Probability Theory

An introduction and beyond

Chapter 2

Finite Probability Fields

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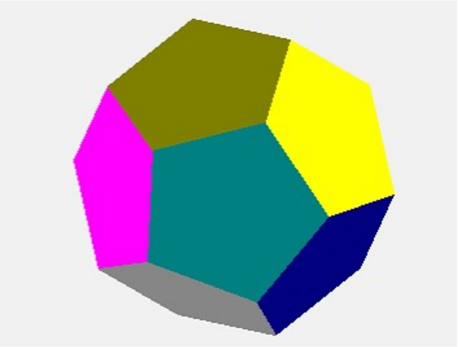

Ole Witt-Hansen (2017) 1997

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1. Introduction

In daily life the concept of probability is heavily used, also when it has nothing to do with the mathematical definition of the word. "The probability that…" is often used synonymous with "there is a chance that…", or " It might.."

In the mathematical theory, however, probability is always a proper fraction or a decimal number between 0 and 1, or what is the same, as a percentage less than or equal to 100%.

When one uses the concept of probability, it should, however, always connected with an uncertainty whether an event will occur or not, e.g. the chance that you win at the roulette or the probability that the metro will run without delay tomorrow.
In each case the use of the term "probability" is an estimate, however, based on some insight, whether an event will occur or not

In mathematics the of concept of probability forms a "model" of an "experiment" the outcome of which is partly unknown, within some definite boundaries.

For example: The probability of getting 6 eyes when rolling a dice or the probability to get 3 of a kind in a hand of poker.

In the mathematical theory one is actually not concerned with establishing the probabilities for the outcome of an real life experiment, but once the probabilities are given (right or not), the purpose of probability theoty is to draw some logical consequences. The conclusions are often far reaching and often unexpected. On this issue, we shall be occupied in the following.

But first we shall establish the mathematical model of a stochastic experiment. This could for example be a model of real life experiments as rolling a dice, or picking a card from a deck of cards.

2. A finite probability field

A finite probability field (U,P) is an abstraction of a *stochastic experiment.* It consists of a finite set U, and a function P defined on U called the *probability* (or *probability function*), having the following properties:

- $U = \{u_1, u_2, u_3, \ldots, u_n\}$, called the *outcome space,* is the set of outcomes of an experiment, and the elements in U, are called *outcomes*.

- $P(u)$ is denoted the *probability* of an outcome (occurs)

- For all $u \in U$ applies: $0 \leq P(u) \leq 1$ and : $P(u_1) + P(u_2) + P(u_3) + \ldots + P(u_n) = 1$

This may be formulated as follows:

- The probability of an outcome from an abstract experiment is always a number between 0 and 1, and the sum of the probabilities is equal to 1.

These are the only conditions necessary to establish a *finite probability field*.

The mathematical part of the theory is, however, neither concerned with establishing the initial probabilities of an experiment, nor it has anything to do with real life experiences.

Examples

1. When rolling a dice the outcome space is $U = \{1, 2, 3, 4, 5, 6\}$. We only assume that the outcomes have the same probability. According the presuppositions above, we must therefore have: $6 \cdot P(u) = 1$, from which follows that $P(u) = 1/6$ for all outcomes.

2. Throwing two coins (heads or tails). Here we may establish the outcome space. If we abbreviate *h* (*head*) *and* *t* (*tail*), then $U = \{(h, h), (h, t), (t, h), (t, t)\}$. Assuming that these 4 outcomes are equally likely, the probability of each of these outcomes is $P(u) = ¼$.

3. When playing at the roulette in a Casino there are 37 fields numbered 0…36. The outcome space is therefore $U = \{0, 1, 2, 3 \ldots 36\}$. It is the common supposition that the fields have equal probability (and it ought to be so), so according to the presuppositions above, the probability that the ball will end in any of the 37 fields should be $P(u) = 1/37$.

4. When rolling two dices (a green one and a red one), we may establish the outcome space, writing down the outcomes (eyes on the green dice, eyes on the red dice), which are considered to be equally likely.
   With this notation the outcome space becomes

   $$U = \{(1,1), (1,2), \ldots.(1,6), (2,1), (2,2), \ldots(2,6)\ldots..(6,5), (6,6)\}$$

   Since there are 6 possibilities for the green dice as well as for the red dice there must be $6 \cdot 6 = 36$ outcomes, and the probability of each outcome must therefore be $P(u) = 1/36$.

3. Events

An event H is formally defined as a *subset* of the *outcome space*. Since an event is a set, it is always written with a capital letter, whereas an outcome is always written with a lower case letter. If we have an outcome space:

$$U = \{u_1, u_2, u_3, \ldots, u_n\}$$

Then three events might be $A = \{u_2, u_7, u_8\}$, $B = \{u_4\}$, $C = \{u_1, u_7\}$.

Events are most often characterized in words. When you roll a dice, where the outcome space is

$$U = \{1, 2, 3, 4, 5, 6\}$$

one can formulate various events as:

A: Getting an even number of eyes.	$A = \{2, 4, 6\}$
B: The dice does not show 6 eyes.	$B = \{1, 2, 3, 4, 5\}$
C: The eyes are greater than 4.	$C = \{5,6\}$

Intuitively, even in public school, and without knowledge of probability theory, most people would settle for the probabilities of three events mentioned above:

$$P(A) = 3/6 = 1/2, \quad P(B) = 5/6 \text{ and } P(C) = 2/6 = 1/3$$

In probability theory, the challenge is often, however, using combinatorics, to decide which outcomes belong to a certain event. In the previous examples it was trivial, but often this is far from the case.

The probabilities stated in the example, when rolling a dice are perfectly right, and this leads to the following definition, assuming that we have an arbitrary outcome space $U = \{u_1, u_2, u_3, ..., u_n\}$

- By the probability of an *event* $P(H)$ one should understand the sum of probabilities $P(u)$ of the outcomes u that belong to the event H.

So if $U = \{u_1, u_2, u_3, ..., u_n\}$, and $A = \{u_2, u_7, u_8\}$, then $P(A) = P(u_2) + P(u_7) + P(u_8)$

One may easily convince oneself that this definition of an *event* is in full accordance with the probabilities of the events *A, B* and *C* in the "rolling a dice" example above.

Since both the outcome space itself, and the empty set $\varnothing = \{\}$ formally are subsets of U, it leads to two special events.

- U is called the safe *event* , and $P(U) = 1$, according the definition of an *event*.
- $\varnothing$ is called the impossible event, and $P(\varnothing) = 0$, according the definition of an *event*.

3.1 Using summation signs

When operating with *events* in probability theory it is practical, if not necessary, to use the so called summation symbol Σ. The summation symbol is merely an abbreviated way of writing a sum of indexed terms. Below we show some examples.

$$\sum_{n=2}^{10} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{10} \qquad P(H) = \sum_{u \in H} P(u)$$

The first summation sign displays the sum of the fractions from 1/2 to 1/10. $n = 2$ is called the lower limit, and 10 is the upper limit. The meaning of the summation symbol should otherwise be obvious.
In the second summation sign there are no lower or upper limits. Instead it should be understood as we add all probabilities for the outcomes that belong to H.

3.2 Symmetric outcome spaces

In all the examples above we have considered outcome spaces, where all outcomes have the same probability. Such an outcome space is called symmetric

In a symmetric outcome space having n outcomes, the probability of each outcome is

$$P(u) = \frac{1}{n}$$

This follows from the presupposition that the sum of probabilities is 1.

In a *symmetric* outcome space *U*, one calculates, according to the definition of an event, the probability of an event *H,* (having *q* elements), as *q* divided by *n,* (the number of elements in *U*).

$$P(H) = \sum_{u \in H} P(u) = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \dots \frac{1}{n} = \frac{q}{n}$$

In this connection, one has a manner op speaking, since the outcomes belonging to *H* are called *favourable* outcomes (or just favourable) and the outcomes belonging to *U* are called *possible* outcomes (or just possible).
By the same token, we shall from now on, when we are dealing with a symmetric outcome space, write *n*(*U*) for the number of elements in *U* and *n*(*H*) for the number of elements in *H*.

This results in some formulas, which are easy to remember each time one must calculate the probability of an event in a symmetric outcome space.

$$P(H) = \frac{favourable\ outcomes}{posssible\ \ outcomes} \quad or \quad just \quad P(H) = \frac{favourable}{possible}$$

$$P(H) = \frac{n(H)}{n(U)}$$

Examples

1. First we consider drawing a single card from a deck of cards consisting of 52 cards.
We want to find the probabilities of the following events:

A: We draw a spade.
B: We draw a picture card.
C: We draw an ace.

The outcome space is obviously symmetric, and since *n*(*U*) = 52 then *P*(u) = 1/52, and for the three events we have: *n*(*A*) = 13*, n*(*B*) = 12 and *n*(*C*) = 4. Thus we find for the three probabilities:

$$P(A) = \frac{n(A)}{n(U)} = \frac{13}{52} = \frac{1}{4} \qquad P(B) = \frac{n(B)}{n(U)} = \frac{12}{52} = \frac{3}{13} \qquad P(C) = \frac{n(C)}{n(U)} = \frac{4}{52} = \frac{1}{13}$$

2. We now look at a simultaneous roll two dices (a red one and a green one). We wish to find the probabilities *P*(2), *P*(3), …, *P*(12) that the dices together show 2 eyes, 3 eyes,…, 12 eyes. We have previously established the symmetric outcome space as: U = {(1,1), (1,2), ….(1,6), (2,1), (2,2),…(2,6)…..(6,5), (6,6)}.
Since this outcome space is symmetric, and it has 36 elements, The probability of each outcome is *P*(*u*) = 1/36.
The calculation of the probabilities is then reduced to find the number of outcomes in each event.

Outcomes	{(1,1)}	{(1,2), (2,1)}	{(1,3), (3,1), (2,2)}	{(1,4), (4,1), (2,3), (3,2)}
Probability *P*(*u*)	$P(2) = \frac{1}{36}$	$P(3) = \frac{2}{36} = \frac{1}{18}$	$P(4) = \frac{3}{36} = \frac{1}{12}$	$P(5) = \frac{4}{36} = \frac{1}{9}$

Outcomes	{(1,5), (5,1), (2,4), (4,2),(3,3)}	{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)}
Probability *P*(*u*)	$P(6) = \frac{5}{36}$	$P(7) = \frac{6}{36} = \frac{1}{6}$

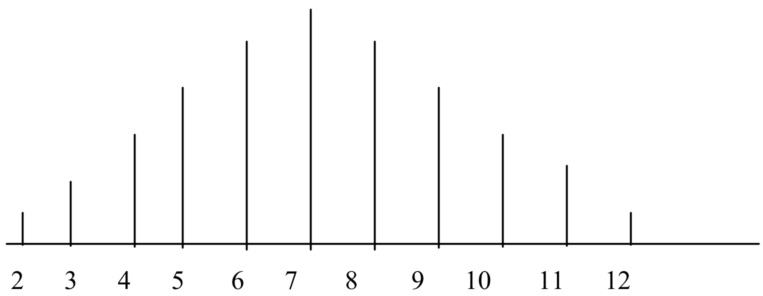
In a quite similar way we find the probabilities $P(8)\ldots P(12)$. Since they are symmetric with respect to $P(7)$

$$P(8)=\frac{5}{36},\quad P(9)=\frac{4}{36}=\frac{1}{9},\quad P(10)=\frac{3}{36}=\frac{1}{12},\quad P(11)=\frac{2}{36}=\frac{1}{18},\quad P(12)=\frac{1}{36}$$

We are then able to define a new (not symmetric) outcome space, shown in the table below. Such a table is often denoted as the probability distribution.

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Finally a "stick diagram" is often used as a graphical representation of the probability distribution.



3. **The birthday problem**:
The birthday problem is actually one of most frequent examples to illustrate elementary probability theory. Suppose we have a class with 24 students. If you make a bet, whether at least two persons in the class have birthday on the same day of the year, what are the odds that you may win the bet?
Said in another way: What is the probability that at least to students in the class have birthday on the same day of the year? We shall generalize the problem, changing the number of student from 24 to *n*.
Solving problems in probability theory it is sometimes easier and more advantageous to calculate the probability of the complementary event to an event.
The complementary event to an event *H* is read as *non H* and it is usually written as *H* with a bar over.
If *H* is the event that at least two persons in the class have birthday on the same day of the year we shall therefore first find the probability of complementary event: That no one has birthday on the same day of the year.
Since the union of an event and the complementary event is equal to *U*, and as they have no common elements, we must have:

$$P(H)+P(\overline{H})=1 \quad \Leftrightarrow \quad P(H)=1-P(\overline{H})$$

With this in mind, we shall calculate the probability that among *n* persons, none have birthday on the same day. Since each person has 365 possibilities of having birthday, and since the persons are supposed to be independent of each other, the outcome space has $365\cdot365\cdot365\cdot\ldots\cdot365 = 365^n$ different possibilities of placing the birthdays on *n* persons. Similarly we find the number of possibilities that no one has birthday on the same day as: $365\cdot364\cdot363\cdot\ldots\cdot(365-n+1)$ (n-factors), since there are 365 possibilities for the first person 365 -1 possibilities for the second person and so on.
Since the outcome space is obviously symmetric, we may calculate the probability *H* of no coincidence i birthday, with the formula:

$$P(\overline{H})=\frac{n(\overline{H})}{n(U)}$$

$$P(\overline{H})=\frac{365\cdot(365-1)\cdot(365-2)\cdot\ldots(365-n+1)}{365\cdot365\cdot365\cdot365\cdot\ldots\cdot365}=\frac{365}{365}\cdot\frac{365-1}{365}\cdot\frac{365-2}{365}\cdot\ldots\frac{365-n+1}{365}$$

Exercising some patience it is certainly possible to calculate this probability, say for $n = 24$, with a pocket calculator or even with a mathematical computer, but an analytic answer to this problem has been given long before the invention of electronic calculators.

We shall now show that it is possible (applying a little mathematics) to give a simple expression for the probability for an arbitrary n.

To do so, we divide the denominator 365 up in the numerator in each of the factors in the expression above, followed by taking the natural logarithm on each side.

$$P(\overline{H})=\frac{365}{365}\cdot\frac{365-1}{365}\cdot\frac{365-2}{365}\cdot\ldots\frac{365-n+1}{365}=1\cdot(1-\frac{1}{365})\cdot(1-\frac{2}{365})\cdot\ldots\cdot(1-\frac{n-1}{365})$$

$$\ln P(\overline{H})=\ln(1-\frac{1}{365})+\ln(1-\frac{2}{365})+\ldots+\ln(1-\frac{n-1}{365})$$

Expanding the natural logarithm to the first order, we have the formula:

$$\ln(1+h)\approx h \quad \textit{for} \quad h<<1$$ (meaning $h < 0.1$) (ln (1.1) =0.095)

Applying this formula to each term in the expression above.

(1.5)
$$\ln P(\overline{H}) = -\frac{1}{365}-\frac{2}{365}-\frac{3}{365}-\ldots\frac{n-1}{365} = -\frac{1}{365}(1+2+3+\ldots(n-1))$$
$$\ln P(\overline{H})=-\frac{1}{365}\frac{n(n-1)}{2}$$

Our aim is then to determine n such that:

$$P(\overline{H})<\frac{1}{2} \quad\Leftrightarrow\quad \ln P(\overline{H})<\ln\frac{1}{2} \quad\Leftrightarrow\quad \ln P(\overline{H})<-0.6931$$

It gives the inequality:

$$-\frac{1}{365}\frac{n(n-1)}{2}<-0.6931 \quad\Leftrightarrow\quad n(n-1)>505.96$$

This is an algebraic inequality of second order:

$$n^2-n-505,963>0$$

The solution to the corresponding quadratic equation is $n=\frac{1\pm45}{2}=\begin{cases}23\\-22\end{cases}$

And it gives the solution to the inequality:

$$n^2-n-505,963>0 \quad\Leftrightarrow\quad n<-22 \;\vee\; n>23$$

Everything so, when the number of persons is larger than 23, the probability for no coincidence of birthday between them is less than 50%, meaning that the probability of a least one coincidence of birthday is greater than 50% when the number of persons are greater than 23.

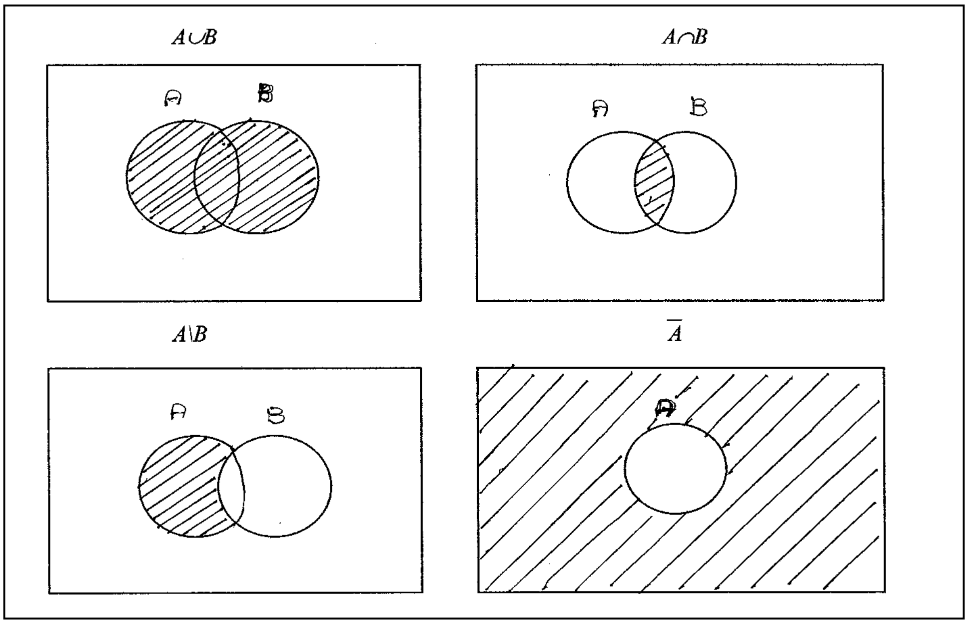
4. Manipulating events and probabilities

For two events, that is, two subsets A and B of a set U, one may, as it is well known, form some new sets. They are:

- *The union* $A\cup B$ are the element which belongs to either A or B.
- *The intersection* $A\cap B$, are the element which belongs to both A and B.
- *The surplus set* $A\backslash B$, are the elements in A, which does not belong to B.

- *The complementary set to* $\overline{A}$ is the elements in *U,* which do not belong to *A*.

These events are also called the "*either…or event*", the "*both …and event*", the "A but not B event" and the *complementary* event. Below are illustrated the four types of composite sets



If *A* and *B* are events in an outcome space *U*, the four composite events $A \cup B$, $A \cap B$, $A \backslash B$, $\overline{A}$ are also events. Within the framework of probability theory we express this as:

The union event $A \cup B$ occurs if at least one of the events *A* or *B* occurs.
The intersection event $A \cap B$ occurs if both *A* and *B* occur.
The surplus event $A \backslash B$ occurs if *A* occurs, but *B* does not.
The complementary event $\overline{A}$ occurs if *A* does not occur.

There are some rules for calculating the probabilities of composite events, and it is most appropriate to write these rules with the help of summation sign, as they imply adding the probabilities of outcomes belonging to an event. Using this notation we have:

$$P(A) = \sum_{u \in A} P(u) \quad , \quad P(B) = \sum_{u \in B} P(u) \quad , \quad P(A \cup B) = \sum_{u \in A \cup B} P(u) \quad ,$$

$$P(A \cap B) = \sum_{u \in A \cap B} P(u) , \qquad P(A \backslash B) = \sum_{u \in A \backslash B} P(u) \quad , \quad P(\overline{A}) = \sum_{u \in CA} P(u)$$

We may then derive the following rules:

$$P(A \cup B) = \sum_{u \in A \cup B} P(u) = \sum_{u \in A} P(u) + \sum_{u \in B} P(u) - \sum_{u \in A \cap B} P(u)$$

When we add the probabilities of the outcomes that either belongs to *A* or belongs to *B*, we will count the outcomes which belong to both *A* and *B* twice, therefore we must subtract these outcomes.
This gives the formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In other words: The probability that either *A* or *B* occur, is the probability that *A* occurs plus the probability that *B* occurs, minus the probability that they occur simultaneously.
If $A \cap B = \emptyset$, so that *A* and *B* have no common outcomes, the two events are said to exclude each other. In that case the formula becomes simpler, since $P(\emptyset) = 0$

$$P(A \cup B) = P(A) + P(B)$$

In the same manner we find:

$$P(\overline{A}) = \sum_{u \in CA} P(u) = \sum_{u \in U} P(u) - \sum_{u \in A} P(u)$$

$$P(\overline{A}) = 1 - P(A)$$

The probability that *A* does not occur is 1 minus the probability that *A* occurs.

$$P(A \setminus B) = \sum_{u \in A \setminus B} P(u) = \sum_{u \in A} P(u) - \sum_{u \in A \cap B} P(u)$$

$$P(A \setminus B) = P(A) - P(A \cap B)$$

The probability that *A* occurs, but *B* does not occur is the probability that *A* occurs minus the probability that *A* and *B* occur simultaneously.

Examples

1. We shall first consider the experiment of drawing a card from a deck of cards. We want to find the probabilities of the following events: *A*: You draw a picture card or a spade. *B*: You do not draw a picture card. *C*: You draw a picture card, but not in hearts. According to the formulas above for the probabilities of composite events, we have, if we notice that there are 12 picture cards, 12 cards of hearts and 3 picture cards of spade.

   $P(A) = P(\text{Picture card}) + P(\text{spade}) - P(\text{Picture card of spades}) = \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52}$

   $P(B) = 1 - P(\text{Picture card}) = 1 - \frac{12}{52} = \frac{40}{52} = \frac{10}{13}$

   $P(C) = P(\text{Picture card}) - P(\text{Picture card of hearts}) = \frac{12}{52} - \frac{3}{52} = \frac{9}{52}$

2. We shall next consider rolling with two dices, and we look at the following events: *A*: The two dices show the same eyes. *B*: The two dices show differe999nt eyes. *C*: The red dice shows more eyes than the green one. *D*: None of the dices show 6 eyes. *E*: We get 6 eyes on at least one of the dices. With the rules stated above, we have:

   $P(A) = \frac{6}{36} = \frac{1}{6}$ and $P(B) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$

   In half of the 36-6 outcomes the red dice shows more eyes than the green dice, so the answer is:

   $P(C) = \frac{1}{2} P(B) = \frac{5}{12} (= \frac{15}{36})$

There are 5·5 =25 outcomes, where none of the dices show 6 eyes, therefore

$$P(D) = \frac{25}{36} \quad \text{and} \quad P(E) = 1 - P(D) = 1 - \frac{25}{36} = \frac{11}{36}$$

5. Samples without replacement

Making a sample without replacement is a classical discipline in probability theory.
Most often it is illustrated by pulling at random, coloured balls from an otherwise closed basket, and calculating the probability of getting various combinations. The technique is however best illustrated by doing some examples. In the examples below, we shall make extensive use of the formula:

$$C(n,q) = \frac{n!}{q!(n-q)!}$$

Which is the number of ways to select a combination of q elements (*q-subset*) from a set having n elements (a *n-set*)

5.1 Examples

1. In a class there are 10 boys and 12 girls, and they must form a committee with 4 members. We shall determine the probability for the following events:

   A: The committee has girls only.
   B: The committee has two boys and two girls.
   C: There is at least one boy in the committee.

   To calculate these probabilities, we shall use the formula for a symmetric outcome space:

   $$P(H) = \frac{\textit{Favourable outcomes}}{\textit{Possible outcomes}} = \frac{n(H)}{n(U)}$$

   The possible outcomes are $C(22,4) = \frac{22!}{4!(22-4)!} = \frac{22 \cdot 21 \cdot 20 \cdot 19}{4 \cdot 3 \cdot 2 \cdot 1} = 11 \cdot 7 \cdot 5 \cdot 19 = 7315$

   And 4 girls can be selected in C(12,4) = 495 different ways. So we have:

   $$P(A) = P(4\,girls) = \frac{C(12,4)}{C(22,4)} = \frac{495}{7315} = 0.0677$$

   In the same manner we get:

   $$P(B) = P(2\text{boys and two girls}) \frac{C(10,2) \cdot C(12,2)}{C(22,4)} = \frac{45 \cdot 66}{7315} = 0.4060$$

   $P(C) = P(\textit{At least one boy}) = 1 - P(4\ \textit{girls}) = 1 - 0.0667 = 0.9333$

3. In a drawer we find disorderly 6 blue, 8 grey and 10 black socks. One persons takes (without controlling the colour) 3 socks hoping that he will get two in the same colour. We shall calculate the probabilities:

   A: You get socks in 3 different colours.
   B: You get either 3 blue, 3 grey or 3 black socks.
   C: You get at least 2 socks in the same colour.

The number of possible outcomes is: $C(6+8+10,3) = C(24,3) = \frac{24 \cdot 23 \cdot 22}{3 \cdot 2 \cdot 1} = 4 \cdot 23 \cdot 22 = 2024$

$$P(A) = P(3\ different\ colours) = \frac{6 \cdot 8 \cdot 10}{C(24,3)} = \frac{480}{2024} = 0.2372$$

$$P(B) = P(3 \text{ of the same kind}) = \frac{C(6,3)+C(8,3)+C(10,3)\cdot}{C(24,3)} = \frac{20+56+120}{2024} = \frac{196}{2024} = 0.0968$$

$P(At\ least\ 2\ in\ the\ same\ colour) = 1 - P(3\ different\ colours) = 1 - 0.2372 = 0.7628$

5.2 Example. Probabilities in Lotto games

In Denmark the weekly online Lotto game is about, in one row to guess 7 numbers out of 36. Besides the 7 drawn lotto numbers, there is also drawn an additional number (an add number).
One may achieve a premium, in the following manners:

1. Premium: Having all the 7 drawn numbers.
2. Premium: Having 6 right numbers plus an add number.
3. Premium: Having 6 out of 7 right numbers
4. Premium: Having 5 out of 7 right numbers.
5. Premium: Having 4 out of 7 right numbers.

We shall begin by calculating the probabilities of these events, when you have filled in one row.

The numbers of different ways one may chose a combination of 7 numbers out of 36 is:

(1.3) $$C(36,7) = \frac{36!}{7!(36-7)!} = \frac{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \cdot 30}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8{,}347{,}680$$

Accordingly the probability $P(7)$ of having the 7 right numbers in one row is:

$$P(7) = \frac{1}{C(36,7)} = p_7 = 1{,}1979 \cdot 10^{-7}$$

When we calculate the probability of having 6 right numbers and one right add number, we reason as follows. The 6 correct numbers can be chosen among the 7 numbers in $C(7,6)=7$ different ways, whereas the add number may be chosen in one way only. Therefore

$$P(6+1\ add) = \frac{C(7,6) \cdot 1}{C(36,7)} = 7\,p_7 = 8.330 \cdot 10^{-7}$$

In the same manner we find, (by the multiplication principle) the probability of having 6 right numbers. The 6 right numbers can be chosen in $C(7,6) = 7$ different ways, and the “wrong” number may be chosen among 36 -7 – 1 = 28 numbers. Therefore:

$$P(6) = \frac{C(7,6) \cdot 28}{C(36,7)} = 196 \cdot p_7 = 2.3479 \cdot 10^{-5}$$

The probability of obtaining 5 right numbers is the number of ways 5 numbers can be selected from 7 numbers, which is $C(7,5) = 21$ times the number of ways we can select 2 numbers from
36-7 =29 numbers, which is $C(29,2)=406$.

$$P(5) = \frac{C(7,5) \cdot C(29,2)}{C(36,7)} = 8526 \cdot p_7 = 1.02 \cdot 10^{-3} = 1.02\ ‰$$

In the same manner we calculate the probability of having 4 right numbers. The number of possibilities are:
$C(7,4) \cdot C(29,3)$

$$P(4) = \frac{C(7,4) \cdot C(29,3)}{C(36,7)} = 127{,}890 \cdot p_7 = 0.0153 = 1.53\%$$

From the calculated probabilities above we see that the probability of having more than 4 correct numbers is immensely small. On the other hand there is still a fair chance of getting 4 right numbers. This is of course a deliberate choice from the administrators of the game since, if a lotto player never wins, most (non addicted) players will stop playing after a certain time.

A lotto coupon has 10 rows, and we shall first calculate the chance *P*(*H*) of winning on at least one of the rows.
We do it by calculating the probability of the complementary event $P(\overline{H})=1-P(H)$.
Since the 10 rows are independent, the probability that none has 4 right numbers is

P(not 4 right numbers on any of the 10 rows) = $(1\text{-}p_4)^{10} = 0.9847^{10} = 0.8557$

The chance of having at least one row with 4 right numbers is therefore:

P(at least on coupon with 4 right numbers) = 1 – 0.8557 = 0.1443.

If we assume that someone buys 10 rows, for 10 weeks, the probability of not winning is $(0.8557)^{10} = 0.4487$. So having bought a coupon 10 weeks the probability of winning is:

P(at least one coupon with 4 right numbers in 10 weeks) = 1- $(0.8557)^{10} = 0.5513$.

But it is worth mentioning that the premium of having 4 right numbers is of the same magnitude or less, than the cost of 10 coupons.

Exercises

1. Let *P* be a probability function belonging to an outcome space *U* = {1,2,3,4,5}. Find *P*(5), if

   a) P(1) = P(2) = 0.1 and P(3) = P(4) = 0.2
   b) P(1) = P(2) = P(3) = P(4) = P(5)
   c) P(1) =0.4 and P(2) = P(3) = P(4) = P(5)

2. Throwing three dices, you should find the probability that the dices do not show three equal.

3. By throwing two dices find the probability that only one show 6 eyes.

4. A car driver must, on his way to work, pass through four traffic lights. It is assumed that there are equal probability of having green and red light.

   a) How many possibilities are there for having red and green light along the route?
   b) What is the possibility of meeting 4 green lights?
   c) What is the possibility of meeting 3 green and 1 red lights?
   d) What is the possibility of meeting 2 green and 2 red lights?

5. Find the probability of getting 13, 12 11 or 10 right, when filling in a pools coupon having 13 rows, each with 3 choices (won, tie, lost).

6. Poker. From a deck of cards 5 cards are drawn. How many possibilities are there?
   How many possibilities are there of getting 5 cards in the same colour (a flush)?

   What is the probability of getting a flush?

7. In a jar there are 6 red and 4 white balls. Three balls are drawn from jar at random. Find the probability that:

   a) Three red balls are drawn.
   b) At least two white balls are drawn.
   c) Exactly one white ball is drawn.

8. (A tough one). At a tasting of wine 5 glasses are placed in a row, where you are supposed to decide where the wine comes from. On the supposition of pure guessing, what is the probability of getting 0, 1, 2, 3, 4 and 5 right. It is a tough one, since you must find out, in how many permutations, where there are one, two, …five elements which correspond to themselves