \neq 1$
$2 - x = -64 = (-4)^3 \Rightarrow x = 66 \Rightarrow x - 1 = 65$, no solution.
The solutions are therefore: $x = 1 \ or \ x = 10$.

95. A very simple quadratic equation $\sqrt{x+5} + \sqrt{20-x} = 7$

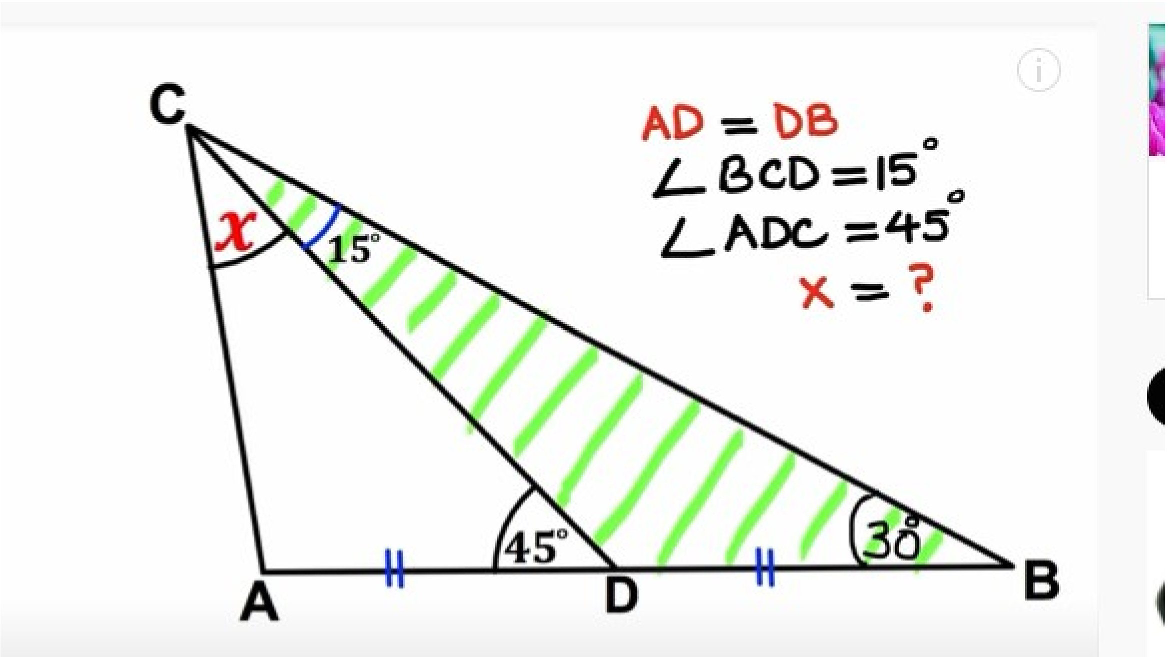
It is the easiest to guess x, so that x +5 and 20 - x become quadratic numbers.

Firstly: $x + 5 = 1$ so $x = -4$, and that gives $20 - x = 24$, and $\sqrt{x+5} + \sqrt{20-x} = 1 + \sqrt{24}$, So: x = -4 is a not solution.

Next x +5 = 4 so x = -1, and that gives 20 – x =21, so x = -1 is not a solution.

Next x +5 = 9 so x = 4, and that gives 20 – x =16, and $\sqrt{x+5}+\sqrt{20-x}=3+4=7$.
So x = 4 is a solution.

Next x +5 = 25 gives x = 20 and x – 1=0 , x = 20 not a solution.
Higher values for x gives a negative argument for the second square root, so the solutions are:



We begin by writing the sine relations for the triangles ABC, BCD, ACD .
From the figure it is clear: $A=180-45-x$; $\angle CDB=180-45=135$; $\angle B=180-135-15=30$

$$Triangle\ ABC: \frac{a}{\sin A}=\frac{c}{\sin(x+15)}=\frac{b}{\sin 30}$$

$$Triangle\ ADC: \frac{\frac{1}{2}c}{\sin x}=\frac{b}{\sin 45}=\frac{|CD|}{\sin A}$$

$$Triangle\ BDC: \frac{a}{\sin 135}=\frac{\frac{1}{2}c}{\sin 15}=\frac{|CD|}{\sin 30}$$

We divide the first sine relation by the third:

$$\frac{\frac{a}{\sin A}}{\frac{a}{\sin 135}}=\frac{\frac{c}{\sin(x+15)}}{\frac{\frac{1}{2}c}{\sin 15}} \Leftrightarrow \frac{\sin 135}{\sin A}=\frac{\sin 15}{\sin(x+15)} \Leftrightarrow \frac{\sin 135}{\sin(x+45)}=\frac{\sin 15}{\sin(x+15)} \Leftrightarrow$$

This may be solved by applying the the addition formulas for sine. However, it is somewhat easier to use the second and the third:

$$\frac{\frac{|CD|}{\sin A}}{\frac{|CD|}{\sin 30}} = \frac{\frac{\frac{1}{2}c}{\sin x}}{\frac{\frac{1}{2}c}{\sin 15}} \Leftrightarrow \frac{\sin 30}{\sin A} = \frac{\sin 15}{\sin x} \Rightarrow$$

$\sin A \sin 15 = \sin 30 \sin x \Leftrightarrow \quad \sin(45 + x)\sin 15 = \sin 30 \sin x$

$\sin(45 + x) = \sin 45 \cos x + \cos 45 \sin x$

$(\sin 45 \cos x + \cos 45 \sin x)\sin 15 = 2\sin 15 \cos 15 \sin x$

Division med $\sin 15 \cos x.$

$\sin 45 + \cos 45 \tan x = 2\cos 15 \tan x \quad \Leftrightarrow (2\cos 15 - \cos 45)\tan x = \sin 45$

$$\tan x = \frac{\sin 45}{(2\cos 15 - \cos 45)} = \frac{1}{3} \quad \Rightarrow \quad x = 18.43$$

96. Do the integral: $\int x^5 \sqrt{x^3 + 1} dx$

$\int x^5 \sqrt{x^3 + 1} dx$. We apply partial integration.

$$\int x^5 \tfrac{2}{3} \frac{1}{3x^2} d(x^3 + 1)^{\frac{3}{2}} = \tfrac{2}{9} \int x^3 d(x^3 + 1)^{\frac{3}{2}} =$$

$$\tfrac{2}{9}((x^3(x^3 + 1)^{\frac{3}{2}} - \tfrac{2}{5}(x^3 + 1)^{\frac{5}{2}}) = \tfrac{2}{9}(x^3 + 1)^{\frac{3}{2}}(x^3 - \tfrac{2}{5}(x^3 + 1) = \tfrac{2}{9}(x^3 + 1)^{\frac{3}{2}}(\tfrac{3}{5}x^3 - \tfrac{2}{5})$$

97. Some trivial equations having square and cubic roots.

$\sqrt{x} + y = 7 \qquad a - b = 11$

$x + \sqrt{y} = 11 \qquad \sqrt{a} + \sqrt{b} = 11$

In the first case, we should think of two Integers , which have the sums above only takes very little time to guess at x =9 and y = 4,since: $\sqrt{9} + 4 = 7$ and $9 + \sqrt{4} = 11$.
The second case is almost as trivial. Find two quadratic numbers having the sum 11. We see that 81 and 4, qualify the second equation, but not the first. But 25 and 26 does, since
$\sqrt{25} + \sqrt{36} = 11$ *and* $36 - 25 = 11$

98. Solve: $3^x + 9^x = 27^x$

$3^x + 9^x = 27^x \quad \Leftrightarrow \quad 3^x + (3^2)^x = (3^3)^x \quad \Leftrightarrow \quad 3^x + (3^x)^2 = (3^x)^3$

We put $y = 3^x$ *and get* $y + y^2 = y^3$ and divide by y to give: $y^2 - y - 1 = 0; \quad d = 1 + 4 = 5$

$$y = \frac{1 \pm \sqrt{5}}{2} \quad \Leftrightarrow \quad 3^x = \frac{1 + \sqrt{5}}{2} \quad \Leftrightarrow \quad x = \frac{\ln\left(\frac{1 + \sqrt{5}}{2}\right)}{3}$$

99. Compute $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n-1}{n!}=?$ **(Tricky)**

We make the rewriting:

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n-1}{n!}= \\ \frac{2-1}{2!}+\frac{3-1}{3!}+\frac{4-1}{4!}+...+\frac{n-1}{n!} =$$

$$\frac{2}{2!}-\frac{1}{2!}+\frac{3}{3!}-\frac{1}{3!}+\frac{4}{4!}-\frac{1}{4!}+...+\frac{n}{n!}-\frac{1}{n!}+...=$$

$$\frac{2}{2!}-\frac{1}{2!}+\frac{3}{3!}-\frac{1}{3!}+\frac{4}{4!}-\frac{1}{4!}+...+\frac{n}{n!}-\frac{1}{n!}+...=$$

$$\frac{2}{2!}+\frac{3}{3!}+\frac{4}{4!}+...+\frac{n}{n!}+...-\left(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{n!}+...'+...\right)$$

$$\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{(n-1)!}+...-\left(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{n!}+...'+...\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n!}-\sum_{n=2}^{\infty}\frac{1}{n!}-=e^1-(e^1-1)=1$$

100. Solve: $y'=\sqrt{x+y}$

$y'=\sqrt{x+y}$. We make the substitution $z=x+y$ and find:

$$y'=\sqrt{z} \quad \Rightarrow \quad dy=\sqrt{z}\,dx \quad dz=dy+dx \quad \Rightarrow \quad dz=\sqrt{z}\,dx+dx=(\sqrt{z}+1)dx$$

$$dz=(\sqrt{z}+1)dx \quad \Rightarrow \frac{dz}{(\sqrt{z}+1)}=dx \quad \Leftrightarrow$$

$$\int\frac{dz}{(\sqrt{z}+1)}=dx \quad \Leftrightarrow \quad \textit{we make the substitition } u=\sqrt{z}+1 \quad \Rightarrow \quad du=\frac{1}{2\sqrt{z}}dz$$ I

$$\int\frac{dz}{\sqrt{z}+1}=dx=\int\frac{u-1}{u}du=\int dx \quad \Leftrightarrow \quad u-\ln u \; = \; x+c \quad \Leftrightarrow$$

$$\sqrt{z}+1-\ln(\sqrt{z}+1)=x \quad \Leftrightarrow \quad \sqrt{x+y}+1-\ln(\sqrt{x+y}+1)=x+c$$

It is, however, not possible to isolate *y*.

101. Solve $x^2+2x+2=\frac{2x^2}{x^2+x+2}$

$x^2+2x+2=\frac{2x^2}{x^2+x+2} \Leftrightarrow (x^2+2x+2)(x^2+x+2)=2x^2 \Leftrightarrow$

$x^4+x^3+2x^2+2x^3+2x^2+4x+2x^2+2x+4=2x^2 \Leftrightarrow$

$x^3(x+1)+2x^2(x+1)+2x(x+1)+4(x+1)=0 \Leftrightarrow$

$(x+1)(x^3+2x^2+2x+4)=0 \Leftrightarrow$

$(x+1)(x^2(x+2)+2(x+2))=(x+1)(x+2)(x^2+2)=0 \Leftrightarrow x=-1 \vee x=-2$

102. Solve the differential equation: $\frac{dy}{dx}=\frac{y}{x}+2\sqrt{\frac{y}{x}}$

$\frac{dy}{dx}=\frac{y}{x}+2\sqrt{\frac{y}{x}}$

We put $u=\frac{y}{x} \Rightarrow dy=udx+xdu$

$dy=udx+xdu=(u+2\sqrt{u})dx \Leftrightarrow$

$xdu=(u+2\sqrt{u}-u)dx \Leftrightarrow$

$xdu=2\sqrt{u}dx \Leftrightarrow$

$\frac{du}{2\sqrt{u}}=\frac{dx}{x} \Leftrightarrow$

$\sqrt{u}=\ln x \Leftrightarrow u=(\ln x)^2 \quad \frac{y}{x}=(\ln x)^2 \Leftrightarrow$

$y=x(\ln x)^2$