Probability Theory
An introduction and beyond

Chapter 1

Combinatorics

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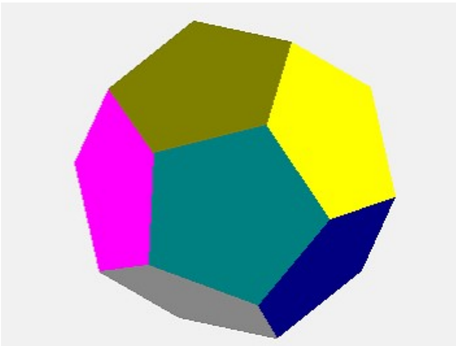

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1. The multiplication and addition principles

In combinatorics one is concerned with methods applied to a systematic counting the number of ways to select a subset of elements from a given finite set.

Combinatorics is founded by two simple but fundamental principles, called the *multiplication principle* and the *addition principle*.
For practical reasons they are often referred to as the "*both… and*" principle and the "*either or*" principle.

Since mathematical theories are axiomatically founded, we shall initially state an obvious fact.

- If we have a set with n different elements, one element can be selected in n different ways.

For example, if we have a class with 24 students a student can be selected in 24 different ways.

However if one must select a student both from the X-class (24 students) and a student from the Y-class (18 students), one may reason as follows:

Each time you has selected a student from the X-class, you may chose a student from the Y-class in 18 different ways. The number of different ways to select the two students is therefore obviously 24 times 18 possibilities: $24 \cdot 18 = 432$ possibilities.

- This is an example of the contents of *the multiplication principle*. If you must select *both* an element from a set of n elements *and* an element from a set with m elements, this can be done in $n \cdot m$ different ways.

The multiplication principle is more informally referred to as the *both…and* principle, since if you can formulate your selection with the words *both…and*, then the number of choices in the selections should be multiplied with each other

Let us next assume that you shall *either* select a student from the X-class *or* a student from the Y-class. In that case you merely make a selection among 24 plus 18 students, and the answer is $24+18 = 42$ possibilities.

- This is an example of the contents of the *addition principle*. If you may select *either* an element from a set of n elements *or* an element from a set of m elements it can be done in $n+m$ different ways.

The *addition principle* is less formally referred to as the *either…or* principle, for the reason that if you can formulate your selection using the words *either…or* you should add the number of choices from the two selections.

Although these examples appear utterly trivial, you may encounter far more complex situations, where you have to decide for yourself, whether the selections are *both…and* or *either…or.*
The two principles can of course trivially be generalized to more than two sets of elements, since you must either *add* the choices of the sets or *multiply* them.

Examples:

1. If you have to choose from a menu, including starter, main dish and dessert, having 4 starters, 7 main dishes and 6 desserts, and if you want the whole menu, it can be done in $4\cdot 7\cdot 6 = 168$ different ways, but if you can't afford a whole menu, and settle for just one dish, the are $4+7+6 = 17$ possibilities.

2. An older model of a lock on a bicycle had 6 keys, each having 3 positions (in, out and neutral). What are the numbers of different combinations to unlock? Well, the first key has 3 positions, and the for each, the next key also has 3 positions, which makes 9, so using the multiplication principle, the number of combinations is $3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3 = 3^6$. (The combination where all the keys are in neutral is not advisable, however).

3. In Denmark we have had for a long time the possibility of playing on the outcomes of national foot-ball games. It is done on a coupon with 13 entries 1 (won), *X* (tie), 2 (lost). What is the number of possibilities when filling in a coupon? Since each entry is filled in independently of the others, and you have to fill in all entries, the multiplication principle applies. So the result is: $3\cdot 3\cdot 3\cdot ...\cdot 3 = 3^{13} = 1.594.323$ possibilities.

2. Permutations

Think of a set having n different elements, (e.g. a class with 24 students). A sorted subset with q elements (think of the last row in the class where $q = 8$) is called a *q-permutation* of a set having n elements, which we shall denote a n-set.
We wish to find a formula, written symbolically $P(n,q)$ or $P_{n,q}$, for the number of different q-permutations, one may form from a n-set.

We start out by making a formula for $P(n,n)$, that is the number of different permutations of a n-set.

We imagine that we have a row of n places (24 sorted seats) and we aim at finding the number of different ways they can be occupied by n elements (e.g. 24 persons).

n	n-1	n-2	…						3	2	1

Since *both* the first *and* the second, and the *nth* place must be occupied, we must apply the multiplication principle. The first place (seat) can by occupied in n (24) different ways, and for each choice, the second place (seat) can be occupied in $n - 1$ (23) different ways and so on, so by the multiplication principle, the first two places (seats) can be occupied in $n(n-1)$ different ways. We may continue this line of argument, so that the n places (seats) can be occupied in $n\cdot(n-1)\cdot(n-2)\cdot ...\cdot 3\cdot 2\cdot 1$ $(24\cdot 23\cdot 22\cdot ...3\cdot 2\cdot 1)$ different ways. This leads to the formula

$$P(n,n) = n\cdot(n-1)\cdot(n-2)\cdot ...\cdot 3\cdot 2\cdot 1 \qquad P(24,24) = 24\cdot 23\cdot 22\cdot ...3\cdot 2\cdot 1$$

- For the product of integers from 1 to n: $n\cdot(n-1)\cdot(n-2)\cdot ...\cdot 3\cdot 2\cdot 1$ the standard symbol $n!$ is used, and it is read *n-factorial.*

To accomplish the general validity of various formulas it is useful to define:

$$0! = 1$$

On most pocket computers you can find e.g. that $6! = 720$ and $10! = 3{,}628{,}800$.
Thus we have shown the formula:

$$P(n,n) = n!$$

Using the factorial notation, it is relatively easy to establish a formula $P(n,q)$, where $n \geq q$, that is the number of q-permutations taken from a n-set.
The number of q-permutations that can be made from a *n-set*, is according to the arguments above:

$$P(n,q) = n(n-1)(n-2)\cdots(n-q+1).$$

Is the then straightforward to establish a formula for $P(n,q)$ for $n \geq q$, since we only have to multiply and divide the result by $(n-q)! = (n-q)(n-q-1)\cdots 3\cdot 2\cdot 1$, since

$$n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-q+1) = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-q+1)\cdot(n-q)!}{(n-q)!} = \frac{n!}{(n-q)!}$$

We thus have the formula:

$$P(n,q) = \frac{n!}{(n-q)!}$$

Notice, that with the definition 0! = 1, the formula is also valid when $n = q$, since then $P(n,n) = n!/0! = n!$

Sometimes it is surprising to realize, how large a number $P(n,q)$ actually is.
For example if we calculate in how many ways can you occupy the last row (8 seats) in a class having 24 students. The answer is:

$$P(24,8) = \frac{24!}{(24-8)!} = 24\cdot 23\cdot 22\cdot\ldots 18\cdot 17 = 2.9654\cdot 10^{10}$$

3. Combinations

If we have a set of n-elements (a *n-set*), you may for example think of a class having 24 students, where you shall select a subset with q elements, a *q-subset.* You may think of appointing a committee with 4 members.

The selection of a *q-subset* from an *n-set* is not a permutation, because the order of the members in the *q-subset* is insignificant. The committee is unsorted.

A *q-subset* selected *from an n-set* is called a *combination*.

- The number of different *q-subset combinations* that can be selected from a *n-set* is denoted $C(n,q)$, or sometimes $C_{n,q}$. We wish to establish a formula for $C(n,q)$.

We notice that trivially: $C(n,1) = n$ and $C(n,n) = 1$

We shall first illustrate the reasoning with the example of selecting a committee of 4 students from a class with 24 students.

We imagine, however, that the committee after the selection must elect a chairman, a referent, a subordinate chairman and a cashier.

The committee can be selected in $C(24,4)$ different ways, but until now we do not know the number $C(24,4)$.
However, once the committee is chosen, the different positions in the committee can be selected in $4{\cdot}3{\cdot}2{\cdot}1$ ways, since there are 4 possibilities for the chairman, then 3 possibilities for the referent…

Since we must select *both* the committee, *and* distribute the positions in the committee, the number of possibilities of selecting the committee, and subsequently a chairman, a referent, a subordinate chairman and a cashier, must be

$$C(24,4)\cdot 4{\cdot}3{\cdot}2{\cdot}1 = C(24,4)\cdot 4!$$

But we might as well have selected the 4 positions directly, and the answer would be: $P(24,4)$, (since the 4 elements in the combination are different).
So when we put the two ways of calculating $P(24,4)$ together, we have:

$$C(24,4)\cdot 4! = P(24,4) \quad \Leftrightarrow \quad C(24,5) = \frac{P(24,4)}{4!} = \frac{24!}{4!(24-4)!}$$

Which is the wanted formula for $C(24,4)$. We shall now repeat the argument with n and q, replacing 24 and 4.

To do so we shall calculate the number of *q-combinations* that can be selected from an *n-set* in two different ways.
The selection of a *q- permutation* (directly) can be done in $P(n,q)$ different ways.
However, we could also do it the other way round, first select a *q-combination,* which can be done in $C(n,q)$ different ways and then permute the q-elements, which can be done in $q!$ different ways.
One way or another, we must find the same number of *q- permutations*. This implies:

$$C(n,q)\cdot q! = P(n,q) \quad \Leftrightarrow \quad C(n,q) = \frac{P(n,q)}{q!} = \frac{n!}{(n-q)!q!}$$

(3.1) $$C(n,q) = \frac{n!}{(n-q)!q!}$$

We have then obtained a general formula, which is also valid for $q = 0$, and $n = q$, because we have set $0! = 1$.
For practical and theoretical use, two formulas are often used, when doing calculation by hand
The two formulas are merely a reduction to avoid $n!$ for large values of n.

(3.2) $$P(n,q) = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-q+1) \quad \text{and} \quad C(n,q) = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-q+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot q}$$

Notice that the last formula has the same number of factors in the numerator and the denominator.

Examples

Lotto: When you play Lotto (in Demark), you must choose 7 numbers out of 36. We want to find the total number of possible Lotto coupons. This is however the same as finding the numbers of 7-subsets from a 36-set. So the answer is simply:

$$C(36,7) = \frac{36!}{(36-7)!7!} = 8.347.680$$

1. In a class having 10 boys and 12 girls they must form a committee of 4, having 2 girls and 2 boys. In how many ways can the committee be formed.
   The 2 boys can be selected in C(10,2) = 45 different ways, and the girls in C(12,2) = 66 different ways. According to the multiplication principle, (since we shall choose *both* the boys *and* the girls), the result is: C(10,2) C(12,2) = 2970

2. From an assembly consisting of 8 woman and 12 men a committee of 5 must be formed. In how many ways can it be done if:

   a) One may choose freely among the 20 members?
   b) The committee must have at least one woman and one man.
   c) The committee must have at least two woman and two men

   a) $C(20,5) = 15.504$

   b) The answer is most easily found if we subtract the number of committees having only woman and the committees having only men.

   $$C(20,5) - C(8,5) - C(12,5) = 15.504 - 56 - 792 = 14.656.$$

   c) There are only 2 possibilities, *either* 2 woman and 3 men *or* 3 woman and 2 men. Therefore we apply both the multiplication principle and the addition principle.

   $$C(12,3)\ C(8,2) + C(12,2)\ C(8,3) = 985$$

Exercises

1. The morse code alphabet consists of the symbols ”•” and ” – ”. A code for a letter or a digit consists of 1 to 5 symbols. How many codes is it possible to create?

2. There are 8 participants in a chess tournament. All must play against all. How many games of chess are there to be played.

3. How many different seating plans are possible, if a company of 7 men and 6 ladies should be seated around a circular table, such that two ladies may not be seated next to each other? How many seating plans are possible, if the company has of 6 men and 6 ladies?

4. In a tasting of 5 types of beer 5 glasses are put in a row op. How many possibilities are there for the place of the glasses?

5. How many 3-ditgit numbers are there, where all the digits are different? How many 3-ditgit numbers are there, where exactly two digits are equal?

6. The registration number (the license plate) of a car (in Denmark) consists of two letters (A-Z) and five digits. E.g. OS 52911 or RU 45710. How many different license plates can be made. Given the two letters, how many plates are there when all the digits are different?

7. As mentioned there are 3^{12} different ways to fill in a coupon (won, tie, lost) playing on the domestic football games. One of the coupons have all 13 correct. How many coupons have 12, 11 10 correct?