Contents

601. Solve for *x*: $\sqrt[3]{4-x^2}+\sqrt{x^2-1}=1$

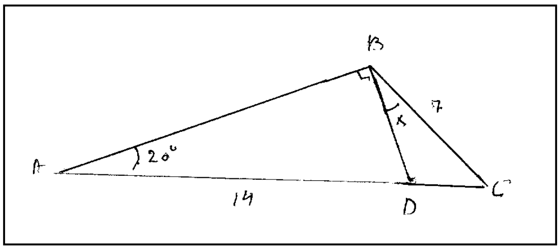
We put: $y=x^2$ and find: $\sqrt[3]{4-y}+\sqrt{y-1}=1$ and we can see that the solution is $y = 5$, so $x=\sqrt{5}$

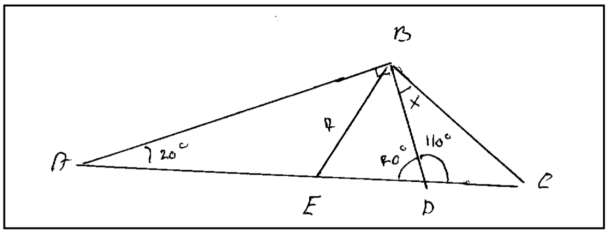
601. Solve for *x*: $\sqrt{x+1}-\sqrt[3]{x}=1$ The solution is obviously $x=8$

602. Determine *x, y* such that: $x^2-y^2=24 \quad and \quad xy=35$

$xy=35$ indicates that a solution could be $x=7 \ \ and \ \ y=5$ and indeed: $7^2-5^2=49-25=24$

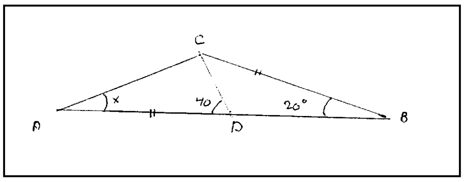
603. Find the angle *x* in the figure below

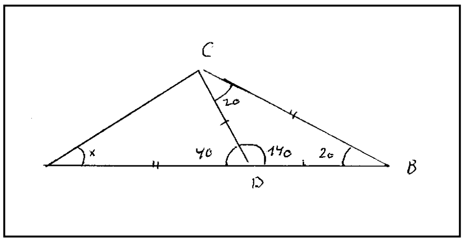


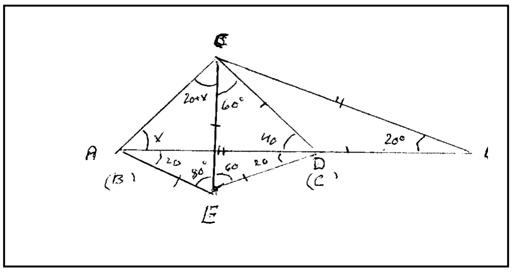


The triangle ABD is a right angle triangle. The side $|BC| = 7$ The angle A is 20^0 and $|AD| = 14$. The angle $D = 90 - A = 70^0$., so the angle $BDC = 110^0$. $|BD|$ is found from:
$|BD|=|AD|\tan 20=14\tan 20=5.10$ A line BE has been drawn such that $|BE|=|BC|=7$ The angles at the base line $|EC|$ are therefore equal. The angle $|BED|$ is found from the sine relations:
$\frac{\sin 70}{|BE|}=\frac{\sin BED}{|BD|} \quad \Rightarrow \quad \sin BED=\frac{0.74\cdot 5.10}{7}= \quad BDE=32.63$
The angle C is also 32.63, and we have that $X=180-(110+32.63)=37.37$

603. Find the angle *x* in the figure below



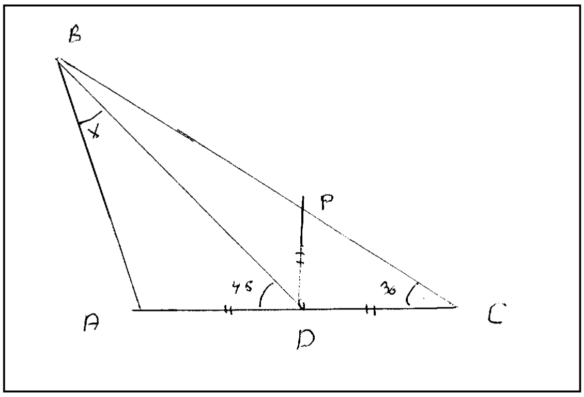


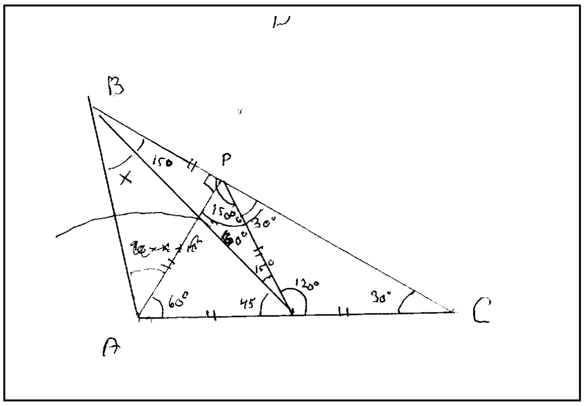


In the triangle ABC the angle $B = 20^0$. A line has been drawn to a point D, such that the angle CDA is 40^0. and such that $|CD|=|DB|$. We use the theorem that the supplement to an angle in a triangle is the sum of the two other angles, that is
$180-C=A+B$. So we have:
$\angle BCD+20^0=40 \quad \Rightarrow \quad \angle BCD=20$
So the triangle BCD is isosceles so that $|CD| = |BD|$.

Since the |*CB*| = |*AD* |. In the third figure, we have moved the triangle *CBD* ,so the sides |*CB*| and |*AD* | are the same.
Since |*CD*| = |*DE*|, the triangle *EDB* is isosceles. The angle *CDE* is $20^0+40^0 = 60^0$ and the angles at the side *CE* is are equal = 60^0. The triangle *CDE* is therefore isosceles. (All sides and angle are equal). The angle *AEC* =140^0 - 60^0 = 80^0. Furthermore the triangle *ACE* is isosceles |*CE*| = |*AE*|.
The angles at the base line *AC* are therefore equal.
$x+20+x+20=180-80 \quad\Rightarrow\quad 2x=60 \quad\Rightarrow\quad x=30$

603. Find the angle *x* in the figure below





Given a triangle *ABC* where the angle *C* = 30^0 and the angle *BDA* = 45^0. Furthermore |*AD*| =|*DC*|.
Since the triangle DPC is isosceles |*DP*|=|*DC*| the angle DPC is also 30^0, and the angle *PDC* is the 180^0 - 60^0 = 120^0
We use the theorem that the supplement to an angle in a triangle is the sum of the two other angles, that is e.g. $180-C=A+B$. Using it on triangle *DPC* we have: The angle *BDP* = 180 – 120 -45 = 15, and the angle *DBC* = 180^0 – 135^0 – 30^0 = 15^0, so the triangle *BDP* is isosceles, so |*PD*| = |*PB*. We draw the line from *A* to *P*. . We then have that triangle *APB* is also isosceles , and the top angle is 180^0 – 60^0 -30^0 = 90^0.
This means that the two angles on the baseline *AB* are equal 45^0 . Therefore
$2(15^0+x)=90^0 \quad\Leftrightarrow\quad x=30^0$

604. Determine *x* and *y* from $x^2+y^2=61 \quad and \quad x-y=11$

$(x-y)^2=x^2+y^2-2xy=121 \quad\Rightarrow\quad 61-2xy=121 \quad\Rightarrow\quad 2xy=-60$

$(x+y)^2=x^2+y^2+2xy=61-60=1 \quad\Rightarrow\quad x+y=\pm1$

$x-y=11 \quad and \quad x+y=1 \quad\Rightarrow\quad x=6 \quad and \quad y=5$

$x-y=11 \quad and \quad x+y=-1 \quad\Rightarrow\quad x=5 \quad and \quad y=6$

604. Determine x^4+y^4 from $x^3+y^3=16 \quad and \quad x+y=4$

We evaluate $4\cdot16=(x+y)(x^3+y^3)=x^4++xy^3+yx^3+y^4=x^4+xy(y^2+x^2)+y^4$

But we need to know: $xy \quad and \quad y^2+x^2$

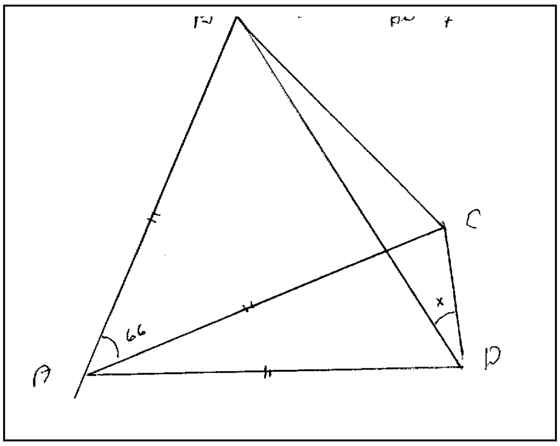
We therefore evaluate $64=(x+y)^3=x^3+y^3+3xy(x+y)=16+12xy \quad\Rightarrow\quad xy=4$

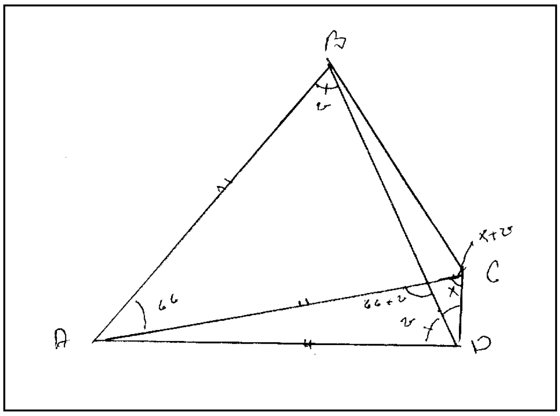
$16=(x+y)^2=x^2+y^2+2xy=x^2+y^2+8 \quad\Rightarrow\quad x^2+y^2=8$

Now we may get: x^4+y^4 from:

$$x^4+xy(y^2+x^2)+y^4=64 \quad \Leftrightarrow \quad x^4+2\cdot 8+y^4=64 \quad \Rightarrow \quad x^4+y^4=48$$

605. Find the angle from the figures below.





O the left is shown on a square ABCD. The sides |AD|, |AC| and |AB| are equal. |AD|=|AC|=|AB|, so the triangles CAD and CAB are isosceles. The angle BAC = 66.
The two angles ABD and BDA are equal to v (Unknown).
From the triangle ACD the angle ADC is v+x, and so is the angle ACD.
E is the crossing point between the lines BD and AC. However the angle AED is the supplement angle to the triangle AEB and it is therefore equal to he sum of the two other angles in triangle AEB that is 66+ v. However the same angle is also supplement to the angle in the triangle: CED and there equal to x + x+ v.

$$66+v=x+x+v \quad \Rightarrow \quad x=33^0$$

606. Determine *x* from: $128x^x=2^{2056}-2^{2055}$

$$2^{2056}-2^{2055}=2^{2055}\cdot 2-2^{2055}=2^{2055}(2-1)=2^{2055} \quad 128=2^7$$

$$128x^x=2^{2056}-2^{2055} \quad \Leftrightarrow \quad x^x=\frac{2^{2055}}{2^7}=2^{2048}, where\ 2048=2^{11}$$

We shall then try to $2^{11}=$write $(2^a)^{2^a}=2^{a\cdot 2^a}$ *where* $a\cdot 2^a=2048=2^{11}$
Since 2048 is a power of 2, so must *a*. It only takes little time to
Verify that *a* = 8. is the solution. So $(2^a)^{2^a}=2^{a\cdot 2^a}=2^{2^3 2^8}=2^{2^{11}}=2^{2048}$

$x^x=(2^8)^{2^8}$ *so* $x=2^8$

607. Determine integer values x, y from $x^5+y^5=211$ *and* $x+y=1$

Clearly *y* must be a negative integer value. Well $2^5=32, 3^5=243$ *and* $4^5=2^{10}=1024$.
So we put our stakes on $x=3$ *and* $y=-2$, and indeed: $x^5+y^5=3^5+(-2)^5=243-32=211$

608. Solve for *x*: $\sin^5 x+\cos^7 x=1$

No complicated algebra is needed since the obvious solution is $x=0$

607. Determine integer values a,b **from** $x^5+y^5=211 \quad and \quad x+y=1$

609. Determine *x* and *y*, such that: $x+y=8 \quad and \quad \frac{1}{x}+\frac{1}{y}=\frac{8}{15}$

This is trivial the solution is: $x=5 \quad and \quad y=3$, since $\frac{1}{3}+\frac{1}{5}=\frac{5+3}{15}=\frac{8}{15}$

610. Determine *a* from $a^3+a^2=7220$ **A qualified guess is 4, since**
$19^3+19^2=6859+361=7220$

611.Solve for *x*: $\sqrt{2^x}\sqrt{7^x}=196 \quad \Leftrightarrow \quad 2^x 7^x=196^2=14^4 \quad \Rightarrow \quad (14)^x=14^4 \quad \Leftrightarrow \quad x=4$

612.Solve for *x*: $x \quad \Leftrightarrow \quad x^{x^2}=2^{1024}$

Well this exercise requires qualified guessing. The solution is x = 16, since:
$(16)^{16^2}=16^{256}=(2^4)^{256}=2^{1024}$

613. Determine integer values of *x, y* such that $x+xy+y=100$

$x+xy+y=100 \quad \Leftrightarrow \quad (x+1)(y+1)-1=100 \quad \Leftrightarrow \quad (x+1)(y+1)=101$
Since 101 allegedly is a prime, we have only the possibilities: $x+1=101 \quad and \quad y+1=1$,or
$x+1=-101 \quad and \quad y+1=-1$ so the solution is x = 100 and y = 0 or x =- 102 and y = -2

613. Determine integer values of *a, b* such that $a^2+ab=21 \quad and \quad b^2+ab=-9$

$a^2+ab=21 \quad and \quad b^2+ab=-9 \quad \Leftrightarrow \quad a(a+b)=21=3\cdot 7 \quad and \quad b(b+a)=-9=-3\cdot 3$
From this we infer that: $\Leftarrow \quad a=3 \quad and \quad a+b=7 \quad or \quad a=7 \quad and \quad a+b=3$
In the former case we find from $a+b=7$ gives b = 4, but this does comply with
$b(b+a)=-9=-3\cdot 3$
The latter where $a=7 \quad and \quad a+b=3$ the solution and gives b = -4. The solution is
therefore $a=7 \quad and \quad b=-4$

613. Determine integer values of *a, b* such that $a^2+b^2=13 \quad and \quad a^3+b^3=35$

Well the first equation can hardly be any other than: $a=2 \quad and \quad b=3$, which also complies with the second equation, since: $a^3+b^3=2^3+3^3 \;=\; 8+27=35$

614. Simplify: $\sqrt{\frac{5353^2-2828^2}{7373+2828}}$

Well, although the appearance of the identity of the two double digits in all the numbers, it took a while before I realized that all such numbers are a multiple of 101. Which is evident when you fist know. For example $5353=53\cdot 101$. Then it is almost trivial, if the problem is properly written.

$$\frac{5353^2-2828^2}{7373+2828}=\frac{(53\cdot101)^2-(28\cdot101)^2}{73\cdot101+28\cdot101}=\frac{(101)^2(53^2-28^2)}{101(73+28)}=\frac{101(53^2-28^2)}{(73+28)}=$$

$$\frac{101(53-28)(53+28)}{(73+28)}=\frac{101(25)(81)}{101}=5^2\cdot9^2$$

So $\sqrt{\dfrac{5353^2-2828^2}{7373+2828}}=45$

615. Determine *a* + *b* from $a^2+b^2=7 \quad and \quad a^3+b^3=10$

$a^3+b^3=(a+b)^3-3ab(a+b)=10$

$a^2+b^2=(a+b)^2-2ab=7 \quad\Rightarrow\quad 2ab=(a+b)^2-7 \quad\Rightarrow$

$(a+b)^3-3ab(a+b)=10 \quad\Leftrightarrow\quad 2(a+b)^3-6ab(a+b)=20 \quad\Rightarrow$

$2(a+b)^3-3((a+b)^2-7)(a+b)=20$

We put $y=(a+b)$ and then we get:

$2y^3-3(y^2-7)y=20 \quad\Leftrightarrow\quad -y^3+21y-20 \quad =0 \quad\Leftrightarrow\quad y^3-21y+20=0$

The integer roots should be found from the integer divisors to 20 that is $\pm1,\pm2,\pm4,\pm5$,

It turns out that: y = -5 is a root, since: -125+105+20=0

To find other roots we make polynomial division with y + 5.

$y+5\,|\,y^3-21y+20\,|\,y^2-5y+4y$

y^3+5y^2

$-5y^2-21y$

$-5y^2-25y$

$4y+20$

$4y+20$

$y^2-5y+4y=0 \qquad d=25-16=9; \quad y=\dfrac{5\pm2}{2} \quad \pm \quad y=\frac{7}{2} \quad or \quad y=\frac{3}{2}$

The integer solution for a + b is a + b =7

616. Determine *a* + *b* + *c* from $ab+c=46 \quad a+bc=64 \;\; and \;\; abc=240$

We can see, that the last equation is satisfied if:

$a=3, \;\; b=10, \;\; and \;\; c=8 \quad or \quad a=4, \;\; b=10, \;\; and \;\; c=6$

If we add the equations: $ab+c=46 \quad and \quad a+bc=64$ we get:

$a+c+ab+bc=110 \quad\Leftrightarrow\quad a+c+b(a+c)=110 \quad\Leftrightarrow\quad (a+c)(b+1)=110=11\cdot10$

For integer solutions, we must therefore have:

$(a+c)(b+1)=11\cdot10 \quad\Leftarrow\quad a+c=11 \;\; and \;\; b+1=10 \quad or \quad a+c=10 \;\; and \;\; b+1=11$

The last possibility has the solution, which comply with: abc = 240.

$a = 4, b = 10,\ and\ \ c = 6$, which gives: $a + b + c = 20;$
$ab + c = 4 \cdot 10 + 6 = 46\ \ and\ \ a + bc = 4 + 6 \cdot 10 = 64$

616. Solve for *x*: $\frac{25^x - 16^x}{25^x + 20^x} = \frac{1}{5}$

$\frac{25^x - 16^x}{25^x + 20^x} = \frac{1}{5} \iff 5(25^x - 16^x) = 25^x + 20^x \iff 5((5^x)^2 - (4^x)^2) = (5^x)^2 + 4^x 5^x$

We divide the equation by $(5^x)^2$ to get:

$$5(1 - \frac{(4^x)^2}{(5^x)^2}) = 1 + \frac{4^x 5^x}{(5^x)^2} \iff 5 - 5\left(\left(\frac{4}{5}\right)^x\right)^2 = 1 + \left(\frac{4}{5}\right)^x$$

We put: $y = \left(\frac{4}{5}\right)^x$ and then we have:

$$5y^2 + y - 4 = 0; \quad d = 1 + 80 = 81 = 9^2; \quad y = \frac{-1 \pm 9}{10} \Rightarrow$$

$$\left(\frac{4}{5}\right)^x = \frac{-1 + 9}{2} \Rightarrow \left(\frac{4}{5}\right)^x = 4 \iff x = \frac{\ln 4}{\ln 4 - \ln 5}$$

617. Determine *x* and *y* from: $x + y = 1\ \ and\ \ x^5 + y^5 = 211$

The solution is: $x = 3\ \ and\ \ y = -2$, since: $x + y = 3 - 2 = 1\ \ and\ \ x^5 + y^5 = 3^5 - 2^5 = 243 - 32 = 211$

618. Solve for *x*: $x^x = 5^{x+25}$

It is rather obvious that the solution must be a power of 5. we try with the power 25.

$$x^x = 5^{x+25} \iff 25^{25} = 5^{2 \cdot 25} = (5^2)^{25} = 25^{25}$$

619. Find the value of the infinite series of radicals: $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{\ ......}}}}}$

This is known as Ramanjuan's famous infinite series. It requires a elegant trick, so it is then no shame, if you can't figure it out for yourself.

We shall for reasons to become obvious consider the function. $f(x) = \sqrt{1 + x(x+2)}$.

It is easy to verify, that the expression in the square root is a perfect square: since:

$1 + x(x+2) = 1 + x^2 + 2x = (x+1)^2$ so: $\sqrt{1 + x(x+2)} = x + 1$

I: $\sqrt{1 + x(x+2)} = x + 1$

In this equation wee replace x by x+1:

II: $\sqrt{1 + (x+1)(x+3)} = x + 2$ we notice that: $1 + (x+1)(x+3) = x^2 + 4x + 4 = (x+2)^2$

The expression (x +2) from II is inserted in I.

III $\sqrt{1+x\sqrt{1+(x+1)(x+3)}}=x+1$

In $\sqrt{1+x(x+2)}=x+1$ we then replace x by x + 2

$\sqrt{1+(x+2)(x+4)}=x+3$, we notice that: $1+(x+2)(x+4)=x^2+6x+9=(x+3)^2$

$\sqrt{1+(x+2)(x+4)}=x+3$ is inserted in III

IV: $\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}}=x+1$

In we replace x by x + 3: $\sqrt{1+(x+3)(x+5)}=x+4$, and this expression is inserted in IV.

$\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)(x+5)}}}}=x+1$

We notice that this recursive insertion can go on …to give:

$$\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)\sqrt{1+(x+4)\sqrt{1+(x+5)\sqrt{1+(x+6)\sqrt{1+...}}}}}}}}=x+1$$

This might be proven by induction, but we abstain from that

But the point is that, if we put x = 2 we get:

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{1+8\sqrt{1+...}}}}}}}}=2+1=3$$

So the repetition of radicals has the sum 3

620. Solve for *x*: $\sqrt{x+1}-\sqrt{\frac{x-1}{x}}=1$

It would be obvious to guess x = 3, 8, but that does not result in a rational number for the second term. So instead we take the square of the equation:

$$\left(\sqrt{x+1}-\sqrt{\frac{x-1}{x}}\right)^2=1^2 \quad\Leftrightarrow\quad (x+1)+\frac{x-1}{x}-2\sqrt{x+1}\sqrt{\frac{x-1}{x}}=1 \quad\Leftrightarrow$$

$$x+\frac{x-1}{x}-2\sqrt{\frac{x^2-1}{x}}=0 \quad\Leftrightarrow\quad \frac{x^2+x-1}{x}-2\sqrt{\frac{x^2-1}{x}}=0 \quad\Leftrightarrow$$

$$\frac{x^2-1}{x}+1-2\sqrt{\frac{x^2-1}{x}}=0$$

We put $y^2 = \frac{x^2-1}{x}$, and then the equation reads: $y^2 + 1 - 2y = 0 \quad \Leftrightarrow \quad (y-1)^2 = 0 \quad \Leftrightarrow \quad y = 1$

We then have $\frac{x^2-1}{x} = 1 \quad \Leftrightarrow \quad x^2 - x - 1 = 0 \qquad x = \frac{1 \pm \sqrt{5}}{2}$

The solution is therefore: $x = \frac{1+\sqrt{5}}{2}$

621. solve for *x*: $x - x^2 - 2x^3 = \frac{1}{3}$

$x - x^2 - 2x^3 = \frac{1}{3} \quad \Leftrightarrow \quad 6x^3 + 3x^2 - 3x + 1 = 0 \quad \Leftrightarrow \quad -6x^3 - 3x^2 + 3x - 1 = 0 \quad \Leftrightarrow$

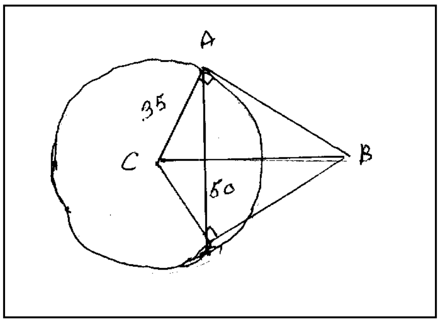
$x^3 - 3x^2 + 3x - 1 = 7x^3$

$(x-1)^3 = x^3 - 1 - 3x(x-1) = x^3 - 1 - 3x^2 + 3x \quad \Leftrightarrow$

So we have:

$(x-1)^3 = 7x^3 \quad \Leftrightarrow \quad x - 1 = \sqrt[3]{7}x \quad \Leftrightarrow \quad x \quad = \frac{1}{1-\sqrt[3]{7}}$

622. Determine $|AB|$ from the figure below.



In the figure to the left the cord $|AD|$ is 50, and the radius $|AC|$ =35. We want to determine the length of AB.
The cord is measured by two times the radius times sine to half of the centre arc it is bend over.

$k = 2R\sin\frac{C}{2}$. From this we find; $\sin\frac{C}{2} = \frac{k}{2R} = \frac{50}{70}$

$\sin\frac{C}{2} = \frac{5}{7}$. Furthermore we have: $\tan\frac{C}{2} = \frac{|AB|}{35}$

So $|AB| = 35\tan\frac{C}{2}$ =35= 35.72

623. Determine integers *a,b,c* such that: $a + \frac{1}{b+\frac{1}{c}} = \frac{37}{16}$

Well we have one equation and three unknowns, so it is clear that we hav to do some qualified guesswork.

If we look at $\frac{1}{b+\frac{1}{c}} = \frac{c}{bc+1}$ then if the denominator should be 16, then bc should be 15, which means that either $b = 3 \quad and \quad c = 5 \quad or \quad b = 5 \quad and \quad c = 3$. This correspond to:

$\frac{c}{bc+1} = \frac{5}{16} \quad or \quad \frac{c}{bc+1} = \frac{3}{16}$

This corresponds to a partition of the rhs. Into $\frac{37}{16} = \frac{32}{16} + \frac{5}{16}$ or $\frac{37}{16} = \frac{34}{16} + \frac{3}{16}$

Since *a* is unbound, we may chose it so the equation is fulfilled, only so a is an integer.

$a = \frac{32}{16} = 2:$ *and* $\frac{1}{b + \frac{1}{c}} = \frac{5}{16} \Leftrightarrow \frac{c}{bc+1} = \frac{5}{16}$

If $a = 2$, $c = 5$ then b=3 does the trick.

623. Solve for *X*: $(x-3)^6 = (3-x)^x$ **The solutions are obviously:** $x = 2$ *or* $x = 6$

624. Solve for *x*: $4^{5^x} = 5^{4^x}$

Since the base of the exponentials are different….there is really no choice than using logarithms.

625. Simplify: $\sqrt[6]{9+4\sqrt{5}}$

$\sqrt[6]{9+4\sqrt{5}} = \sqrt[3]{\sqrt{9+4\sqrt{5}}}$ First we want to simplify $\sqrt{9+4\sqrt{5}}$ writing it as $a + b\sqrt{5}$, so:

$(a+b\sqrt{5})^2 = 9+4\sqrt{5}$. $(a+b\sqrt{5})^2 = a^2 + 5b^2 + 2ab\sqrt{5}$

Easy to see that it works if: a= 2, and b = 1, since $a^2 + 5b^2 + 2ab\sqrt{5} = 4+5+4\sqrt{5} = 9+4\sqrt{5} =$
The next step is only a little harder.

$\sqrt[3]{\sqrt{9+4\sqrt{5}}} = \sqrt[3]{2+\sqrt{5}}$. So $(a+b\sqrt{5})^3$ proportional to: $2+\sqrt{5}$.

$(a+b\sqrt{5})^3 = a^3 + 5\sqrt{5}b^3 + 3ab\sqrt{5}(a+b\sqrt{5}) = a^3 + 15ab^2 + \sqrt{5}(5b^3 + 3a^2b)$

It is not straightforward to make a guess for a and b, but the appearance of second and third powers suggest that they should be small. We therefore make an first with $a = b = 1$.

$(a+b\sqrt{5})^3 = a^3 + 5\sqrt{5}b^3 + 3ab\sqrt{5}(a+b\sqrt{5}) = a^3 + 15ab^2 + \sqrt{5}(5b^3 + 3a^2b) =$

$1 + 15 + \sqrt{5}(5+3) = (16+8\sqrt{5}) = 8(2+\sqrt{5}).$

Therefore we have:

$\sqrt[6]{9+4\sqrt{5}} = \sqrt[3]{\sqrt{9+4\sqrt{5}}} = \sqrt[3]{2+\sqrt{5}} = \frac{1}{2}(2+\sqrt{5})$

626. Solve for *x*: $x^3 + 3x^2 - 12x + 8 = 0$

Possible integer roots should be found among the divisors to 8: $\pm 1, \pm 2, \pm 4, \pm 8$

We can see that x = 1 is a root, since: 1 + 3 – 12 + 8 =0.

We therefore make polynomial division with x -1.

$x - 1 \mid x^3 + 3x^2 - 12x + 8 \mid x^2 + 4x - 8$

$x^3 - x^2$

$4x^2 - 12x$

$4x^2 - 4x$

$-8x + 8$

$-8x + 8$

$x^2 + 4x - 8 = 0;$ $d = 16 + 32 = 48 = 3 \cdot 16;$ $x = \frac{-4 \pm 4\sqrt{3}}{2} = -2 \pm 2\sqrt{3}$

627. Solve for : 7^{x-y}, when given that: $7^{x+1}+7^{y}=13\sqrt{7}$ *and* $7^{x}+7^{y+1}=3\sqrt{7}$

$7^{x+1}+7^{y}=13\sqrt{7} \quad \Leftrightarrow \quad 7\cdot 7^{x}+7^{y}=13\sqrt{7}$

$7^{x}+7^{y+1}=3\sqrt{7} \quad \Leftrightarrow \quad 7^{x}+7\cdot 7^{y}=3\sqrt{7}$

We divide the first equation with the second:

$$\frac{7\cdot 7^{x}+7^{y}}{7^{x}+7\cdot 7^{y}}=\tfrac{13}{3}$$

Then we divide nominator and denominator by on the left side by 7^{y}.

$$\frac{7\cdot\frac{7^{x}}{7^{y}}+1}{\frac{7^{x}}{7^{y}}+7}=\tfrac{13}{3} \quad \Leftrightarrow \quad \frac{7\cdot 7^{x-y}+1}{7^{x-y}+7}=\tfrac{13}{3}$$

And we put $a=\frac{7^{x}}{7^{y}}=7^{x-y}$: $\frac{7a+1}{a+7}=\tfrac{13}{3} \quad \Leftrightarrow 21a+3=13a+91 \quad \Leftrightarrow \quad 9a=88 \quad \Leftrightarrow \quad a=\tfrac{88}{8}=11$

the result is: $7^{x-y}=11$

628. Calculate the infinite sequence of square roots: $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}+\sqrt{\frac{1}{16}+\sqrt{\frac{1}{256}+}}}}$ **...**

We shall show that if we multiply the expression by $\sqrt{2}$ all the numbers in the square roots will be the same, since they are the square of the presiding number, and they all are the square of ½.

$$\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}+\sqrt{\frac{1}{16}+\sqrt{\frac{1}{256}+...}}}}=\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2^2}+\sqrt{\frac{1}{2^4}+\sqrt{\frac{1}{2^8}+...}}}}$$

$$\sqrt{2}\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}+\sqrt{\frac{1}{16}+\sqrt{\frac{1}{256}+...}}}}=\sqrt{2\cdot\frac{1}{2}+2\sqrt{\frac{1}{4}+\sqrt{\frac{1}{16}+\sqrt{\frac{1}{256}+...}}}}=$$

$$\sqrt{2\cdot\frac{1}{2}+\sqrt{\frac{1}{4}\cdot 2^2+\sqrt{\frac{1}{16}\cdot 2^4+\sqrt{\frac{1}{256}\cdot 2^8+...}}}}=$$

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ }...}}}}=$$

So we are left with a infinite sequence of square roots of exactly the same kind:

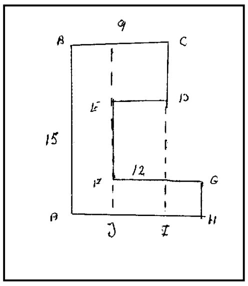
$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ }...}}}}=$ The value of which we denote s.

$s=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ }\ ...}}}}=$ Since this is infinite, we may write:

$s=\sqrt{1+s} \quad \Leftrightarrow \quad s^2=1+s \quad \Leftrightarrow \quad s^2-s-1=0; \quad d=1+4=5 \qquad s\frac{1+\sqrt{5}}{2}$

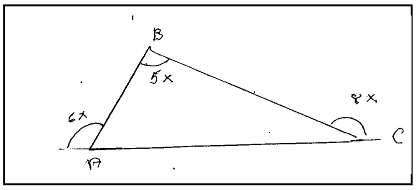
So the original expression $\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}+\sqrt{\frac{1}{16}+\sqrt{\frac{1}{256}+\ \dots}}}}=\frac{s}{\sqrt{2}}=\frac{1+\sqrt{5}}{2\sqrt{2}}=\frac{\sqrt{2}+\sqrt{10}}{4}$

629. Find the perimeter of the figure shown below.



In the figure to the left all angles are 90^0. The length of $|AB|=9$, the length of $|BC|=9$, the length of $|FG|=12$. We shall find the perimeter of the figure.
In the vertical direction, we have $|CD|+|EF|+|GH|=15$.
In the horizontal direction, we have: $9-|ED|+12-|ED|=9$ so $|ED|=6$
The perimeter is then:
Horizontally: $9+|ED|+12+9-|ED|=30$
Vertically: $15+15=30$.
So the perimeter is 60

630. Determine "the angle" X in the figure shown below



In a triangle the sum of the angles are equal to 180^0
In the figure we have:
$A+B+C=180-6x+5x+180-8x=360-9x$

$360-9x=180 \quad \Rightarrow \quad 9x=180 \quad \Leftrightarrow \quad x=20^0$

631. Solve for *x* and *y*: $x^{\log y}+y^{\log x}=2 \quad and \quad x^{\log x}+y^{\log y}=11$

We shall apply the rule: $a^{\log b}=b^{\log a}$ You can see that this rule is valid, if you take the logarithm on both sides:
$\log b\log a=\log a\log b$. So:

$x^{\log y}+y^{\log x}=2x^{\log y}=2 \quad \Rightarrow \quad x^{\log y}=1 \quad \Leftrightarrow \quad x=1 \quad \vee \quad y=1$

$x=1 \quad \Rightarrow \quad 1+y^{\log y}=11 \quad \Leftrightarrow \quad y^{\log y}=10 \quad \Leftrightarrow \quad \log y^{\log y}=\log 10 \quad \Leftrightarrow$

$(\log y)^2=1 \quad \Leftrightarrow \quad \log y=1 \quad \vee \quad \log y=-1 \quad \Leftrightarrow \quad y=10 \quad \vee \quad y=\frac{1}{10}$

Since the equations are completely symmetric in *x* and *y*, we have a similar solution, where *x* and *y* are permuted.

632. Solve for integer values *a* and *b*. $\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}=\frac{1}{\sqrt{20}}$

$\sqrt{20}=2\sqrt{5}$. Since 5 is a prime, $\sqrt{a}$ and $\sqrt{b}$ must both be a multiples of $\sqrt{5}$. If we call these factors *x* and *y*, We then have an equation.

$\frac{1}{x\sqrt{5}}+\frac{1}{y\sqrt{5}}=\frac{1}{2\sqrt{5}} \quad \Leftrightarrow \quad \frac{1}{x}+\frac{1}{y}=\frac{1}{2}$

There are however only two solutions $(x, y) = (4,4)$ and $(x, y) = (3,6)$, since:

$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ and $\frac{1}{3}+\frac{1}{6}=\frac{2}{3}+\frac{1}{6}=\frac{1}{2}$.

This correspond to. $\sqrt{a}=\sqrt{b}=\sqrt{4^2\cdot 5}$ *and* $\sqrt{a}=\sqrt{3^2\cdot 5}\quad=\sqrt{b}=\sqrt{6^2\cdot 5}$

633. Solve for integer values *a* and *b*. $\sqrt{a}+\sqrt{b}=\sqrt{2023}$

We notice that: $\sqrt{2023}=\sqrt{289\cdot 7}=17\cdot\sqrt{7}$, so the equation reads:

$\sqrt{a}+\sqrt{b}=17\sqrt{7}$

Since 7 is a prime, then $\sqrt{a}$ *and* $\sqrt{b}$ must each be a multiple of $\sqrt{7}$. If we denote the factors to $\sqrt{a}$ *and* $\sqrt{b}$ *by* x *and* y the equation reads:

$x\sqrt{7}+y\sqrt{7}=17\sqrt{7} \quad\Leftrightarrow\quad x+y=17$

All non negative pair of integers that fulfils this equation are a solution to the original equation, since: $x\sqrt{7}+y\sqrt{7}=17\sqrt{7} \quad\Leftrightarrow\quad \sqrt{7x^2}+\sqrt{7y^2}=17\sqrt{7}$

$(x,y)=(0,17),(1,16),(2,15),(3,14),(4,13),(5,12),(6,11),(7,10),(8,9)$

634. Simplify: $\sqrt{12321+12456\cdot 12234}$

Well, first we notice that $12321=111^2$ therefore 111 must be? A key number. That is however not so straight forward until you discover that 12456-111= 12234 + 111=12345, so

$$\sqrt{12321+12456\cdot 12234}=\sqrt{111^2+(12345-111)\cdot(12345+111)}=\sqrt{111^2+(12345^2-111^2)}=12345$$

635. Solve the differential equation: $y'y''=y'''$

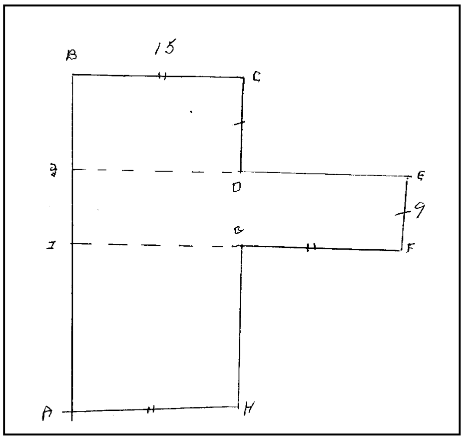
$y'y''=y''' \quad\Leftrightarrow\quad \frac{1}{2}((y')^2)'=y''' \quad\Leftrightarrow\quad \frac{1}{2}(y')^2=y''$

We put: $z=y'$ and find:

$$\tfrac{1}{2}z^2=z' \quad\Leftrightarrow\quad \frac{dz}{z^2}=\tfrac{1}{2}dx \quad\Leftrightarrow\quad -z^{-1}=\tfrac{1}{2}x+c \quad\Leftrightarrow\quad z=-\frac{1}{\frac{1}{2}x+c}$$

$$z=y' \quad\Leftrightarrow\quad \frac{dy}{dx}=-\frac{1}{\frac{1}{2}x+c} \quad\Leftrightarrow\quad y=2\ln(\tfrac{1}{2}x+c)+c_1$$

636. Find the perimeter and area of the geometrical figure below



To the left is shown a geometrical figure with corners *ABCDEFGH.*

Given: $|AC| = |GF| =|AC| =15$ and $|EF| = |CD| = 9$

The perimeter is $2\cdot 15+2\cdot(15+18)+2\cdot 15=126$

The area is found as the 3 rectangles *JCBD, JEFI* and *AIGH.*

$A=15\cdot 9+9\cdot 30+15\cdot 15=630$

637. Solve for *x*: $x^{\log x+1}=2$

$x^{\log x+1}=2 \quad \Leftrightarrow \quad (\log x+1)\log x=\log 2 \quad \Leftrightarrow \quad (\log x)^2+\log x=\log 2$

We put: $y=\log x$ and then the equation reads:

$y^2+y-\log 2=0; \quad d=1+4\log 2; \quad \Rightarrow$

$$y=\frac{-1\pm\sqrt{1+4\log 2}}{2}=\frac{-\log 10\pm\sqrt{\log 10+4\log 2}}{2}=\frac{-\log 10\pm\sqrt{\log 160}}{2}$$

$$\log x=\frac{-\log 10\pm\sqrt{\log 160}}{2} \qquad x=10^{\frac{-\log 10\pm\sqrt{\log 160}}{2}}\ ???$$

678. Find the sum of the infinite fraction. $s=2\sqrt{2}+\cfrac{1}{2\sqrt{2}+\cfrac{1}{2\sqrt{2}+}}$

$$s=2\sqrt{2}+\cfrac{1}{2\sqrt{2}+\cfrac{1}{2\sqrt{2}+}}=2\sqrt{2}+\frac{1}{s} \quad \Rightarrow \quad s^2-2\sqrt{2}-1=0; \quad d=8+4=12$$

$$s=\frac{2\sqrt{2}+2\sqrt{3}}{2}=\sqrt{2}+\sqrt{3}$$

679. Simplify: $\sqrt{17-\sqrt{77}-\sqrt{78-2\sqrt{77}}}$

We shall first look at $78-2\sqrt{77}$, which, we shall try to write as:

$78-2\sqrt{77}=(a-b\sqrt{77})^2=a^2+77b^2-2ab\sqrt{77}$,

Which is easily be fulfilled if $a=b=1$, so $78-2\sqrt{77}=(1-\sqrt{77})^2$

$$\sqrt{17-\sqrt{77}-\sqrt{78-2\sqrt{77}}}=\sqrt{17-\sqrt{77}-(1-\sqrt{77})}=\sqrt{16}=4$$

679. Determine integer *a* such that: $a^3+a^2=7220$

Well: $\sqrt[3]{7220}=19.37$, so we will start with a = 19, and indeed:

$a^3+a^2=68759+361=7220$

680. Solve for *x*: $\sqrt{x+1}-\sqrt[3]{x}=2$ **The solution is obviously *x* =8, since:**

$\sqrt{8+1}-\sqrt[3]{8}=2$

681. Determine non negative integers *a,b,c,d* such that $a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}=\frac{87}{38}$

Trying to manipulate the lhs, does not lead anywhere, after all we have one equations with four unknowns. Instead we shall manipulate the rhs. so it has the same structure as the lhs.

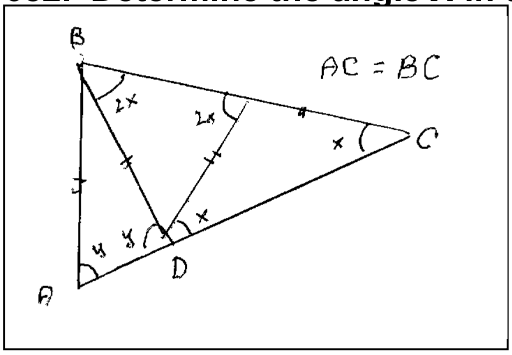
$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}=\frac{87}{38}=2+\frac{11}{38}$, Since a is an integer, then a =2, and $\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}=\frac{11}{38}$ $\Leftrightarrow$

$b+\cfrac{1}{c+\cfrac{1}{d}}=\frac{38}{11}=3+\frac{5}{11}$ From which follows: $b=3$ *and* $c+\frac{1}{d}=\frac{11}{5}=2+\frac{1}{5}$

From which it follows that: $c=2$ *and* $d=5$.

So $a=2, b=3, c=2,$ *and* $d=5$

682. Determine the angle *X* in the figure below.



The angle $C = X$. F is the point between B and C.

$|FC| = |FD| = |BD| = |AB|$, and $|BC| = |AC|$

$\angle CDF = \angle C = X$. $\angle CFD = 180 - 2X \Rightarrow$

$\angle DFB = \angle DBF = 2X \Rightarrow \angle BDF = 180 - 4X$

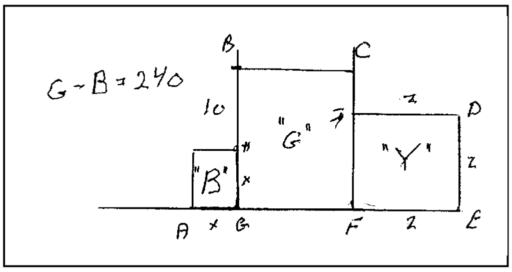
$\angle Y$ is the complement (180 minus) to

$Y = 180 - (X + (180 - 4X)) = 3X,$ *and* $\angle ABD = 180 - 6x$.

Since $|BC| = |AC|$ we must have: $A + B + C = 180$

$3X + 3X + X = 180 \Rightarrow X = \frac{180}{7}$

683. Evaluate the area of the "Yellow" square in the figure below, when the area of the Green minus the blue square is 240.



All squares are true squares, that is, all sides are equal in each square. It is given that $|GE| = 28$ and $|BH| = 10$.

Area of Green minus Area of Blue=

$(28-z)(28-z)-x^2 = 240$ and $28 - z = 10 + x \Rightarrow$

$(10+x)(10+x)-x^2 = 240 \Leftrightarrow$

$100 + x^2 + 20x - x^2 = 240 \quad x = 7 \Rightarrow$

$z = 28 - 10 - x = 11$

So the area of the yellow square is 121.

684. Simplify: $\left(\frac{1}{64}\right)^{-\frac{6}{4}}$

$\left(\frac{1}{64}\right)^{-\frac{6}{4}} = (64)^{\frac{3}{2}} = \left(\sqrt{64}\right)^3 = 8^3 = 512$

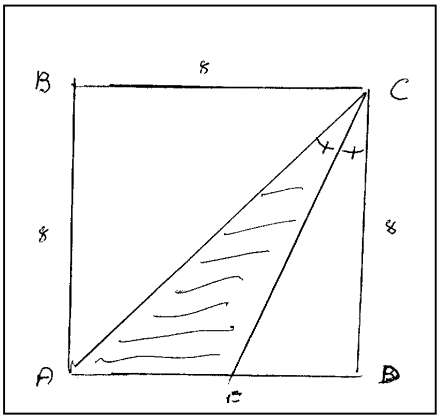
685. Solve for *x*: $(x^2-x-1)^{x+2}=1$. The solution is trivial:

$(x^2-x-1)^{x+2}=1 \quad \Leftrightarrow \quad x+2=0 \quad or \quad x^2-x-1=1 \quad \Leftrightarrow \quad x=-2 \quad or$

$$(x^2-x-1)^{x+2}=1 \quad \Leftrightarrow \quad x+2=0 \quad or \quad x^2-x-1=1 \quad \Leftrightarrow \quad x=-2 \quad or$$

$$x^2-x-2=0; \quad d=1+8=9; \quad x=\frac{1\pm 3}{2} \quad \Leftrightarrow \quad x=2 \quad or \quad x=-1$$

686. Determine the area of the shaded area inscribed in the square below.



The figure shows a triangle *ACE* inscribed in a square *ABCD* with side length 8. A line is drawn from *C* to a point *E*, such that the angle *CAE* = *CED*.

T(*ACE*)= *T*(*ACD*)- *T*(*ECD*).

The angle *ACD* = 90/2 = 45. so the angle X=*ECD*= *ACE*=22½. $T(ACD)=\frac{1}{2}8\cdot 8=32$

$|ED|=8\cdot\tan X$

$T(ECD)=\frac{1}{2}8\cdot|ED|=32\tan X$

$T(ACE)=T(ACD)-T(ECD)=32-32\tan X=$

$32(1-\tan 22\tfrac{1}{2})=32(1-\frac{\sqrt{2}}{2+\sqrt{2}})\approx 18.7$

$$\tan\tfrac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\sqrt{\frac{1-\cos x}{2}}}{\sqrt{\frac{1+\cos x}{2}}}=\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}=\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}.$$

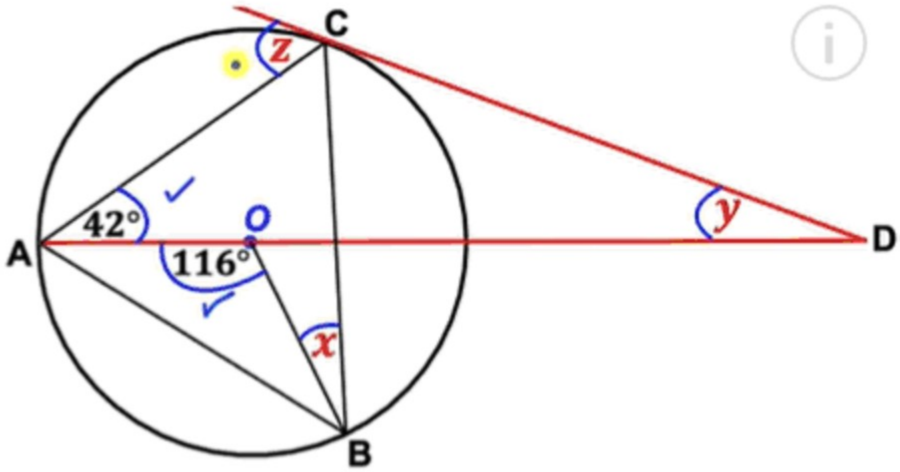
$$\tan\tfrac{45}{2}=\frac{\sqrt{1-\cos 45}}{\sqrt{1+\cos 45}}=\frac{\sqrt{1-\frac{\sqrt{2}}{2}}}{\sqrt{1+\frac{\sqrt{2}}{2}}}=\frac{\sqrt{1-\frac{1}{2}}}{1+\frac{\sqrt{2}}{2}}=\frac{\sqrt{2}}{2+\sqrt{2}}$$

687. Determine $x_1^{\ 3}+x_2^{\ 3}+...x_n^{\ 3}$ **from** $x_1+x_2+...x_n=19$ **and** $x_1^{\ 2}+x_2^{\ 2}+...x_n^{\ 2}=99$

We solve it first for *n* =2.

$$(x_1+x_2)(x_1^{\ 2}+x_2^{\ 2})=19\cdot 99=x_1^{\ 3}+x_2^{\ 3}+2x_1x_2(x_1+x_2)$$

688. Determine the angles X, Y, Z from the figure below



To find the angles, we notice the following geometrical theorems:
A centre angle is measured by the arc it covers. E.g. $\angle O = A\widehat{B} = 116$
$\angle C = \frac{1}{2} A\widehat{B} = \frac{1}{2}116 = 58$

Since ΔAOB is isosceles $|OA| = |OB|$, we have: $OAB = OBA = \frac{1}{2}(180 - AOB) = 32$
$A = 42 + 32 = 74$
Using the triangle: ABC, we may write;

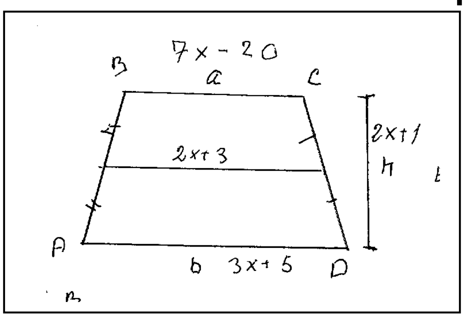
$A + B + C = 180 \quad \Leftrightarrow \quad 74 + 32 + X + 58 = 180 \quad \Rightarrow X = 16 \quad \Rightarrow \quad B = 32 + 16 = 48$

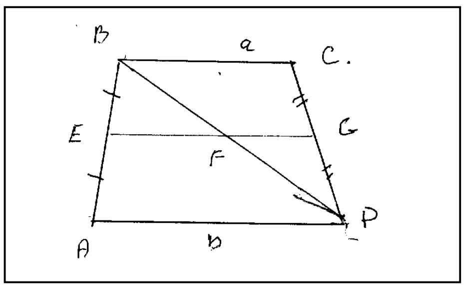
A secant angle or a tangent-secant angle is measured by the half the difference of the arc's it spans.

$B = Z = \frac{1}{2} A\widehat{C} = 48 \quad \Rightarrow \quad Z = 48$

$Y = \frac{1}{2}(A\widehat{C} - C\widehat{E}) = Z - 42 = 48 - 42 = 6 \quad Y = 6$

689. Find the area of the trapezoid





Given the trapezoid above. First we shall find x. Here we shall apply a small theorem from elementary geometry.
A midpoint transversal (a line connecting the midpoint of two sides in a triangle is measured by half the length of the baseline in the triangle.

In the figure to the right we have sketched the trapezoid, and drawn a line *BD*, which separate the trapezoid into two triangles *ABD* and *BCD*. Here *EG* is a midpoint transversal in both of them.
It therefore follows that |*EF*| = ½|*AD*| and |*FG*| = ½|*BC*| Then
|*EF*|+|*FG*|= |*EG*|=½(|*AD*|+|*BC*|)
So the midpoint line is half the sum of the parallel sides in the trapezoid. This we shall apply to find *x*.

$2x+3=\frac{1}{2}(7x-20+3x+5) \quad \Leftrightarrow \quad 4x+6=10x-15 \quad x=\frac{21}{6}=\frac{7}{2}$

The area of a trapezoid =½height times the sum of the parallel side = height times midpoint line.

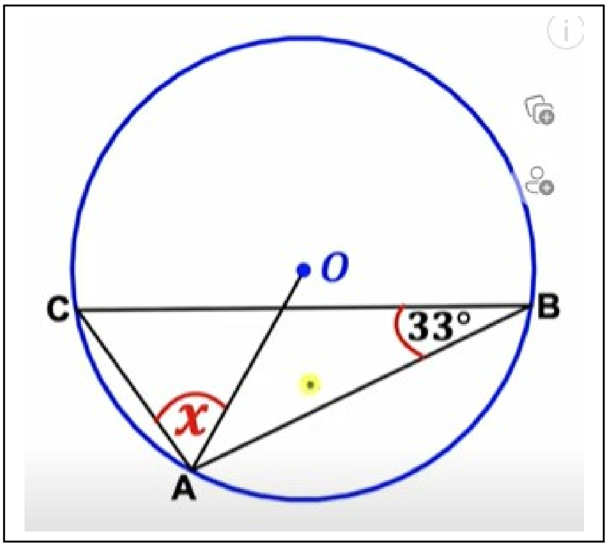
$$T=(2x+1)(2x+3)=(7+1)(7+3)=80$$

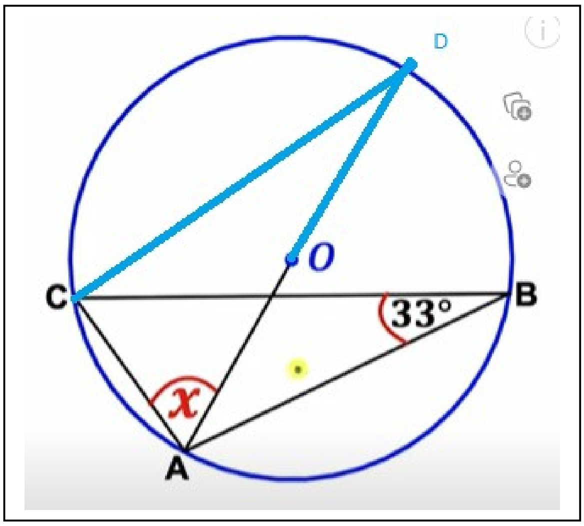
690. Determine integer values *a* and *b* from: $a+ab+b=666$

$a+ab+b=666 \quad \Leftrightarrow \quad (a+1)(b+1)-1=666 \quad \Leftrightarrow \quad (a+1)(b+1)=667$ We have $667=23\cdot 29$
If we require non negative solutions, and both 23 and 29 are primes, we must therefore have:

$a+1=23 \ \ and \ \ b+1=29 \quad \Leftrightarrow \quad a=22 \ \ and \ \ b=28$, and $22+22\cdot 28+28=666$

691. Determine the angle *X* from the figure to the left.





We notice a theorem from geometry:
A periphery angle is measured by half the angle it spans on the circle.
In the figure to the right is drawn two lines from the centre *AO* is prolonged to *D*, and the line *DC* is drawn.
Since *B* and *D* spans the same arc then the angle *ADC* = the angle *B* = 33.
AD = is a diagonal and the angle *ACD* therefore spans 180, so *ACD* = 90.
In the triangle *ACD, D* = 33, *C* = 90, and *CAD* = *X,*
So *X* = 180 – 90 – 33 = 67.

692 . Determine integer values of *a + b* from:

$(a+1)(b+1)(a+b)=2022 \quad and \; a^3+b^3=1933$
$(a+1)(b+1)(a+b)=2022 \quad \Leftrightarrow \quad (a+ab+b+1)(a+b)=2\cdot 3\cdot 337$ (337 is a prime)

If we claim integers, then we must conclude: $(a+ab+b+1)=337 \;\; and \;\; (a+b)=6$.

But there is no way that: $(a+b)=6 \;\; can\ give \;\; a^3+b^3=1933$.

The solution is allegedly $a+b=19$, but this is also strange since $\frac{2022}{19}$ is not an integer, and therefore *a* and *b* cannot be integers.

And $a^3+b^3=1933$ is not possible for any integers $a+b=19$

We may try with an algebraic solution:

$(a+1)(b+1)(a+b)=2022 \quad and \; a^3+b^3=1933$
$(a+1)(b+1)(a+b)=2022 \quad \Leftrightarrow \quad (a+ab+b+1)(a+b)=2022$

We put $a+b=y$, and get:

$(y+ab+1)y=2022 \quad \Rightarrow \quad aby=2022-y^2-y$

$a^3+b^3=(a+b)^3-3ab(a+b)=1933 \quad \Leftrightarrow \quad y^3-3aby=1933$ and $aby=2022-y^2-y$

$y^3-3(2022-y^2-y)=1933 \quad \Leftrightarrow \quad y^3+3y^2+3y=3\cdot 2022+1933 \quad \Leftrightarrow \quad y^3+3y^2+3y=7999$
$y^3+3y^2+3y+1=7999+1 \quad \Leftrightarrow \quad (y+1)^3=8000 \quad \Leftrightarrow \quad y+1=20 \quad \Leftrightarrow \quad y=19 \quad \Rightarrow \quad a+b=19$

However, if we solve $a+b=19$ and $a^3+b^3=1933$ numerically we find:
$a=7.5432 \quad and\ b=11.4568$, so although $a+b=19$, *a* and *b* are not integers!!!!

693. Determine *x and* y from: $(1+x)(1+y)(x+y)=2016 \quad and \quad x^3+y^3=1216$
$(1+x)(1+y)(x+y)=2016 \quad \Leftrightarrow \quad (x+y+xy+1)(x+y)=2016$
$2016=2^5\cdot 3^2\cdot 7$, so there many different ways to factorize, but $x^3+y^3=1216$ has the solution
x = 10 and *y* = 6, since: $10^3+6^3=1216$, and 16 is an divisor in 2016.

So we could hope that: *x* = 10 and *y* = 6 is the solutions, also because there are no other integers that satisfy: $x^3+y^3=1216$, which is seen by inspection of $x=11,12,13$
The problem with "the solution" is however that:

$(x+y+xy+1)(x+y)=(10+6+60+1)(16)=1232 \;\; not \;\; 2016$

I think that there is a error in the problem, so it should read:

$(1+x)(1+y)(x+y)=2016 \quad and \quad x^3+y^3=1232$

$(1+x)(1+y)(x+y)=2016 \quad \Leftrightarrow \quad (x+y+xy+1)(x+y)=2016$

However we may try to solve it algebraically by putting: $z=x+y$
$(x+y+xy+1)(x+y)=2016 \quad \Rightarrow \qquad (z+xy+1)z=2016 \quad \Leftrightarrow \quad z^2+xyz+z=2016$
$x^3+y^3=1216=(x+y)^3-3xy(x+y) \quad \Rightarrow \quad z^3-3xyz=1216$
$xyz=2016-z-z^2$ when inserted in $z^3-3xyz=1216$ gives:

$z^3=1216+3(2016-z^2-z) \quad \Leftrightarrow \quad z^3+3z^2+3z=1216+3\cdot 2016 \quad \Leftrightarrow$
$z^3+3z^2+3z=7264 \quad \Leftrightarrow z^3+3z^2+3z+1=7265 \quad \Leftrightarrow \quad (z+1)^3=7265$???

However 7265 is not a cubic number, so the stated problem must be wrong:

However, if we change $(1+x)(1+y)(x+y)=1232 \quad and \quad x^3+y^3=1216$, we get the solution .
$x+y=16$

694. Solve for *x*. $x^x=3^{1215}$ **:** $1215=5\cdot 3^5, so \quad 3^{1215}=(3^5)^{3^5}$ *so the solution is* $x=3^5$

695. Solve for complex values: $m^4+4=0$

It is obvious to guess at: $m=1+i \quad (i+1)^2=(1-1+2i)=2i; \quad \Rightarrow \quad (2i)^2=-4$,

But $\pm 1\pm i$ are all solutions.

696. Calculate the sum of the sequence $\frac{1}{5}+\frac{1}{45}+\frac{1}{117}+\frac{1}{221}+\frac{1}{357}+\frac{1}{525}$

$$\frac{1}{5}+\frac{1}{45}+\frac{1}{117}+\frac{1}{221}+\frac{1}{357}+\frac{1}{525}$$

We shall first look at $\frac{1}{45}$

$$\frac{1}{45}=\frac{1}{5\cdot 9}=\frac{1}{5(5+4)}=\frac{1}{4}(\frac{1}{5}-\frac{1}{9})=\frac{1}{4}(\frac{9}{5}-\frac{5}{9})=\frac{1}{4}(\frac{4}{45})=\frac{1}{45}$$

And then we observe that we may do the same with the other terms

$$\frac{1}{117}=\frac{1}{9\cdot 13}=\frac{1}{9\cdot(9+4)}=\frac{1}{4}(\frac{1}{9}-\frac{1}{13})$$

$$\frac{1}{221}=\frac{1}{13\cdot 17}=\frac{1}{4}(\frac{1}{13}-\frac{1}{17})$$
$$\frac{1}{357}=\frac{1}{17\cdot 21}=\frac{1}{4}(\frac{1}{17}-\frac{1}{21})$$

$$\frac{1}{525}=\frac{1}{25\cdot 21}=\frac{1}{4}(\frac{1}{21}-\frac{1}{25})$$

And finally

$$\frac{1}{5}=\frac{1}{1\cdot 5}=\frac{1}{4}(\frac{1}{1}-\frac{1}{5})$$

So we have: $\frac{1}{5}+\frac{1}{45}+\frac{1}{117}+\frac{1}{221}+\frac{1}{357}+\frac{1}{525}=$

$$\frac{1}{4}(\frac{1}{1}-\frac{1}{5})+\frac{1}{4}(\frac{1}{5}-\frac{1}{9})+\frac{1}{4}(\frac{1}{9}-\frac{1}{13})+\frac{1}{4}(\frac{1}{13}-\frac{1}{17})+\frac{1}{4}(\frac{1}{17}-\frac{1}{21})+\frac{1}{4}(\frac{1}{21}-\frac{1}{25})=\frac{1}{4}(1-\frac{1}{25})=\frac{1}{4}\frac{24}{25}=\frac{6}{25}$$

This is however a telescopic sum, and the only thing that remains is the first and the last term.

686. Given that: $5^x=\frac{1}{2}^y=10^5$ Determine the value of: $\frac{1}{x}+\frac{1}{y}$

$$5^x=\frac{1}{2}^y=10^5 \quad\Rightarrow\quad 5^x=10^5 \Leftrightarrow \quad 5=10^{\frac{5}{x}} \quad and \quad \frac{1}{2}^y=10^5 \quad\Leftrightarrow\quad \frac{1}{2}=10^{\frac{5}{y}} \quad\Rightarrow$$
$$\frac{5}{2}=10^{5(\frac{1}{x}+\frac{1}{y})} \quad\Rightarrow\quad \log\frac{5}{2}=5(\frac{1}{x}+\frac{1}{y}) \quad\Rightarrow\quad (\frac{1}{x}+\frac{1}{y})=\frac{1}{5}\log\frac{5}{2} \quad\Rightarrow\quad (\frac{1}{x}+\frac{1}{y})=\log\sqrt[5]{\frac{5}{2}}$$

697. Find the sum of the series: $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+....$

To determine the sum, we evaluate the integral:

$\int(1+x+x^2+...)dx=x+\frac{1}{2}x^2+\frac{1}{3}x^3+...$ and we can see that:

$$\frac{1}{x}\int(1+x+x^2+...)dx=\frac{1}{x}(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4...)=1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+....$$

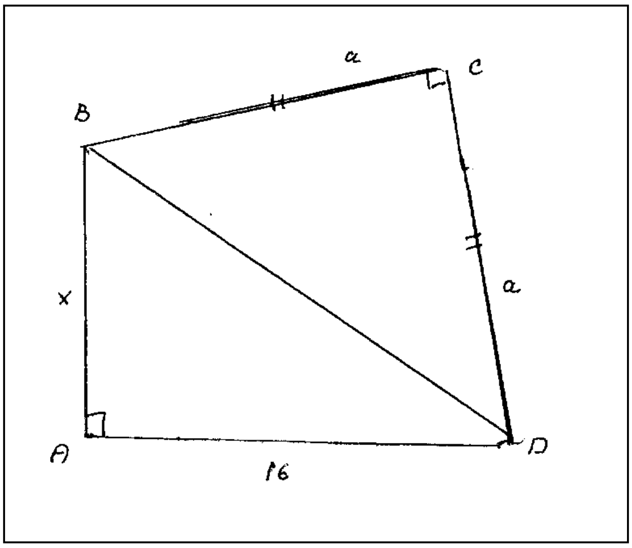
However the integrand is a infinite geometric series, having the sum formula (for $x<1$)

$S=\frac{1}{1-x}$ which is what we find for our sum.

$$\int(1+x+x^2+...)dx=\int\frac{1}{1-x}dx=-\ln(1-x)$$

And the series: $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+....=\frac{1}{x}\int(1+x+x^2+...)dx=\frac{1}{x}\int\frac{1}{1-x}dx=-\frac{\ln(1-x)}{x}$

698. Determine the sides in the square shown below



The area of the square: $T = 169$.
The sides $|BC|=|CD| = a$
The area of the square is the area of the two right angle triangles:

$$T = 169 = \tfrac{1}{2}a^2 + \tfrac{1}{2}16x$$

The two right angle triangles ABD and BCD
Have the common hypotenuse BD, therefore:

$$2a^2 = x^2 + 16^2$$

$169 = \frac{1}{2}a^2 + \frac{1}{2}16x \Rightarrow 2a^2 + 32x = 676$
$2a^2 = 676 - 32x$ is inserted in $2a^2 = x^2 + 16^2$
$x^2 + 16^2 = 676 - 32x \Leftrightarrow x^2 + 32x - 420 = 0$

$x^2 + 32x - 420 = 0; \quad d = 32^2 + 4 \cdot 420 = 2704 = 52^2$

$x = \dfrac{-32 \pm 52}{2} = 10 \qquad a^2 = 2T - 16x = 338 - 160 = 178 \Rightarrow a = \sqrt{178}$

699. Solve for *x*: $\dfrac{x^2+2}{x^4+x^2+1} = 2$??

$\dfrac{x^2+2}{x^4+x^2+1} = 2 \Leftrightarrow x^2 + 2 = 2(x^4 + x^2 + 1) \Leftrightarrow$
$2x^4 + 2x^2 + 2 - (x^2+2) = 0 \Leftrightarrow 2x^4 + x^2 = 0 \Leftrightarrow x^2(2x^2+1) \Leftrightarrow x = 0$

700. Determine *a, b* and *c*, such that $2^a + 2^b + 2^c = 2320$

I may be resolved by qualified guesswork, but we shall make it easier, if we divide the equation by 2 as long as it is a even number.
$2320 = 2 \cdot 1160 = 2 \cdot 2 \cdot 580 = 2 \cdot 2 \cdot 2 \cdot 290 = 2^4 \cdot 145$
Since 2320 is the sum of powers of 2, it must be an even number until one of the powers $2^a + 2^b + 2^c$ is one.
If we therefore solve $2^a + 2^b + 2^c = 2320 \Leftrightarrow 2^{a-4} + 2^{b-4} + 2^{c-4} = 145$, one of the powers must be one.
In principle it is the same problem, but the numbers are much easier to handle.
$2^7 = 128$. $145 - 128 = 17 = 16 + 1$, so we have the equation; $2^7 + 2^4 + 2^0 = 128 + 16 + 1 = 145$
And we get the original equation by multiplying by 2^4: $2^{11} + 2^8 + 2^4 = 2048 + 64 + 16 = 2320$

So $a = 11$, $b = 8$ and $c = 4$

701. Determine *a, b* and *c*, such that: $2^a + 4^b + 8^c = 328$

$2^a + 4^b + 8^c = 328 \Leftrightarrow 2^a + 2^{2b} + 2^{3c} = 328$

For *a*, *b*, *c* positive 328 must be a power of 2. We therefore successively divide the equation by 2. $328 = 2 \cdot 164 = 2 \cdot 2 \cdot 82 = 2 \cdot 2 \cdot 2 \cdot 41$ We then have

$2^{a-3} + 2^{2b-3} + 2^{3c-3} = 41 = 32 + 8 + 1 = 2^5 + 2^3 + 2^0 \Rightarrow a - 3 = 5, 2b - 3 = 3$ *and* $3c - 3 = 0 \Rightarrow$
$a = 8,\ b = 3,\ c = 1$

702. Solve for *x*: $\left(\frac{1}{x}\right)^x = 4^{x+\frac{1}{16}}$

$\left(\frac{1}{x}\right)^x = 4^{x+\frac{1}{16}}$ The equation cannot be solved analytically, but $x = \frac{1}{16}$ seem to do the trick, since:

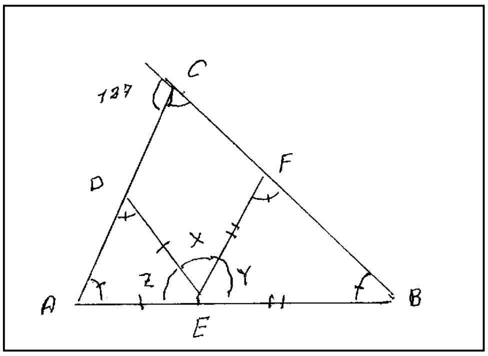
$$\left(\frac{1}{\frac{1}{16}}\right)^{\frac{1}{16}} = 4^{\frac{1}{16}+\frac{1}{16}} \Leftrightarrow 16^{\frac{1}{16}} = 4^{\frac{2}{16}} \Leftrightarrow 16^{\frac{1}{16}} = (4^2)^{\frac{1}{16}} = 16^{\frac{1}{16}}$$

703. Solve for *x*: $x \cdot 3125^x = 1$

$3125 = 5 \cdot 625 = 5 \cdot 25^2 = 5^5$ So we have: $x \cdot 5^{5x} = 1$ And it sees to have the solution $x = \frac{1}{5}$ since

$$\frac{1}{5} \cdot 5^{5 \cdot \frac{1}{5}} = 1 \Leftrightarrow \frac{1}{5} \cdot 5 = 1$$

704. Determine the angle *X* in the figure below.



From the figure we may establish some relations

1. $C = 180^0 - 127^0 = 53^0$
2. $A + B = 180 - C = 127^0$
3. Since ΔEFB is isosceles have: $F = B$
4. Since ΔAED is isosceles we have: $A = D$
5. $X + Y$ is supplementary angle to $Z = AED$, so $X + Y$
   $X + Y = 180 - Z = A + D = 2A$.
   $X + Y = 2A$.
6. $X + Z$ is supplementary angle to $Y = BEF$, so
   $X + Z = 180 - Y = B + F = 2B$
7. The sum of the angels in *DCFE* is 360^0. So
8. $X + 180 - D + 180 - F + 53 = 360 \Rightarrow$
   $X - A - B + 53 = 0$

We thus have the equations:

1. $A + B = 127^0$ 2. $X + Y = 2A$ 3. $X + Z = 2B$ 4. $X + 53 = A + B$ $X + Y + Z = 180$

From the fourth equation we find: $X = A + B - 53 = 127 - 53 = 74$.
But could also be found 1+2+3:

$$2X + Y + Z = 2(A + B) \Leftrightarrow X + 180 = 254 \Leftrightarrow X = 254 - 180 = 74$$

Since we have 5 unknowns B, C, X, Y, Z but only 3 independent equations it is in principle not possible to determine them all.

705. Determine *x* from: $x^{21} + x^{14} = 36$

. $x^{21} + x^{14} = 36 \quad \Leftrightarrow \quad (x^7)^3 + (x^7)^2 = 36$

We put $x^7 = y \quad (y)^3 + (y)^2 = 36$, which is seen to have the solution $y = 3$, since $3^3 + 3^2 = 36$,

So the solution is $x = \sqrt[7]{3}$

704. Find the value of the integral $\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx$

We call the integral: $\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx$ for (A)

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = \int_0^{\frac{\pi}{2}} x d\ln(\sin x) = [x\ln(\sin x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \ln(\sin x)dx = -\int_0^{\frac{\pi}{2}} \ln(\sin x)dx$$

We call the integral: $\int_0^{\frac{\pi}{2}} \ln(\sin x)dx$ for (B) ,

and we therefore have A = -B

There is a theorem: $\int_0^a f(x)dx = \int_0^a f(a-x)dx = [-F(a-x)]_0^a = -(F(0) - F(a)) = F(a) - F(0)$

We make the substitution: $x = \frac{\pi}{2} - u$ in B. This gives:

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)dx = -\int_{\frac{\pi}{2}}^0 \ln(\sin(\tfrac{\pi}{2} - u)du = -\int_{\frac{\pi}{2}}^0 \ln(\cos u)du = \int_0^{\frac{\pi}{2}} \ln(\cos u)du = \int_0^{\frac{\pi}{2}} \ln(\cos x)dx = \text{B:}$$

Then we add: $\int_0^{\frac{\pi}{2}} \ln(\sin x)dx + \int_0^{\frac{\pi}{2}} \ln(\cos x)dx = 2\text{B}$

$$2B = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx + \int_0^{\frac{\pi}{2}} \ln(\cos x)dx = \int_0^{\frac{\pi}{2}} (\ln(\sin x) + \ln(\cos x))dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)dx$$

And make use of the logarithmic rule for the integrands $\ln(ab) = \ln a + \ln b$: $\sin 2x = 2\sin x \cos x$

$$2\text{B} = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)dx = \int_0^{\frac{\pi}{2}} \ln(\tfrac{1}{2}\sin 2x)dx \text{ followed by } u = 2x$$

2B=$\frac{1}{2}\int_0^{\pi}\ln(\frac{1}{2}\sin(u)du = \frac{1}{2}\int_0^{\pi}\ln(\sin(u)du - \frac{1}{2}\left[x\ln\frac{1}{2}\right]_0^{\pi} = \frac{1}{2}\int_0^{\pi}\ln(\sin(u)du - \frac{\pi}{2}\ln 2 = \frac{1}{2}\int_0^{\pi}\ln(\sin(x)dx - \frac{\pi}{2}\ln 2$

Since sin is an even function: $\sin(\pi - x) = \sin x$, we have: $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_{\frac{\pi}{2}}^{\pi} f(\sin x)dx$

2B=$\frac{1}{2}\int_0^{\pi}\ln(\sin(x)dx - \frac{\pi}{2}\ln 2 = \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}}\ln(\sin(x)dx - \frac{\pi}{2}\ln 2 = \int_0^{\frac{\pi}{2}}\ln(\sin(x)dx - \frac{\pi}{2}\ln 2 = B - \frac{\pi}{2}\ln 2$

$B = \int_0^{\frac{\pi}{2}}\ln(\sin x)dx = -\frac{\pi}{2}\ln 2$ = -A

$\int_0^{\frac{\pi}{2}}\frac{x}{\tan x}dx = \frac{\pi}{2}\ln 2$

705. Determine the value of i^i

$i^i = e^{\ln i^i} = e^{i\ln i}$

So we have to find $x = \ln i$.

$x = \ln i \quad \Leftrightarrow \quad i = e^x \quad and \quad e^{ix} = cox + i\sin x \quad \Rightarrow \quad x = \frac{\pi}{2}, so \quad \ln i = i\frac{\pi}{2}$

$i^i = e^{i\ln i} = e^{-\frac{\pi}{2}}$