On the construction of ovals
And
The golden ratio

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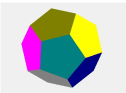

Ole Witt-Hansen 2008

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1. Ovals in Rome

In the spring of 2008 I was functioning as the math teacher for a class on a study tour to Rome.
My task was to discover and reveal the mathematics behind art and architecture.
If you consult the Internet seeking for mathematics together with Rome and architecture, you get several thousand hits, but you get at least as many if you seek on mysteries together with Rome and architecture. (And the are many duplicates)!
Most of the references (if not all) are, however, enlightening, descriptive, without any mathematics. So, as it has happened many times before, I had to do the mathematics myself.

My choice became: Construction of geometrical ovals and the golden ratio.

Geometrical constructions with dividers and rulers have otherwise disappeared completely from the training in mathematics in the Danish high schools. A bit distressing, since it was the geometrical constructions and geometrical proofs that originally raised my interest for mathematics in the ninth grade of grammar school.
In the mid sixties high school geometrical construction was still a part of the curriculum, and the advantage was that it gave an introduction to the significance of a mathematical proof.

The golden ratio has, in later years got a high attention in the Danish high school, (for incomprehensible reasons, since it neither comprises an exact derivation of the number $\frac{\sqrt{5}+1}{2} \approx 1{,}618$, nor the construction of Euclid).
I used to tell my students that the approximate value of the golden ratio is so easy to remember, since the digits correspond to the start of the Thirty-year war in Europe (1618-1648). But nowadays the students have even heard of this war.

Dan Brown refers in his books to 1.618 as a divine number. What a plebeian! Of course a sleazy approximate decimal number can never be divine!

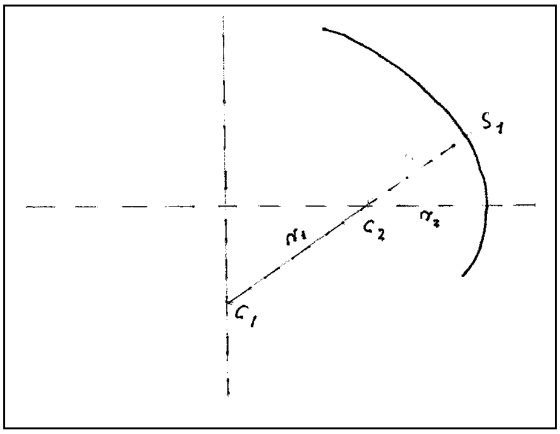
Ovals have been applied in architecture and decorations since antiquity. There exists ovals e.g. Cassinis, Descartes and Lamé's ovals that may be described as "geometrical locations" (as is the case for the circle or the ellipse), but what one usually referred to as ovals, and are seen in numerous renaissance buildings are, however, not geometrical locations, as the conic sections for example, but rather geometrical constructions composed by two pair of circular sections having different radii.

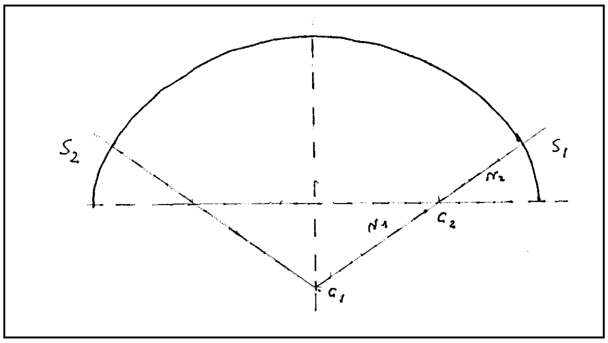
2. Geometrical ovals

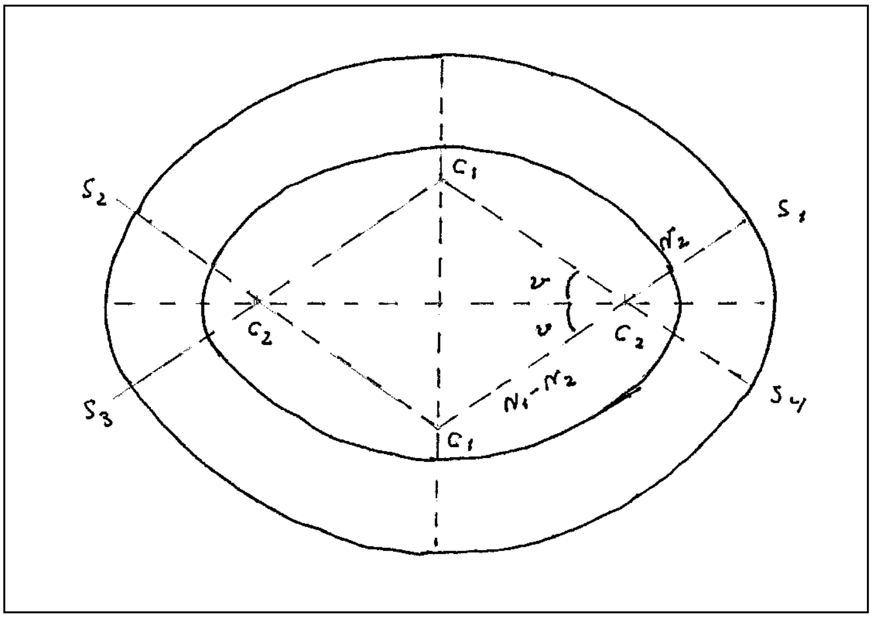
In the literature an on the internet one may find several ingenious methods (with a scent of gothic mystery) to construct certain ovals. From a mathematical point of view, however, the construction of ovals rely on a on a rather trivial geometric fact.
If you should put two circular arcs having radii r_1 and r_2 together to form a differentiable curve, then the centre of the two circles must necessarily lie on the same radial line, ensuring that they have a common tangent in the point, where they connect, and this is actually the only condition necessary to construct an oval.

Below are shown 3 figures, which lead to the construction of an oval.
In the first figure two arcs having centres on the same radial live and having radii r_1 and r_2 are connected at S_1.

When you add the second connecting point S_2, lying symmetric with respect a vertical line, you should continue the larger circular arc to S_2 . The radius of the smaller circle must be unchanged and its centre should lie on a line through the centre C_1 and S_2.
If such a line is drawn, and the smaller circular arc is drawn starting at S_2, then we have constructed half an oval. The lower half of the oval may then be copied from the upper half.







From the figure above, we can realize that the oval also may be constructed more easily from a rhombus which scales (length/width) the oval. We simply draw the diagonals of the rhombus and prolong the sides.
The corners of the rhombus are then the centres of the 4 circular arcs, while the diagonals in the rhombus are the symmetry lines of the oval.
From the figure it is possible to find an expression for half the major axis and half the minor axis.

$$a = r_2 + (r_1 - r_2)\cos v \quad \textit{and} \quad b = r_1 - (r_1 - r_2)\sin v$$

The radii in the circles may be chosen rather arbitrarily, as long as the radius of the big circle arc is lager than the least diameter in the rhombus.

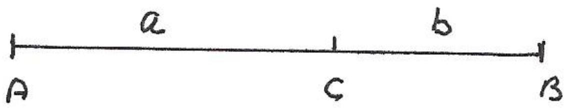
In the figure above, we have drawn two ovals from the same rhombus. What we see is that there is the overall same distance between the two ovals. This is no surprise, however, since the arcs in the two ovals are parts of concentric circles.
This is, however, not a trivial detail, since building an arena there should preferably be the same distance between the rows of the seats round the arena.
This is however *not* the case for two ellipses having the same symmetry axes, and for this reason all arenas back from the antiquity have an oval shape and not an elliptic shape.

For the eye it is almost impossible to distinguish an oval from an ellipse, but when ovals are often preferred in architecture it is for the reason that it is possible to draw many concentric ovals, as already mentioned.
The most famous oval square in Europe is possibly the Peters square at the Vatican in Rome, where there is raised several marble columns in a row along a radius of the oval shape.
If you are sitting in one of the centres of the oval, you will, however, only see one column, because they are placed in a row of at concentric ovals
The other most famous example is of course the arena of Colosseum at the centre of Rome.

3. The golden ratio

The golden ratio is the designation for dividing a line segment having the length $a + b$ into two pieces a and b, which satisfy the relation:

$$\frac{a}{b} = \frac{a+b}{a}$$



The equation can be rewritten as: $\frac{a}{b} = 1 + \frac{b}{a}$.

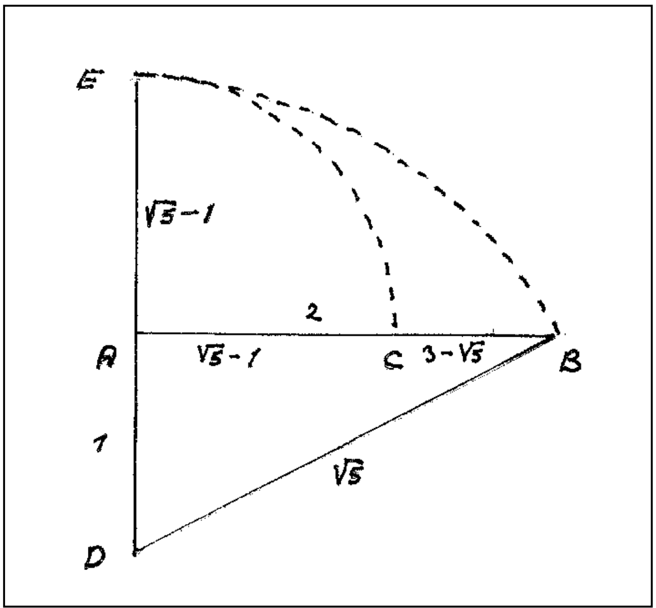
And if we put $\frac{a}{b} = x$, then the golden ratio will satisfy the equation:

$$x = 1 + \frac{1}{x} \iff x^2 - x - 1 = 0 \iff x = \frac{1 \pm \sqrt{5}}{2}.$$

The ratio between the two pieces is : $\frac{a}{b} = x = \frac{1+\sqrt{5}}{2} \approx 1{,}618$.

3.1 A geometrical construction of the golden ratio

The geometrical division of a line section in the golden ratio is attributed to Euclid, and it is not entirely trivial.



The line segment is denoted *AB*. Without restriction we may put $|AB| = 2$.
In the point *A* there is raised a normal to *AB* and the piece 1 is allocated downwards.
The endpoint of the line piece is *D*.
ΔABD is a right angled triangle with the sides (cathetus) 1 and 2. We therefore have:

$$|BD| = \sqrt{1^2+2^2} = \sqrt{5}\,.$$

With centre in *D* and radius $\sqrt{5}$ is drawn a circle. The point where the circle intersects the normal in *A* is *E*. We can see that $|AE| = \sqrt{5}-1$.
Having centre in *A* and radius $|AE|$ is drawn a circle. The intersection with *AB* is *C*.
The assertion is that the point *C* divides *AB* in the golden ratio. We have:

$$|AC| = |AE| = \sqrt{5}-1 \quad \text{and consequently} \quad |CB| = 2 - |AC| = 3-\sqrt{5}\,.$$

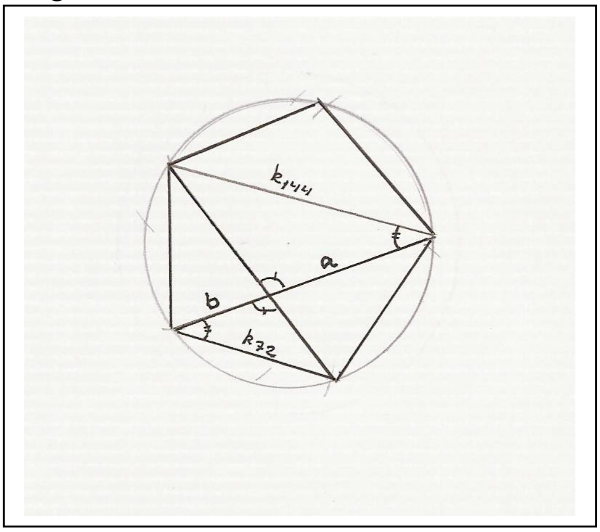
We then evaluate the ratio:

$$\frac{|AC|}{|CB|} = \frac{\sqrt{5}-1}{3-\sqrt{5}} = \frac{(\sqrt{5}-1)(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{5-3+2\sqrt{5}}{3^2-5} = \frac{2+2\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2}$$

Which, we recognize as the golden ratio.

3.2 appearances in mathematics of the golden ratio.

The golden ratio pops up in numerous connections in many sciences (often speculatively), but here we settle for two examples from mathematics.
One may show geometrically that the two diagonals in a pentagon divide each other in a ratio that is the golden ratio.



The side in a pentagon is a chord which spans an angle 72^0 in the circumscribed circle. According to the chord formula:

$$k = 2R\sin\tfrac{v}{2}$$

The chord is therefore: $k_{72} = 2R\cdot\sin 36$.
The diagonal in the pentagon spans over 144^0 and therefore it has the length.

$$k_{144} = 2R\cdot\sin 72.$$

A diagonal in a pentagon is (due to the symmetry) parallel to the opposite side.

The two triangles shown in the figure is therefore even angled, so that the ratio between k_{144} and k_{72} is the same as the ratio between the two pieces a and b.

$$\frac{a}{b}=\frac{k_{144}}{k_{72}} \Leftrightarrow \frac{a}{b}=\frac{2R\sin 72}{2R\sin 36}=\frac{\cos(90-72)}{\sin 36}=\frac{\cos 18}{2\sin 18\cos 18}=\frac{1}{2\sin 18}$$

We have above applied the formula: $\sin 2v = 2\cdot\sin v\cdot\cos v$, but $\sin 18^0$ is already known from the derivation of the length of the chord k_{10} (the chord in a regular decagon).

$\sin 18=\frac{\sqrt{5}-1}{4}$. Inserting this above, we find:

$$\frac{a}{b}=\frac{k_{144}}{k_{72}}=\frac{1}{2\sin 18}=\frac{2}{\sqrt{5}-1}=\frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}=\frac{2(\sqrt{5}+1)}{5-1}=\frac{\sqrt{5}+1}{2}$$ (The golden ratio)

3.3 Connection to the Fibonacci numbers

One may derive the following expression for the nth Fibonacci number:

$$a_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Where $\frac{1+\sqrt{5}}{2}\approx 1{,}618$ is the golden ratio.

In fact one may show that:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1+\sqrt{5}}{2}$$

If we put

$$b=\frac{1+\sqrt{5}}{2} \quad\text{and}\quad c=\frac{1-\sqrt{5}}{2}.$$

Then it follows, since $c<b$.

$$\frac{a_{n+1}}{a_n}=\frac{b^{n+1}-c^{n+1}}{b^n-c^n}=\frac{b-c\left(\frac{c}{b}\right)^n}{1-\left(\frac{c}{b}\right)^n}\to b=\frac{1+\sqrt{5}}{2}\quad \textit{for}\ n\to\infty \qquad \text{since}\ \left(\frac{c}{b}\right)^n\to 0\quad \textit{for}\ n\to\infty$$