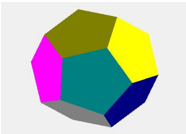
Quaternions

A mathematical monster

This is an article from my home page: www.olewitthansen.dk



Ole Witt-Hansen 1981 (2018)

Contents

1. Introduction

We shall in the following use the standard notation for logical operators.

$a \in L$, $a \notin L$: reads : a belongs to the set L, (a does not belong to the set L)
"$\Rightarrow$" reads : if...then or implies.
"$\Leftrightarrow$" reads : if and only if, or logical equivalent.
"$\wedge$" reads : (logical) and
"$\vee$" reads : (logical) or
$\forall x$: ....reads : for all (for every) x implies .....
$\exists x$: ....reads : There exists a x for which....applies

1.1 Composition

The structure of the numbers is characterized by giving two arbitrary, e.g. rational numbers one may form new numbers, written as $a + b$ or $a \cdot b$, which are also rational numbers.
We express this, as *addition* and *multiplication* are *compositions* within the set of rational numbers.

Given a set M ,which may be (almost) anything, one may define a *composition* *"*"* (star), within the set *M,* which is a mapping (a function) from M x M into M.
If *a, b, c* belongs to *M,* then instead of writing $f(a,b) = c$, we shall write:

(1.1) $$a * b = c$$

A composition "*" is thus a calculation rule, which to two numbers $a \in M, b \in M$ ascribe exactly one element $c \in M$, where $c = a*b$

1.2 Algebraic body

An algebraic body L, (which we shall refer to as a *body*), is a group with two compositions, which we shall refer to as addition (+) and multiplication (·) that fulfils the following conditions:

1. L is a commutative group with respect to addition
2. $L\backslash\{0\}$ is a commutative group with respect to multiplication
3. The distributive law applies.

$$\forall a,b,c \in L : a(b+c) = ab + ac$$

1.3 Division algebra over the body of real numbers

A division algebra over the real numbers is an extension of the real numbers to numbers represented by higher dimension: (a,b), (a,b,c), (a,b,c,d), etc.
This may also be represented as a vector space with n dimensions over the real numbers $(V, +, R)$, Assuming that there is also defined a multiplication composition $\times$ in V, then the vector space $(V, +, \times, R)$ form a division algebra.

For simplicity the vectors are written with ordinary low letters *u,v,w*, from the end of the alphabet and real numbers with Greek letters or low letters from the top of the alphabet.
The vector space organized in this way forms an algebra, if it complies with the following conditions:

$$\forall\, u,v,w \in V : u\times(v+w) = u\times v + u\times w$$
$$\forall\, u,v,w \in V : (v+w)\times w = u\times w + v\times w$$
$$\forall\, u,v,w \in V : (\alpha u)\times w = (\alpha u)\times w = u\times(\alpha v) = \alpha(u\times v)$$

Specifically if the algebra forms a body, it is called a *division algebra*.

The two dimensional division algebra is now known as the complex number system, where Cardano in 1545, first published an article on the subject.

We now intend to prove, that apart from the real numbers, there exists only two division algebras over the real numbers, namely a two dimensional (the complex number system) and a four dimensional, called the *quaternion* number system.

The proof of this assertion is due W.R. Hamilton, who first published a proof in 1843.
The proof is completed in several steps. We assume that $(V,+,\cdot,R)$ is a division algebra of dimension n. We shall in the following assume that $n > 1$.

Let x be an arbitrary element in V. The $n + 1$ elements: $1, x, x^2, \ldots x^n$ must be linear dependent, since the vector space has only n dimensions. So there exists $n + 1$ real numbers $(\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_n)$, which are not all zero such that:

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n = 0$$

According to the fundamental theorem of algebra, any polynomial having real coefficients can be written as a product of real first degree polynomials, and real second degree polynomials without real roots. For any element x, we may therefore establish an equation.

$$\sum_{\nu=0}^{m} \alpha_\nu x^\nu = \alpha_m \prod_{\nu=1}^{p} (x^2 + 2\beta_\nu + \gamma_\nu) \prod_{\nu=1}^{q} (x - \delta_\nu)$$

Here $\alpha_m \neq 0$ *and* $\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu \in R$. Since the zero rule ($a\cdot b = 0 \Leftrightarrow a = 0 \vee b = 0$) applies in a division algebra, we may conclude that x is a root in either a first degree equation or in a quadratic equation. In the first case x must be a real number. For every $x \notin R$ *exists* $\beta, \gamma \in R$, such that:

$$x^2 + 2\beta x + \gamma = (x+\beta)^2 + \gamma - \beta^2 = 0 \quad \text{where} \quad \beta^2 - \gamma < 0$$

Let *V'* denote the set of the elements $x' \in V'$, which have a non positive square. We denote this set $R_{0-} = R_{\_} \cup \{0\}$, so that

$$V' = \{x' \in V \,|\, x'^2 \in R_{0-}\}.$$

We then have: $R \cap V' = \{0\}$

We then claim that every $x \in V$ have only one unique representation:

$$x = \varsigma + x', \quad \varsigma \in R, \quad x' \in V'.$$

For $x \notin R$, we may conclude that $x + \beta$ belongs to *V'*, which shows that $x = x' - \beta$ is a unique representation of *x*.
We now claim that *V'* is a subspace in (*V*, +, *R*). We bypass the formal proof, since it is rather obvious and not so interesting.

For *n* = 2 the vector space (*V*, +, · *R*) is isomorphic with (*C*,+, ·).
Since *R* is a 1-dim subspace, and *V'* is a subspace which has only 0 in common with *R*, then *V'* must be one dimensional. If $x' \neq 0$ is an element, and *i*, that is a base for *V'*, the also $i = (-x'^2)^{-\frac{1}{2}} x'$ be a base for *V'*. Every element *x* in *V* has accordingly one and only one representation:

$$x = \varsigma + \eta i \quad where \quad \varsigma, \eta \in R \quad and \quad i^2 = -1$$

Hereby is proven that any 2-dim division algebra over *R* is isomorphic with the body of complex numbers.
In the following, we shall assume, that *n* > 2.

For any $x', y' \in V'$: $x'y' + y'x' \in R$. Since $(x'+y')^2 \in R$, which can be seen from:

$$x'y' + y'x' = (x'+y')^2 - x'^2 - y'^2$$

If $x_1', x_2', ..., x_r' \in V'$ are linear independent, then $(1, x_1', x_2', ..., x_r')$ are also linear independent. (We bypass a formal proof)
Let $x', y' \in V'$ be linear independent. Then there exists then a $\rho \in R$, so that for

$$y'' = y' + \rho x'$$

applies:

$$x'y'' + y''x' = 0$$

So there is exactly one real number ρ, so that:

$$x'(y' + \rho x') + (y' + \rho x')x' = x'y' + y'x' + 2\rho x'^2 = 0$$

The elements *x'* and *y''* are presumably also linear independent.

The elements:

$$i = (-x'^2)^{-\frac{1}{2}} x' \quad \text{and} \quad j = (-y'^2)^{-\frac{1}{2}} y'$$

are therefore linear independent, and

$$i^2 = -1, \quad j^2 = -1, \quad ij = -ji$$

1.4 Construction of the body of quaternion

If we put

$$k = ij$$

Then we get:

$$k^2 = ijij = -ijji = i^2 = -1$$
$$ki = iji = -i^2 j = j\ , \qquad ik = i^2 j = -j$$
$$kj = ij^2 = -i \quad , \qquad jk = jij = -ij^2 = i$$

The elements *i, j, k* are linear independent and from the previous, we infer that also the elements: 1, *i, j, k* are also linear independent, when $n > 3$. This follows from:

$$\rho i + \sigma j + \tau k = 0$$

where the coefficients are real numbers.

When multiplying by *k* from the right

$$\rho ik + \sigma jk + \tau k^2 = 0 \quad \Leftrightarrow \quad \rho ik + \sigma jk - \tau = 0 \quad \Leftrightarrow \quad -\rho j + \sigma i - \tau = 0$$

Since *i* and *j* and also 1, *i*, *j*, are linear independent then this is only possible if $\rho = \sigma = \tau = 0$.

The dimension of *V* cannot be larger than 4.

In other words the introduced elements, *i, j, k* form a basis for *V'*. To prove this, we consider an arbitrary element $z' \in V'$. According to VIII, we have:

$$(1)\ \ iz' + z'i = \rho \quad (2)\quad jz' + z'j = \sigma \quad (3)\quad kz' + z'k = \tau$$

If we multiply (1) with *j* from the left, using (2), we have:

$$jiz' + \sigma i - z'ji = -kz' + \sigma i + z'k = \rho j$$

If we then eliminate *kz'* using (3) we have:

$$2z'k = \rho j - \sigma i + \tau$$

followed by multiplication with *k* from the right.

$$-2z' = \rho i + \sigma j + \tau k$$

This equation shows, however, that any element *z'* belonging to *V'* ,that is, to the subspace spanned by *i, j* and *k.* and this subspace must therefore be identical to *V'*.

What we have proven is: That the necessary conditions for the existence of a division algebra $(V, +, \cdot, R)$ having dimension $n > 2$, is that $n = 4$, and that $(V, +, R)$ has a base (1, *i, j, k)*, where "1" is the one element of the algebra, and for which:

$$i^2 = j^2 = k^2 = -1,$$
$$jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k$$

Because of the distributive law and the laws that apply for the real numbers the product of two numbers, $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$ may be calculated directly:

$$ab = (a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k)$$

Following the rules above, we find:

$$ab = a_0 b_0 + a_0 b_1 i + a_0 b_2 j + a_0 b_3 k + a_1 b_0 i + a_1 b_1 i^2 + a_1 b_2 ij + a_1 b_3 ik + a_2 b_0 j + a_2 b_1 ji + a_2 b_2 j^2 + a_2 b_3 jk + a_3 b_0 k + a_3 b_1 ki + a_3 b_2 kj + a_3 b_3 k^2$$

$$ab = a_0 b_0 + a_0 b_1 i + a_0 b_2 j + a_0 b_3 k + a_1 b_0 i - a_1 b_1 + a_1 b_2 k - a_1 b_3 j + a_2 b_0 j - a_2 b_1 k - a_2 b_2 + a_2 b_3 i + a_3 b_0 k + a_3 b_1 j - a_3 b_2 i - a_3 b_3$$

$$ab = (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i + (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1) j + (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0) k$$

This shows that the defined multiplication is unique, when a base of the sort considered has been chosen.
Thus we conclude that apart from isomorphism there exists only one four dimensional division algebra over *R*.

1.5 Existence of the quaternion body

Formally we still need to justify that there actually exists such one.

In the 4-dim vector space (V_4, +, *R*), is chosen a base. The product *ab* of two vectors:

$$(a_0, a_1, a_2, a_3) \quad and \quad (b_0, b_1, b_2, b_3)$$

Is defined by the coordinate set:

$$\begin{aligned}(&a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3,\\ &a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2,\\ &a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1,\\ &a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0)\end{aligned}$$

It is almost trivial to verify that the vector space (V_4, +, ·, *R*) with this multiplication forms a division algebra. Furthermore it has a one element, namely (1,0,0,0), which is easily verified.

For $i=(0,1,0,0)$, $j=(0,0,1,0)$, $k=(0,0,0,1)$, we can verify the relations below, using the definition of the product.

$$i^2 = j^2 = k^2 = -1,$$
$$jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k$$

The element:

$$a = a_0 + a_1 i + a_2 j + a_3 k$$

Is called a *quaternion*. The product of two quaternion may be calculated by using the multiplication formula above and applying the distributive law.
The algebra is associative, but it is only necessary to show it for the base element, because of the distributive law. Some examples:

$$i(jk) = i^2 = -1 \qquad (ij)k = k^2 = -1$$
$$i(ij) = ik = -j \qquad (ii)j = i^2 j = -j$$

Notice, however, that the algebra is not commutative, since for example: $ki \neq ik$

Finally, we shall show that every non zero quaternion has an inverse element.
To do so we invent the conjugate element, as it is the case or the complex numbers.

The conjugate $\overline{a}$ to $\quad a = a_0 + a_1 i + a_2 j + a_3 k$
is defined as $\quad \overline{a} = a_0 - a_1 i - a_2 j - a_3 k$

Obviously we have: $\overline{\overline{a}} = a$, and by doing the calculation, we find, by replacing the b_m *m=1,2,3* by $-a_m$ in the second factor:

$$ab = \begin{matrix}(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3, \\ a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2, \\ a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1, \\ a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\end{matrix} \qquad a\overline{a} = \begin{matrix}(a_0a_0 - a_{11} - a_2b_2 - a_3b_3, \\ a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2, \\ a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1, \\ a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\end{matrix} = (a_0{}^2 + a_1{}^2 + a_2{}^2 + a_3{}^2, 0,0,0)$$

$$\overline{a}a = a\overline{a} = a_0{}^2 + a_1{}^2 + a_2{}^2 + a_3{}^2$$

This number is denoted the square of the length of the quaternion and it is also written $|a|^2$.

For a non zero $|a|$ is non zero. We then have:

$$a\frac{\overline{a}}{|a|^2} = \frac{\overline{a}}{|a|^2}a = 1$$

Which shows that $\frac{\overline{a}}{|a|^2}$ is the inverse element to a.

Hereby we have shown that the introduced quaternion algebra is a division algebra, and specifically an algebraic body, which we shall refer to as the *quaternion body* (*K, +,· ,R)*.

For $a,b \in K$, $a \neq 0$, each of the equations

$$ax = b \quad , \quad ya = b$$

Has exactly one solution: $x = a^{-1}b$, $y = ba^{-1}$, which are in general different from each other.