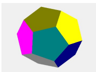
Perturbation Theory



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1. Conventional nondegenrate perturbation theory

Consider a general eigenvalue problem

(1,1) $$Lu = \lambda u$$

Where L can be a regular matrix or a linear differential operator, and u is the eigenvector.
It is well known that for any regular linear operator has a set of mutual orthogonal eigenvectors u_j that form a base for the space of functions. Mutual orthogonal means that $u_i u_j = 0$, for $i \neq j$

If $u_1, u_2, u_3....$ is an orthonormal, $u_j^{\;2} = 1$ base of eigenvector the for (almost) any function ψ, we have:

$$\psi = a_1 u_1 + a_2 u_2 + a_3 u_3 + ...$$

The j'th term is found by taking the dot product with u_j, which gives: $a_j = u_j \cdot \psi / u_j^{\;2} = u_j \cdot \psi$, since $u_i u_j = 0$, for $i \neq j$, and $u_j^{\;2} = 1$

Suppose we have a differential operator L^0, where the eigenvalues $\lambda_n^{\;0}$ and eigenfunctions $u_n^{\;0}$ are known, and we have an operator L that is only slightly different from an operator L_0 .
We then have the two equations:

(1,2) $$L^0 u_n^{\;0} = \lambda_n^{\;0} u_n^{\;0} \qquad \text{and} \qquad Lu = \lambda u$$

We wish to find an approximate solution to the second equation. We put

(1.3) $$L = L^0 + Q\,,$$

where Δ is supposed to be "small" compared to L. And let us assume that there exists a number δ less than one that in some manner represent the order of magnitude of $|Q/L|$

Then we assume that the eigenvalue λ, may be written as a series:

(1,4) $$\lambda_n = \lambda_n^{\;0} + \lambda_n^{\;(1)} + \lambda_n^{\;(2)} + ...$$

The upper index indicate the order of the approximation, like 1, δ , $\delta^{\;2}$ ....

And the eigenfunctioins may have a series expansion:

(1.5) $$u_n = u_n^{\;0} + \sum a_{mn}^{(1)} u_m^0 + \sum a_{mn}^{(2)} u_m^0 + ...$$

Notice that $u_n^{\;0}$ is omitted from the sums,

Where again the upper index indicate the order of approximation δ , $\delta^{\;2}$ ....
We then substitute (1,4) and (1.5) into (1,3).

(1,6) $(L^0+Q)(u_n{}^0+\sum a_{mn}^{(1)}u_m^0+\sum a_{mn}^{(2)}u_m^0)=(\lambda_n{}^0+\lambda_n{}^{(1)}+\lambda_n{}^{(2)}+\ldots)(u_n{}^0+\sum a_{mn}^{(1)}u_m^0+\sum a_{mn}^{(2)}u_m^0)$

The first order terms give:

From our suppositions we have $L^0u_n{}^0=\lambda_n{}^0u_n{}^0$, so we are left with:

(1.7) $Qu_n{}^0+L^0(\sum a_{mn}^{(1)}u_m^0+\sum a_{mn}^{(2)}u_m^0)=(\lambda_n{}^{(1)}+\lambda_n{}^{(2)}+\ldots)(u_n{}^0+\sum a_{mn}^{(1)}u_m^0+\sum a_{mn}^{(2)}u_m^0)$

When we evaluate the expression after doing the operation with L^0, we obtain

$$Qu_n{}^0+\sum a_{mn}^{(1)}\lambda_m^0u_m^0+\sum a_{mn}^{(2)}\lambda_m^1u_m^0+\ldots=(\lambda_n{}^{(0)}+\lambda_n{}^{(1)}+\lambda_n{}^{(2)}+\ldots)(u_n{}^0+\sum a_{mn}^{(1)}u_m^0+\sum a_{mn}^{(2)}u_m^0+\ldots)$$

Keeping only the first order term we find:

(1.8) $$Qu_n{}^0+\sum a_{mn}^{(1)}\lambda_m^0u_m^0=\lambda_n{}^{(0)}u_n{}^0+\lambda_n{}^{(0)}\sum a_{mn}^{(1)}u_m^0$$

If we dot both sides with u_n^0 we obtain, since $u_n^0\cdot u_m^0=0\quad for\ \ n\neq m$

(1.9) $$u_n{}^0\cdot Qu_n{}^0+u_n{}^0\cdot\sum a_{mn}^{(1)}\lambda_m^0u_m^0=\lambda_n{}^{(1)}u_n{}^0+\lambda_n{}^{(1)}\sum a_{mn}^{(1)}u_m^0$$

All the dot products $u_n^0\cdot u_m^0=0\ \ for\ \ n\neq m$, and we are left with.

(1.10) $$u_n{}^0\cdot Qu_n{}^0=\lambda_n^{(1)}$$

This is a very simple result. The first order correction to the eigenvalue can then be written as

$$\lambda_n^{(1)}=u_n{}^0\cdot Qu_n{}^0+O(Q^2)$$

Where "O" is read order of. To Find the first order corrected eigenvector, we dot the first order equation (1.8) with $u_0^p,\quad p\neq n$

$$u_m^0\cdot Qu_n{}^0+u_m^0\cdot\sum a_{mn}^{(1)}\lambda_m^0u_m^0=u_m^0\cdot\lambda_n{}^{(1)}u_n{}^0+u_m^0\cdot\lambda_n{}^{(1)}\sum a_{mn}^{(1)}u_m^0$$

Which gives

$$u_p^0\cdot Qu_n{}^0+u_p^0\cdot\sum a_{mn}^{(1)}\lambda_m^0u_m^0=u_p^0\cdot\lambda_n{}^{(1)}u_n{}^0+u_p^0\cdot\lambda_n{}^{(1)}\sum a_{mn}^{(1)}u_m^0$$

$$u_p^0\cdot Qu_n{}^0+a_{pn}^{(1)}\lambda_p^0=\lambda_n{}^{(0)}a_{pn}^{(1)}\quad\Leftrightarrow$$

$$u_p^0\cdot Qu_n{}^0=(\lambda_n^0-\lambda_p{}^{(0)})a_{pn}^{(1)}\quad\Leftrightarrow\quad\text{Assuming that }\lambda_n^0\neq\lambda_p{}^{(0)}$$

$$a_{pn}^{(1)}=\frac{u_p^0\cdot Qu_n{}^0}{\lambda_n^0-\lambda_p{}^{(0)}}$$

We can see from (1,9) and (1.10) that the perturbation Q may be considered small the numbers $u_p^0 Q u_n^0$ are considered small compared to $\lambda_n^0 - \lambda_p{}^{(0)}$.

(1.11) **Mathematical example.**

Let there be given a matrix operator $L = \begin{pmatrix} 2+\varepsilon & 1-\varepsilon \\ 1-\varepsilon & 2+2\varepsilon \end{pmatrix}$ *where* $\varepsilon << 1$, and

$L^0 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and therefore $Q = \varepsilon \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

The eigenvalues λ_1^0, λ_2^0, are given by the determinant:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad (2-\lambda)^2 - 1 = 0 \quad \Leftrightarrow \quad \lambda = 1 \vee \lambda = 3$$

This correspond to the eigenvectors: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix} = 3\begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix}$

$$2u_1^{(1)} + u_2^{(1)} = u_1^{(1)} \quad \wedge \quad u_1^{(1)} + 2u_2^{(1)} = u_2^{(1)} \qquad \vee \qquad 2u_1^{(2)} + u_2^{(2)} = 3u_2^{(2)} \quad \wedge \quad u_1^{(2)} + 2u_2^{(2)} = 3u_2^{(2)}$$

$$u_1^{(1)} + u_2^{(1)} = 0 \ \wedge \ u_1^{(1)} + u_2^{(1)} = 0 \qquad \vee \qquad 2u_1^{(2)} - 2u_2^{(2)} = 0 \ \wedge \ u_1^{(2)} - u_2^{(2)} = 0$$

They have the solution (1,-1) and (1,1) and normalized:

From the general formalism, we get: $\begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Then by the general formalism:

$$\lambda_1^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\varepsilon\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\varepsilon}{2}\begin{pmatrix} 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \tfrac{5}{2}\varepsilon$$

$$\lambda_2^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\varepsilon\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\varepsilon}{2}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \tfrac{1}{2}\varepsilon$$

So that

$$\lambda_1 = 3 + \tfrac{5}{2}\varepsilon + \ldots \qquad\qquad \lambda_2 = 1 + \tfrac{1}{2}\varepsilon + \ldots$$

Also

$$a_{21}^{(1)} = \frac{u_2 Q u_1}{\lambda_2^0 - \lambda_1^0} = \frac{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\varepsilon\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}}{3-1} = -\frac{\varepsilon}{4}$$

$$a_{12}^{(2)} = \frac{u_1 Q u_2}{\lambda_1^0 - \lambda_2^0} = \frac{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\varepsilon\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}}{1-3} = \frac{\varepsilon}{4}$$

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} - \frac{\varepsilon}{4}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} \approx \frac{1}{\sqrt{2}}\begin{pmatrix}1-\frac{\varepsilon}{4}\\ 1+\frac{\varepsilon}{4}\end{pmatrix}$$

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} + \frac{\varepsilon}{4}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \approx \frac{1}{\sqrt{2}}\begin{pmatrix}1+\frac{\varepsilon}{4}\\ -1+\frac{\varepsilon}{4}\end{pmatrix}$$

Reference: Mathews and Walker Mathematical Methods of Physics