Contents

501. Determine *x, y* from the equation: $\frac{x^3 - y^3}{x^2 - y^2} = 1$

Well $x^2 \neq x^3$ except for $x = 0$ *or* $x = \pm 1$, so it evident that $x = 1$ and $y = 0$ is a solution.

But we may give a more formal proof:

$$\frac{x^3 - y^3}{x^2 - y^2} = \frac{(x-y)^3 + 3xy(x-y)}{(x-y)(x+y)} = \frac{(x-y)^2 + 3xy}{(x+y)} = 1$$

$$(x-y)^2 + 3xy - (x+y) = 0 \quad \Leftrightarrow \quad (x+y)^2 - xy - (x+y) \quad \Leftrightarrow \quad (x+y)((x+y)-1) - xy = 0$$

$$(x+y)((x+y)-1) = xy \quad \Leftarrow \quad x+y = x \quad and \quad x+y-1 = y \quad \Leftrightarrow \quad x = 1 \quad and \quad y = 0$$

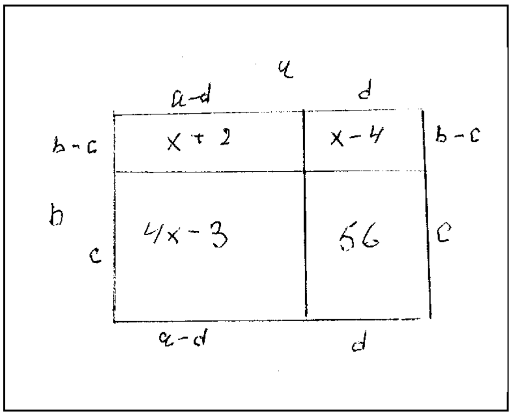
502. Solve for *x*: $x^{\frac{\log(\log(x)}{\log x}} = 2$

We put $y = \log x$ then we have:

$$x^{\frac{\log y}{y}} = 2 \quad \Leftrightarrow \left(10^{\log x}\right)^{\frac{\log y}{y}} = 2 \quad \Leftrightarrow \quad \left(10^{y}\right)^{\frac{\log y}{y}} = 2 \Leftrightarrow$$

$$10^{\log y} = 2 \quad \Leftrightarrow \quad y = 2 \quad \log x = 2 \quad x = 100$$

503. calculate *x* from the rectangle below.



The rectangle below is divided into 4 smaller rectangle, their sides are shown in the figure. Besides *x* we have 4 unknowns *a, b, c, d*. Since we have 1 equation for each of the smaller rectangles, we have 4 equations, so it should in principle be possible to solve

$I: cd = 56 \quad and \; II: \; c(a-d) = 4x - 3$.

$III: \; d(b-c) = x - 4 \quad and \quad IV; \quad (a-d)(b-c) = x + 2$

From I + II: $V: \; ac - 56 = 4x - 3$

From III+IV:

$$VI: \; \frac{(a-d)}{d} = \frac{x+2}{x-4} \quad \Leftrightarrow \quad (x-4)(a-d) = (x+2)d$$

From II+V:

$$\frac{(a-d)(x-4)}{c(a-d)} = \frac{(x+2)d}{4x-3} \quad \Leftrightarrow \quad \Leftrightarrow$$

$$(x-4)(4x-3) = (x+2)56 \quad \Leftrightarrow \quad 4x^2 - 3x - 16x + 12 = 56x + 112 \quad \Leftrightarrow$$

$$4x^2 - 75x - 100 = 0; \quad d = 75^2 + 1600 = 7225 = 85^2$$

$$x = \frac{75 \pm 85}{8} \quad \Leftrightarrow \quad x = \frac{160}{8} = 20 \quad or \quad x = -10/8$$

$$x = 20$$

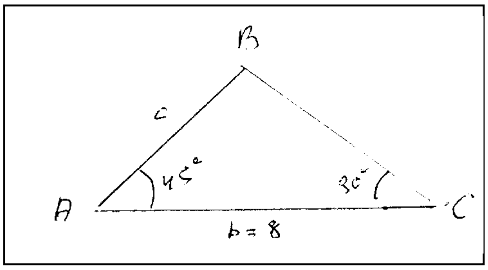
504. Determine *a* and *b* from: $a^2 + 2ab + b = 44$

$a^2 + 2ab + b = 44 \Leftrightarrow a^2 + b(2a+1) = 44 \Leftrightarrow 2a^2 + 2b(2a+1) = 88 \Leftrightarrow$

$a(2a+1) - a + 2b(2a+1) = 88 \Leftrightarrow 2a(2a+1) - (2a+1) + 4b(2a+1) = 176 - 1 \Leftrightarrow$

$(2a+1)(2a-1+4b) = 175 = 7 \cdot 25 \Leftarrow 2a+1 = 7 \;\; and \;\; 2a-1+4b = 25 \Leftrightarrow a = 3 \;\; and \;\; b = 5$

505. Determine the side *c* in the triangle shown below.



The angle *B* is 180-(45+30) = 105.

We apply the sine relations on the triangle *ABC*.

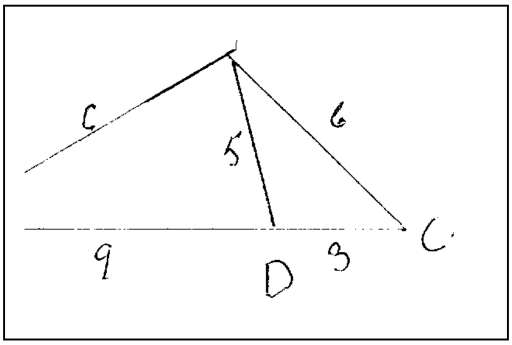
$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow c = \frac{b \sin C}{\sin B}$

sin(180-(45+30))= sin(45+30) $= \sin 45 \cos 30 + \cos 45 \sin 30$

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3} + 1)$

$c = \frac{8 \cdot \frac{1}{2}}{\frac{\sqrt{2}}{4}(\sqrt{3}+1)} = \frac{16}{\sqrt{2}(\sqrt{3}+1)}$

506. Determine the side *c* in the triangle shown below.



We apply the cosine relations on the triangles: *DBC* and *ABC*

$c^2 = a^2 + b^2 - 2ab\cos C \Leftrightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{3^2 + 6^2 - 5^2}{2 \cdot 3 \cdot 6} = \frac{5}{9}$

$c^2 = 12^2 + 6^2 - 2 \cdot 6 \cdot 12 \cos C$

$c^2 = 144 + 36 - 144 \cdot \frac{5}{9} = 180 - 80 = 100$

$c = 10$

507. Determine $8^{5x-6y+7}$ **from** $2^x = 3 \;\; and \;\; 8^y = 72$

$8^y = 72 \Leftrightarrow 2^{3y} = 2^3 \cdot 3^2 = 2^3 \cdot 2^{2x} = 2^{2x+3} \Rightarrow 3y = 2x + 3 \Leftrightarrow 6y = 4x + 6$

$8^{5x-6y+7} = 8^{5x-4x-6+7} = 8^{x+1} = (2^x)^3 \cdot 8 = 3^3 \cdot 8 = 27 \cdot 8 = 216$

508. Determine $\frac{1}{a} + \frac{1}{b}$ **from:** $a^3 + b^3 = 10 \;\; and \;\; a + b = 7$

$a^3 + b^3 = 10 \Leftrightarrow (a+b)^3 - 3ab(a+b) = 10 \Leftrightarrow 7^3 - 3ab \cdot 7 = 10 \Leftrightarrow 21ab = 333 \;\; ab = \frac{111}{7}$

$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{49}{111}$

509. Determine *m* and *n* from: $3^m - 2^n = 1$ **The solution is *m* = *n* = 1**

510. Solve for *x*: $x^3 = (x-1)^3$

$x^3 = (x-1)^3 \Leftrightarrow x = x-1$ which has no solution, but: $x^2 = (x-1)^2$ has the solution $x = \frac{1}{2}$

511. Determine *x* from: $3^{3x} - 3^x = 720$ **The solution is *x* = 2**

512. Sole for *x*: $\log_x 3 = \log_{81} x$

$\log_x 3 = y \quad \Leftrightarrow \quad 3 = x^y \; and \; \log_{81} x = y \quad \Leftrightarrow$

$x = 81^y \quad \Rightarrow \quad 3 = (81^y)^{\,y} = 3^{4y^2} \quad \Rightarrow \quad 4y^2 = 1 \quad y = \frac{1}{2} \quad or \quad y = -\frac{1}{2}$

$3 = x^{\frac{1}{2}} \quad \Rightarrow \quad x = 3^2 \quad and \quad 3 = x^{-\frac{1}{2}} \quad \Rightarrow \quad x = \dfrac{1}{3^2}$

513. Determine *x* and *y* from: $x\sqrt{y} + y\sqrt{x} = 182 \;\; and \;\; x\sqrt{x} + y\sqrt{y} = 183$

We put: $\sqrt{x} = a \quad and \quad \sqrt{y} = b$

$a^2b + b^2a = 182 \quad and \quad a^3 + b^3 = 183 \quad \Leftrightarrow \quad ab(a+b) = 182 \quad and \quad (a+b)^3 - 3ab(a+b) = 183 \quad \Rightarrow$

$(a+b)^3 - 3 \cdot 182 = 183 \quad \Leftrightarrow \quad (a+b)^3 = 729 = 9^3 \quad \Rightarrow \quad a+b = 9$

$ab = \dfrac{182}{9} \quad \Rightarrow \quad b = \dfrac{182}{9a} \quad \Rightarrow \quad a + \dfrac{182}{9a} = 9 \quad \Leftrightarrow \quad 9a^2 - 81a + 182 = 0 \quad ; \quad d = 81^2 - 36 \cdot 182 = 9$

$a = \dfrac{81 \pm 3}{18} \Rightarrow \quad a = \dfrac{14}{3} \quad or \quad a = \dfrac{13}{3} \quad and \quad b = 9 - \dfrac{14}{3} = \dfrac{13}{3} \quad and \quad b = 9 - \dfrac{13}{3} = \dfrac{14}{3}$

$x = a^2 \quad \Rightarrow \quad x = \dfrac{196}{9} \quad and \quad y = \dfrac{169}{9} \quad or \quad x = \dfrac{169}{9} \quad and \quad y = \dfrac{196}{9}$

514. Determine *x* and *y* from: $x^2 - y^2 = 28$ **- An easy one**

$x^2 - y^2 = 28 \quad \Leftrightarrow \quad (x-y)(x+y) = 29$

Since 29 is a prime, we must have: $(x-y) = 1 \quad and \quad (x+y) = 29$ which is easily solved to give:

$2y = 30 \quad \Leftrightarrow \quad y = 15 \quad and \quad x = 14$

515. Solve for x: $3^x = 5^{x^2+3x}$

$3^x = 5^{x^2+3x} \quad \Leftrightarrow \quad x\ln 3 = (x^2+3x)\ln 5 \quad \Leftrightarrow \quad x^2 \ln 5 + x(3\ln 5 - \ln 3) \quad \Leftrightarrow \quad x(x\ln 5 + 3\ln 5 - \ln 3) = 0 \quad \Leftrightarrow$

$x = 0 \quad or \quad x = \dfrac{3\ln 5 - \ln 3}{\ln 5} \quad ????$

516. Determine integer solution to $a^2 + b^2 + c^2 = 294$

The solution is $a = 14, b = c = 7$, since: $14^2 + 7^2 + 7^2 = 196 + 49 + 49 = 294$

517. Determine: $\frac{1}{x}+\frac{1}{y}$ **from** $2^x=(\frac{1}{5})^y=100$

$2^x=100 \quad \Leftrightarrow \quad 2=100^{\frac{1}{x}} \quad and \quad (\frac{1}{5})^y=100 \quad \Leftrightarrow \quad \frac{1}{5}=100^{\frac{1}{y}} \quad \Leftrightarrow \quad 5=100^{-\frac{1}{y}}$

$2\cdot 5=100^{\frac{1}{x}-\frac{1}{y}} \quad \Leftrightarrow \quad 10^2=(100^{\frac{1}{x}-\frac{1}{y}})^2 \quad \Leftrightarrow \quad 10^2=10^{2(\frac{1}{x}-\frac{1}{y})} \quad \Leftrightarrow \quad \frac{1}{x}-\frac{1}{y}=1$

518. Determine *x* and *y* from: $x^2-xy=14 \quad and \quad y^2+xy=60$

Actually we need only one of the two equations to determine x and y-

$x^2-xy=14 \quad \Leftrightarrow \quad x(x-y)=2\cdot 7 \quad \Leftarrow \quad x=7 \quad and \quad x-y=2 \quad \Leftrightarrow \quad x=7 \quad and \quad y=5$

We can see that the second equation is fulfilled with this choice:

$y^2+xy=60 \quad \Leftrightarrow \quad y(y+x)=5\cdot 12=60$

519. Determine *a* /*b* from: $a+b=6\sqrt{ab}$

$a+b=6\sqrt{ab} \quad \Leftrightarrow \quad \sqrt{a}^2-6\sqrt{b}\sqrt{a}+b=0 \quad d=36b-4b=32b$

$\sqrt{a}=\frac{6\sqrt{b}\pm\sqrt{32}\sqrt{b}}{2}=\sqrt{b}(3\pm\sqrt{8}) \quad \Leftrightarrow \frac{\sqrt{a}}{\sqrt{b}}=(3\pm\sqrt{8}) \quad \Rightarrow \quad \frac{a}{b}=(3\pm\sqrt{8})^2$

520. Determine *a* from: $a^2-a^3=392$**. The solution is *a* = -7**

$(-7)^2-(-7)^3=49+343=392$

519. Determine *a, b, c* such that: $a+b+c=9 \quad and \quad a^2+b^2+c^2=29$

The solution is $a=2, b=3, c=4$ since:

$a+b+c=2+3+4=9 \quad and \quad a^2+b^2+c^2=4+9+16=29$

521. Determine (if possible) integers *a, b, c* such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$

This is a strange exercise, since for positive integers: $\frac{1}{a}>\frac{1}{a+b+c}$. Also:

$(a+b+c)\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ The possibilities for which: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ are

$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1 \quad and \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1 \qquad \frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$

So it is evident that: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ is impossible, however, if we allow negative integers, there are three solutions, evidently: $a=-b, \ or \ a=-c \ or \ b=-c$.

We aim to prove that but first we shall look at only two variables. $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$

$\frac{1}{a}+\frac{1}{b}=1$ has the only solution $a=b=2$, this can be seen if we rewrite it as:

$\frac{1}{b}=1-\frac{1}{a} \quad \Leftrightarrow \quad \frac{1}{b}=\frac{a-1}{a} \quad \Leftrightarrow \quad b=\frac{a}{a-1}$ If a -1 is a divisor in a then the only possibility is a = 2.

Again: $\frac{1}{a}>\frac{1}{a+b}$ so it is not possible for positive integers. We have:

$\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b} \quad \Leftrightarrow \quad \frac{a+b}{ab}=\frac{1}{a+b} \quad \Leftrightarrow \quad (a+b)^2=ab$ Which is impossible.

We shall now make a similar proof for $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b+c}-\frac{1}{c}=\frac{c-(a+b+c)}{(a+b+c)c} \quad \Leftrightarrow \quad \frac{a+b}{ab}=\frac{-a-b}{(a+b+c)c} \quad \Leftrightarrow \quad \frac{-a-b}{(a+b+c)c}-\frac{a+b}{ab}=0 \quad \Leftrightarrow$$

$$\frac{a+b}{(a+b+c)c}+\frac{a+b}{ab}=0 \quad \Leftrightarrow \quad \frac{(a+b)ab}{(a+b+c)abc}+\frac{(a+b)(a+b+c)c}{(a+b+c)abc}=0 \quad \Leftrightarrow$$

$$\frac{(a+b)ab+(a+b)(a+b+c)c}{(a+b+c)abc}=0 \quad \Leftrightarrow \quad \frac{(a+b)(ab+(a+b+c)c}{(a+b+c)abc}=0 \quad \Leftrightarrow$$

$$\frac{(a+b)(ab+ac+bc+cc)}{(a+b+c)abc}=0 \quad \Leftrightarrow \quad \frac{(a+b)(a(b+c)+c(b+c)}{(a+b+c)abc}=0 \quad \Leftrightarrow$$

$$\frac{(a+b)(b+c)(a+c)}{(a+b+c)abc}=0 \quad \Leftrightarrow \quad a+b=0 \ \ or \ \ b+c=0 \quad or \quad a+c=0$$

522. Determine *x* from: $x^2-4=94\cdot96\cdot98\cdot100$

$x^2-4=94\cdot96\cdot98\cdot100 \quad \Leftrightarrow \quad (x-2)(x+2)=(96-2)(96+2)(98-2)(98+2) \quad \Leftarrow$

$x-2=(96-2)(98-2) \quad and \quad x+2=(96+2)(98+2) \quad \Rightarrow$

$2x= \ (96-2)(98-2)+(96+2)(98+2) \quad \Rightarrow$

$2x=96\cdot98-2\cdot98-2\cdot96+4+96\cdot98+2\cdot98+2\cdot96+4=2\cdot96\cdot98+8$

$x=96\cdot98+4$

523. Calculate the sum of the following infinite expressions:

$s=\sqrt{1+\sqrt{1+\sqrt{1+...}}}$, $\quad s=\sqrt{1+2\sqrt{1+2\sqrt{1+...}}}$, $\quad s=\sqrt{1+n\sqrt{1+n\sqrt{1+...}}}$ $\quad s=\sqrt{1+2\sqrt{1+3\sqrt{1+...}}}$

$s=\sqrt{1+\sqrt{2+\sqrt{3+...}}}$

$s=\sqrt{1+\sqrt{1+\sqrt{1+...}}}=\sqrt{1+s} \quad \Leftrightarrow \quad s^2=1+s \quad \Leftrightarrow \quad s^2-s-1=0; \quad d=1+5=5 \quad s=\frac{1+\sqrt{5}}{2};$

Known as the golden ratio.

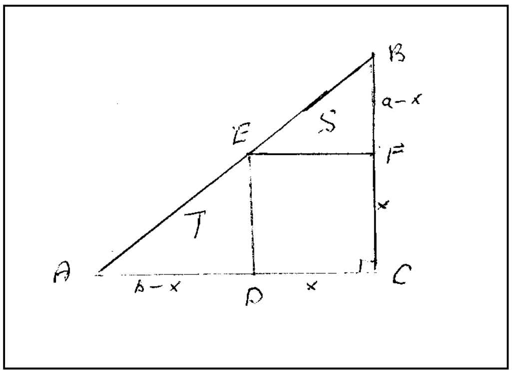
$s = \sqrt{1+2\sqrt{1+2\sqrt{1+...}}} \quad \Leftrightarrow \quad s = \sqrt{1+2s} \quad \Leftrightarrow \quad s^2 - 2s - 1 = 0 \quad s = \frac{2+\sqrt{8}}{2} = 1+\sqrt{2}$

$s = \sqrt{1+n\sqrt{1+n\sqrt{1+...}}} \Leftrightarrow \quad s = \sqrt{1+ns} \quad \Leftrightarrow \quad s^2 - ns - 1 = 0 \quad s = \frac{n+\sqrt{n^2+4}}{2}$

$s = \sqrt{1+2\sqrt{1+3\sqrt{1+...}}}$ is much harder, but we may establish a recursion relation:

We denote: $s_n = n\sqrt{1+s_{n+1}} \quad \Leftrightarrow \quad s_{n+1} = \frac{s_n^{\ 2}}{n^2} - 1$

524. Determine the square expressed by the areas *S* and *T*.



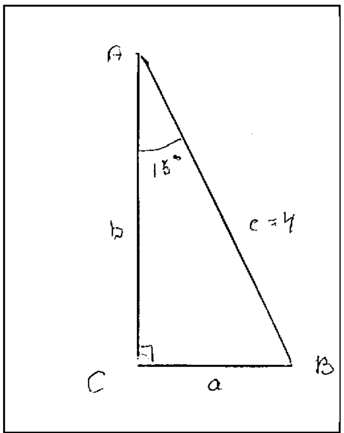
We shall determine the area of the square with side x, expressed by the areas *S* and *T*. With the notation of the figure we have the isosceles triangles AED and EBF .

We can then establish:

$\frac{b-x}{x} = \frac{x}{a-x} \quad and \quad x(a-x) = 2S \quad and \quad x(b-x) = 2T \quad \Rightarrow$

$\frac{2T}{x^2} = \frac{x^2}{2S} \quad \Rightarrow \quad x^4 = 4ST \quad \Rightarrow \quad x^2 = 2\sqrt{ST}$

524. Determine the area of the triangle shown below.



From the triangle follows: $B = 75^0$. From the formulas for the right angle Triangle follows:

$\tan A = \frac{a}{b} \quad \Rightarrow \quad a = b\tan 15$

$a^2 + b^2 = c^2 \quad \Rightarrow \quad (b\tan 15)^2 + b^2 = 16 \quad \Rightarrow b^2(1+\tan^2 15) = 16$

$T = \frac{1}{2}ab = \frac{1}{2}\tan 15 b^2 = \frac{1}{2}\tan 15 \frac{16}{1+\tan^2 15}$

525. Determine *x* from the equation: $49^x - 42^x = 36^x$

$49^x - 42^x = 36^x \quad \Leftrightarrow \quad (7^x)^2 - 6^x \cdot 7^x - (6^x)^2 = 0$

We divide the equation by $6^x \cdot 7^x$ to get: $\left(\frac{7}{6}\right)^x - 1 - \left(\frac{6}{7}\right)^x = 0$

We put: $y = \left(\frac{7}{6}\right)^x$ and then we have:

$$y-1-\frac{1}{y}=0 \quad \Leftrightarrow \quad y^2-y-1=0 \quad d=5 \quad y=\frac{1\pm\sqrt{5}}{2}$$

$$x=\frac{\ln\left(\frac{1+\sqrt{5}}{2}\right)}{\ln 7-\ln 6}$$

526. Solve for *x*: $2^x=4x$ **The solution is evidently *x* = 4.**

527. Determine *x* and *y* from: $x^2-xy=12 \quad and \quad y^2+xy=40$

Actually we need only one of the two equations to determine x and y.

$x^2-xy=12 \quad \Leftrightarrow \quad x(x-y)=2\cdot 6=3\cdot 4 \quad \Leftarrow \quad x=4 \quad and \quad x-y=3 \quad \Leftrightarrow \quad x=4 \quad and \quad y=1$

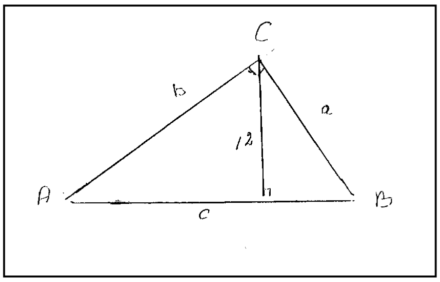
But the second equation is not fulfilled with this choice. So we try

$$x^2-xy=12 \quad \Leftrightarrow \quad x(x-y)=2\cdot 6 \quad \Leftarrow \quad x=6 \quad and \quad x-y=2 \quad \Leftrightarrow \quad x=6 \quad and \quad y=4$$

We can see that the second equation is fulfilled with this choice, since:

$$y^2+xy=16+24=40 \quad \Leftrightarrow \quad y(y+x)=5\cdot 12=60$$

526. Determine the sides *a, b, c* in the triangle shown below.



Furthermore the perimeter is $a+b+c=60$,
The area $2T$ can be calculated in two ways as: $ab=12c$

$a+b=60-c \quad \Rightarrow \quad (a+b)^2=(60-c)^2 \quad \Leftrightarrow$

$a^2+b^2+2ab=3600+c^2+120c$

Since $a^2+b^2=c^2$ and $ab=12c$, we find:

$$c^2+24c=3600+c^2-120c \quad \Leftrightarrow 144c=3600 \quad \Leftrightarrow \quad c=\frac{3600}{144}=25$$

527. Solve for *x* and *y*: $\ln x\ln y=21 \quad and \quad xy=e^{10}$

$\ln x\ln y=21 \quad and \quad xy=e^{10} \quad \Leftrightarrow \ln x\ln y=21 \quad and \quad \ln x+\ln y=10$

We put $a=\ln x \quad and \quad b=\ln b$ and find;

$$ab=21 \quad and \quad a+b=10 \quad \Leftrightarrow \quad a+\frac{21}{a}-10=0 \quad \Leftrightarrow \quad a^2-10a+21=0 \quad d=100-84=16$$

$a=\frac{10\pm 4}{2} \quad \Leftrightarrow \quad a=7 \quad or \quad a=3 \quad \Rightarrow \quad \ln x=7 \quad and \quad \ln y=3 \Leftrightarrow \quad x=e^7 \quad and \quad y=e^3 \quad or$

$\ln x=3 \quad and \quad \ln y=7 \quad \Leftrightarrow \quad y=e^7 \quad and \quad x=e^3$

528. Solve the differential equation: $f'(x) = f^{-1}(x)$

When a function has an inverse function, we have:

$$y = f(x) \quad \Leftrightarrow \quad x = f^{-1}(y) \quad and \quad (f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{or}$$

$$(f^{-1}(y))' = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

We replace y with x.

$$(f^{-1}(x))' = \frac{1}{f'(y)} = \frac{1}{(f^{-1}(x))'}$$

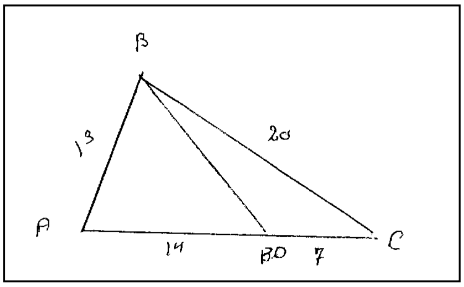
If $f'(x) = f^{-1}(x)$ We then get:

$$(f'(x))' = \frac{1}{f'(f^{-1}(x))} \quad \Leftrightarrow \quad f''(x) = \frac{1}{f'(f'(x))} \quad \Leftrightarrow \quad f''(x) = \frac{1}{f''(x)} \quad \Leftrightarrow$$

$$f''(x)f''(x) = 1 \quad \Leftrightarrow f''(x)^2 = 1 \quad \Leftrightarrow \quad f''(x) = \pm 1 \quad \Leftrightarrow$$

$$f'(x) = \pm x + c \quad \Leftrightarrow \quad f(x) = \pm(\tfrac{1}{2}x^2 + cx) + d$$

529. Find the length of |BD| from the triangle below.



We may either write two expression for cos$A$ *or cosC,* we choose cos$A$.
The cosine-relation for a triangle ABC is

$$c^2 = a^2 + b^2 - 2ab\cos C \quad or$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

From the triangles ABD and ABC: we find:

If we solve for $|BD|^2$ we find: $|BD|^2 = 225 \Rightarrow |BD| = 15$

530. Simplify: $\dfrac{\sqrt{15} + \sqrt{35} + \sqrt{21} + 5}{\sqrt{15} + 2\sqrt{5} + \sqrt{7}}$

$$\frac{\sqrt{15} + \sqrt{35} + \sqrt{21} + 5}{\sqrt{15} + 2\sqrt{5} + \sqrt{7}} = \frac{\sqrt{3}\sqrt{5} + \sqrt{5}\sqrt{7} + \sqrt{3}\sqrt{7} + \sqrt{5}\sqrt{5}}{\sqrt{3}\sqrt{5} + 2\sqrt{5} + \sqrt{7}} =$$

$$\frac{\sqrt{3}(\sqrt{5} + \sqrt{7}) + \sqrt{5}(\sqrt{7} + \sqrt{5})}{(\sqrt{3} + \sqrt{7}) + (\sqrt{5} + \sqrt{7})} = \frac{(\sqrt{5} + \sqrt{7})(\sqrt{3} + \sqrt{5})}{(\sqrt{3} + \sqrt{7}) + (\sqrt{5} + \sqrt{7})} = \frac{ab}{a + b}$$

531. Solve for *x*. $x^3 - 4x^2 + 5x - 2 = 0$

Although it is very easy two guess a root, we shall introduce the theory by guessing roots.

1. Factorizing a quadratic polynomial.

A number r is said to be a *root* in a polynomial $p(x)$ if $p(r) = 0$.

Determining the roots of a polynomial is the same as finding the intersection of $p(x)$ with the x-axis. We shall then show some theorems about the roots in a quadratic polynomial, which as we know, may have two roots if $d > 0$, only one root if $d = 0$ or no roots if $d < 0$.

First we look at the case where $d > 0$, where the polynomial has two roots r_1 and r_2.

r_1 and r_2 are roots in: $p(x) = ax^2 + bx + c \quad \Leftrightarrow \quad ax^2 + bx + c = 0 \quad \Leftrightarrow$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \Leftrightarrow \quad x = r_1 \ \vee \ x = r_2 \quad \Leftrightarrow$$

$$x - r_1 = 0 \ \vee \ x - r_2 = 0 \quad \Leftrightarrow \quad (x - r_1)(x - r_2) = 0$$

By multiplying the parentheses and collecting the terms, we have:

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

If we compare it to: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, we see that we must have:

$$r_1 + r_2 = -\frac{b}{a} \quad og \quad r_1 \cdot r_2 = \frac{c}{a}$$

This can be formulated as:

In the ordered (by decreasing powers of x) and reduced (the coefficient to x^2 is 1) quadratic equation, the sum of the roots is equal to the coefficient to x with opposite sign, and the product of the roots is equal to the last term in the equation.

This theorem is often used to guess the roots in a quadratic equation

Example

1) Guess the roots in the quadratic equation: $x^2 + 2x - 15$. We should think of two numbers having the sum -2 and the product -15. The only possibility is 3 and -5 (Since the equation can have at most 2 roots)

2) If the roots are not integral numbers it is only a little more difficult.
   $x^2 + \frac{3}{2}x - 1 = 0$. We can see that: $-2 + \frac{1}{2} = -\frac{3}{2}$ *and* $-2 \cdot \frac{1}{2} = -1$, so the roots are -2 and $\frac{1}{2}$.

We have established above that:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x - r_1)(x - r_2)$$

Multiplying the equation by a, we find:

$$ax^2 + bx + c = a(x - r_1)(x - r_2)$$

This is called the *factorization* of the quadratic polynomial, and it is a special case of a more comprehensive theorem about factorization of higher degree polynomials .

2. A theorem about the sum and product of the roots in a third degree polynomial.

A third degree polynomial can be written as: $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

The polynomial is said to be normalized if $a_3 = 1$ in which case we have:

$p(x) = x^3 + a_2x^2 + a_1x + a_0$

Clearly, if α is a root in the normalized polynomial, so it is in the polynomial itself.

Let us assume that the polynomial has three roots: $\alpha_1, \alpha_2, \alpha_3$ then:

$$x^3 + a_2x^2 + a_1x + a_0 = 0 \quad \Leftrightarrow \quad x = \alpha_1 \quad \vee \quad x = \alpha_2 \quad \vee \quad x = \alpha_3 \quad \Leftrightarrow$$

$$x - \alpha_1 = 0 \quad \vee \quad x - \alpha_2 = 0 \quad \vee \quad x - \alpha_3 = 0 \quad \Leftrightarrow$$

$$(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) = 0 \quad \Leftrightarrow$$

$$(x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2)(x - \alpha_3) = x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)x - \alpha_1\alpha_2\alpha_3$$

Which leads to the theorem:

In a normalized and ordered 3. degree equation, having 3 roots, the sum of the roots is equal to the coefficient to x^2 with opposite sign. And the product of the roots is the constant term with opposite sign.

2.1 Example

A normalized polynomial having the roots $\alpha_1 = 1,\ \alpha_2 = -2,\ \alpha_3 = 3$ *can be written as:*

$p(x) = x^3 - 2x^2 - 5x + 6$

It can be verified directly that $\alpha_1 = 1,\ \alpha_2 = -2,\ \alpha_3 = 3$ *are roots, and that*

$\alpha_1 + \alpha_2 + \alpha_3 = 2$ *and* $\alpha_1\alpha_2\alpha_3 = -6$.

A similar theorem can be obtained from polynomial of higher degree, but they are not interesting, since it does not help much to guessing the roots.

However, as it is shown in the section of polynomials that integral roots are divisors in the constant term of the polynomial.

$x^3 - 4x^2 + 5x - 2 = 0$ if the roots are denoted: r_1, r_2, r_3 we have according to the above theorem:

$r_1 + r_2 + r_3 = 4$ *and* $r_1 \cdot r_2 \cdot r_3 = 2$ which has the only solution $r_1 = 2,\ r_2 = 1,\ r_3 = 1$.

552. Determine *a, b, c* such that: $a + b + c = 9$ *and* $a^2 + b^2 + c^2 = 27$

The solution is $a = b = c = 3$.

553. Solve for *x*: $x^{x^{x^3}} = 3$ **The solution is** $\sqrt[3]{3}$ **since:** $(\sqrt[3]{3})^3 = 3$

553a. Solve for *x*: $\left(\frac{x}{6}\right)^x = 6^{6^2}$ **The solution is:** $x = 6^2$**, since** $\left(\frac{6^2}{6}\right)^{6^2} = 6^{6^2}$

554. Solve for *x*: $(2+\sqrt{3})^x + (2-\sqrt{3})^x = 4$

$(2+\sqrt{3})^x\,(2-\sqrt{3})^x = (2^2 - \sqrt{3}^2)^x = 1^x = 1$

We put $a = (2+\sqrt{3})^x$ *and* $b = (2-\sqrt{3})^x$ and then we have:

$a+b=4 \quad and \quad ab=1 \quad \Rightarrow \quad a+\frac{1}{a}=4 \quad \Leftrightarrow \quad a^2-4a+1=0 \;\; ;d=16-4=12$

$a=\frac{4\pm 2\sqrt{3}}{2} \quad \Leftrightarrow \quad a=2+\sqrt{3} \;\; or \;\; a=2-\sqrt{3} \quad \Rightarrow \quad (2+\sqrt{3})^x = 2+3\sqrt{3} \quad \Leftrightarrow \quad x=1 \;\; or$

$(2-\sqrt{3})^x = 2-3\sqrt{3} \quad \Leftrightarrow \quad x=1$

255. Solve for *x*: $2^{\frac{3x-2}{x-1}} + 3^{\frac{2x-1}{x-1}} = 43$

$2^{\frac{3x-2}{x-1}} + 3^{\frac{2x-1}{x-1}} = 43$ We put $a = \frac{3x-2}{x-1}$ *and* $b = \frac{3x-2}{x-1}$, and then we have: $2^a + 3^b = 43$

The last equation evidently has the solution $a=4 \;\; and \;\; b=3$

$\frac{3x-2}{x-1}=4 \quad and \quad \frac{2x-1}{x-1}=3 \quad \Leftrightarrow \quad 3x-2=4x-4 \quad and \quad 2x-1=3x-3 \quad x=2 \quad and \quad x=2$

556. Determine the sides a,b,c **in a right angle triangle with the perimeter** $a+b+c=24$ **and the area** $T=\frac{1}{2}ab=24$**.**

We have:

$a+b+c=24 \;\Rightarrow\; (a+b)^2=(2\Leftrightarrow 4-c)^2 \quad \Leftrightarrow \quad a^2+b^2+\;2ab=\;576+c^2-48c \quad \Leftrightarrow$

$c^2+96=576+c^2-48c \quad \Leftrightarrow \quad 48c=480 \quad \Leftrightarrow \quad c=10. \quad a+b+c=24 \quad \Leftrightarrow \quad a+\frac{48}{a}+10=24 \quad \Leftrightarrow$ 257.

$a^2-14a+48=0 \quad ;d=196-192=4. \quad a=\frac{14\pm 2}{2} \quad a=8 \;\; or \;\; a=6 \quad \Rightarrow \quad b=24-a-c$

$b=6 \;\; or \;\; b=8 \quad c=10$

Solve for x: $\left(\frac{x}{6}\right)^x = 6^{6^2}$ **The solution is:** $x=6^2$**, since:** $\left(\frac{6^2}{6}\right)^{6^2} = 6^{6^2}$

558. Determine *a* and *b*, such that: $5^a + 5^b = 130$**. The solution is:** $a = 3 \quad and \quad b = 1$

559. Solve for *x*: $\cos x - \cos 2x = \frac{1}{2}$ **We have:** $\cos 2x = 2\cos^2 x - 1$ **So we have:**

$\cos x - 2\cos^2 x + 1 = \frac{1}{2} \quad \Leftrightarrow \quad 2\cos^2 x - \cos x - \frac{1}{2} = 0 \quad we \;\; put \;\; y = \cos x$

$2y^2 - y - \frac{1}{2} = 0 \qquad d = 1 + 4 =$

$5 \quad y = \dfrac{1 \pm \sqrt{5}}{4} \quad \Leftrightarrow \quad y = \cos x = \dfrac{1+\sqrt{5}}{4} \quad or \quad y = \cos x = \dfrac{1-\sqrt{5}}{4}$

$x = \pm 0.6283 + 2p \quad or \quad x = \pm 1.8850 + 2p$

560. Solve for *x*: $\dfrac{1}{x} + \dfrac{1}{x+4} = x + 2$

$\dfrac{1}{x} + \dfrac{1}{x+4} = x + 2 \quad \Leftrightarrow \quad x + 4 + x = x(x+2)(x+4) \quad \Leftrightarrow \quad 2(x+2) = x(x+2)(x+4) \quad \Leftrightarrow$

$x + 2 = 0 \;\vee\; x^2 + 4x = 2 \quad \Leftrightarrow \quad x^2 + 4x - 2 = 0 \;;\; d = 16 + 8 = 24; \;\; x = \dfrac{-4 \pm 2\sqrt{6}}{2} \quad \Leftrightarrow$

$x = -2 \quad \vee \quad x = -2 \pm \sqrt{6}$

561. Solve for *x*: $x^{x^6} = 144$**. The solution is:** $x = \sqrt[6]{12}$ **, since:**

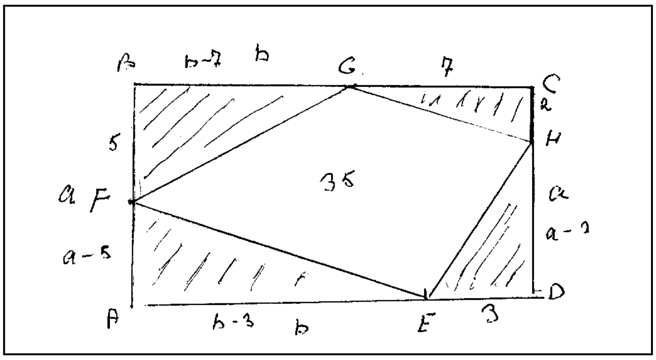
$(\sqrt[3]{12})^{\sqrt[6]{12}^6} = (\sqrt[6]{12})^{12} = ((\sqrt[6]{12})^6)^2 = 12^2 = 144$

562. solve for *x*: $7^{x+8} = 8^{x+7}$

$7^{x+8} = 8^{x+7} \quad \Leftrightarrow \quad \left(\dfrac{7}{8}\right)^x = \left(\dfrac{8}{7}\right)^7 \dfrac{1}{7} \quad \Leftrightarrow \quad \left(\dfrac{7}{8}\right)^{x-7} = \dfrac{1}{7} \quad \Leftrightarrow \quad x - 7 = -\dfrac{\ln 7}{\ln 7 - \ln 8} \quad \Leftrightarrow$

$x = 7 - \dfrac{\ln 7}{\ln 7 - \ln 8}$

562. Find the total area of the 4 triangles in the figure shown below



Our aim is to find the total area of all the shaded triangles. We do by expressing that the area of the rectangle is the area of triangle plus the area=35 of the inscribed square.

$2ab = 5(b-7) + 7 \cdot 2 + 3(a-2) + (b-3)(a-5) + 70$

$2ab = 5b - 35 + 14 + 3a - 6 + ab - 5b - 3a + 15 + 70$

$2ab = 35 + 14 - 6 + ab + 15 + 70 \quad \Rightarrow$

$ab = 64 = 8 \cdot 8 = 4 \cdot 16 = 2 \cdot 32$

Which have the solutions:

$a = 8 \;\; and \;\; b = 8 \;\; or \;\; a = 16 \;\; and \;\; b = 4 \;\; or \;\; a = 32 \;\; and \;\; b = 2$

The collected sum of the rectangles is 64 - 35= 29

563. Determine *a* from: $a^3 + a^2 = 392$ **The solution is *a* = 7 since:** $343 + 49 = 392$

263. Determine $a^{-1} + b^{-1}$ from: $a^3 + b^3 = 10 \quad and \quad a + b = 7$

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b) = 10 \quad \Rightarrow \quad 343 - 21ab = 10 \quad \Leftrightarrow 21ab = 333 \quad \Rightarrow \quad ab = \frac{111}{7}$$

$$a^{-1} + b^{-1} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{49}{111}$$

564. Solve the integrals: $\int \frac{1}{\sin x} dx$; $\int \frac{x}{\sin x} dx$; $\int \frac{x}{1+x^4} dx$; $\int \frac{x^2}{1+x^4} dx$

$$\int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}} dx = \int \frac{1}{2\cos^2\frac{x}{2}\tan\frac{x}{2}} dx = \ln|\tan\tfrac{x}{2}|$$

$$((\tan^{-1} x)' = \frac{1}{1+x^2} \quad \text{so} \quad \int \frac{x}{1+x^4} dx = \tfrac{1}{2}\tan^{-1} x^2$$

565. Determine *a, b, c* from: $a^2 + b^2 + c^2 = 1000$ **The solution is** $a = 30,\ b = 6,\ c = 8$

Since: $a^2 + b^2 + c^2 = 30^2 + 6^2 + 8^2 = 900 + 36 + 64 = 1000$

566. Simplify $\sqrt{30 + 12\sqrt{6}}$

We try to write $30 + 12\sqrt{6}$ as $(a\sqrt{3} + b\sqrt{2})^2 = 3a^2 + 2b^2 + 2ab\sqrt{6}$

Since $2ab = 12$ and one possible solution is $a = 2 \quad and \quad b = 3$ then we have:

$3a^2 + 2b^2 + 2ab\sqrt{6} = 12 + 18 + 12\sqrt{6} = 30 + 12\sqrt{6}$

So $\sqrt{30 + 12\sqrt{6}} = 2\sqrt{3} + 3\sqrt{2})$

567. Solve for *x*: $9^x + 33^x = 121^x$

$9^x + 33^x = 121^x \quad \Leftrightarrow \quad (3^x)^x + 3^x \cdot 11^x = (11^x)^2$

We divide the equation by $3^x \cdot 11^x$

$(\frac{3}{11})^x + 1 = (\frac{11}{3})^x$ We put: $y = (\frac{3}{11})^x$ and find:

$$y + 1 - \frac{1}{y} = 0 \quad \Leftrightarrow \quad y^2 + y - 1 = 0 \quad ; \quad d = 1 + 4 = 5 \quad y = \frac{-1 \pm \sqrt{5}}{2}$$

$$(\frac{3}{11})^x = \frac{-1+\sqrt{5}}{2} \quad \Leftrightarrow \quad x = \frac{\ln(\frac{-1+\sqrt{5}}{2})}{\ln 3 - \ln 11}$$

567a. Solve for *x*: $169^x + 143^x = 121^x$

$169^x + 143^x = 121^x \quad \Leftrightarrow \quad (13^x)^2 + 11^x \cdot 13^x = (11^x)^2$

We divide the equation by $13^x \cdot 11^x$, and the en we have: $(\frac{13}{11})^x + 1 = (\frac{11}{13})^x$

We put: $y = (\frac{13}{11})^x$ and find:

$y + 1 - \frac{1}{y} = 0 \quad \Leftrightarrow \quad y^2 + y - 1 = 0 \quad ; \quad d = 1 + 4 = 5 \quad y = \frac{-1 \pm \sqrt{5}}{2}$

$(\frac{13}{11})^x = \frac{-1 + \sqrt{5}}{2} \quad \Leftrightarrow \quad x = \frac{\ln(\frac{-1+\sqrt{5}}{2})}{\ln 13 - \ln 11}$

569. Solve for *x*: $x^{x^2} = 2$ **The solution is:** $x = \sqrt{2}$**, since** $(\sqrt{2})^{\sqrt{2}^2} = \sqrt{2}^2 = 2$

570. Calculate the value of the infinite expression: $\sqrt{2\sqrt{4\sqrt{8...}}}$

The expression may be written: $\sqrt{2\sqrt{2^2\sqrt{2^3...}}}$ To find the sum, we multiply the square root with $\frac{1}{2}$.

$\frac{1}{2}\sqrt{2\sqrt{2^2\sqrt{2^3...}}} = \sqrt{2\cdot(\frac{1}{2})^2\sqrt{2^2\sqrt{2^3...}}} = \sqrt{2\sqrt{2^2\cdot(\frac{1}{2})^4...}} = \sqrt{2\sqrt{2^2\cdot(\frac{1}{2})(\frac{1}{2})^3\sqrt{2^3...}}} =$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2(\frac{1}{2})^4\sqrt{2^4}...}}} =$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^8...}}}} =$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^3(\frac{1}{2})^5\sqrt{2^5}...}}}}$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^3(\frac{1}{2})^5\sqrt{2^5}...}}}} =$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^3\sqrt{2^5(\frac{1}{2})^{10}}...}}}} =$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^3\sqrt{2^5(\frac{1}{2})^4(\frac{1}{2})^6\sqrt{2^6}}...}}}} =$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^3(\frac{1}{2})^5\sqrt{2^5(\frac{1}{2})^4\sqrt{2^6(\frac{1}{2})^{12}}}...}}}}$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^3(\frac{1}{2})^5\sqrt{2^5(\frac{1}{2})^4\sqrt{2^6(\frac{1}{2})^5(\frac{1}{2})^7}}...}}}}$

$\sqrt{2\sqrt{2^2\cdot(\frac{1}{2})\sqrt{2^3\cdot(\frac{1}{2})^2\sqrt{2^4\cdot(\frac{1}{2})^3(\frac{1}{2})^5\sqrt{2^5(\frac{1}{2})^4\sqrt{2^6(\frac{1}{2})^5\sqrt{2^7(\frac{1}{2})^{14}}}}...}}}}$

What we see is that all the numbers in the radicals are equal to 2.
We therefore have an infinite radical expression:

$s = \sqrt{2\sqrt{2\sqrt{2....}}}$ and we therefore have:

$s = \sqrt{2s} \quad \Rightarrow \quad s^2 = 2s \quad so \quad s = 2$ But we multiplied with one half so $s = 4$

571. Solve for *x* and *y*.

$2x^2 - x + y - 2xy = 17$

We rewrite the expression as: $(2x-1)(x-y) = 17$

Since 17 is a prime then for integer solutions we must have:

$(2x-1) = 17 \quad and \quad (x-y) = 1 \quad or \quad (2x-1) = 1 \quad and \quad (x-y) = 17$

$(2x-1) = 17 \quad \Rightarrow \quad x = 9 \quad and \quad (x-y) = 1 \quad \Rightarrow \quad y = 8 \quad or$

$(2x-1) = 1 \quad \Rightarrow \quad x = 1 \quad and \quad (x-y) = 17 \quad \Rightarrow \quad y = -16$

572. Determine *a,b,c,d* from: $a+b+c+d = 60: \; a+b = 20; \quad b+c = 30; \; c+a = 40;$

$b+c = 30 \; and \; a+b = 20 \quad \Rightarrow \quad c-a = 10$

$c-a = 10 \quad and \quad c+a = 40 \quad \Rightarrow \quad 2c = 50 \quad \Rightarrow \quad c = 25$

$a = 15; \quad b = 5: \quad a+b+c+d = 60 \quad \Rightarrow \quad d = 15$

573. Solve for *x*: $x^{x^5} = 100$. The solution is: $x = \sqrt[5]{10}$, since;

$(\sqrt[5]{10})^{\sqrt[5]{10}^5} = (\sqrt[5]{10})^{10} = ((\sqrt[5]{10})^5)^2 = (10)^2 = 100$

574. Simplify: $\dfrac{2^{33}+2^{22}+2^{11}}{2^{33}-1}$

According to the formula for a quotient series: $s = a_0 \dfrac{q^n - 1}{q-1}$ we have:

$$\frac{2^{33}+2^{22}+2^{11}}{2^{33}-1} = \frac{2^{11}}{2^{33}-1}\frac{(2^{11})^3-1}{2^{11}-1} = \frac{2^{11}}{2^{11}-1}$$

576. Solve for *x*: $8^x + 2^x = 68$. The solution is *x* = 2, since: $8^2 + 2^2 = 64 + 4 = 68$

576. Solve for *x*: $x^{x^6} = 144$ The solution is $x = \sqrt[6]{12}$ since: $(\sqrt[6]{12})^{\sqrt[6]{12}^6} = (\sqrt[6]{12})^{12} = 144$

577. Given: $x+y+z = 0 \quad and \quad xyz = 15$ Determine $x^3+y^3+z^3 =$

We do it in two steps: First we notice that: $x+y = -z$

Then we calculate $x^3+y^3 = (x+y)^3 - 3xy(x+y)$

$x^3+y^3+z^3 = (x+y)^3 + z^3 - 3xy(x+y)$

$x^3+y^3+z^3 = -3xy(x+y) + (x+y+z)^3 - 3xyz(x+y+z) \quad \Leftrightarrow$

$x^3+y^3+z^3 = -3xy(x+y) + (x+y+z)((x+y+z)^2 - 3xyz)$

Since $x+y+z = 0$ we have:

$x^3+y^3+z^3 = -3xy(x+y)$ and since $x+y = -z$

$x^3 + y^3 + z^3 = -3xy(x + y) = 3xyz = 45$

578. Determine *x + y* from: $x^3 y + xy^3 = 10 \quad and \quad x^4 + 6x^2 y^2 + y^4 = 41$

$x^3 y + xy^3 = 10 \quad \Leftrightarrow \quad xy(x^2 + y^2) = 10$

$x^4 + 6x^2 y^2 + y^4 = 41 \quad \Leftrightarrow \quad (x^2 + y^2)^2 + 4x^2 y^2 = 41 \quad \Rightarrow$

$(\frac{10}{xy})^2 + 4x^2 y^2 = 41 \quad \Leftrightarrow \quad 100 + 4x^4 y^4 = 41x^2 y^2$

$We\ put\ t = x^2 y^2 \quad and \quad get: \quad 4t^2 - 41t + 100 = 0; \quad d = 41^2 - 1600 = 81$

$t = \frac{41 \pm 9}{2} \quad \Leftrightarrow \quad t = 25 \quad \vee \quad t = 16 \quad \Leftrightarrow \quad x^2 y^2 = 25 \quad \vee \quad x^2 y^2 = 16 \quad \Leftrightarrow \quad xy = 5 \quad \vee \quad xy = 4$

$xy(x^2 + y^2) = 10 \quad \Rightarrow \quad x^2 + y^2 = 2 \quad or \quad x^2 + y^2 = \frac{5}{2}$

$x^2 + y^2 = (x + y)^2 - 2xy = 2 \quad \vee \quad x^2 + y^2 = (x + y)^2 - 2xy = \frac{5}{2} \quad \Rightarrow$

$(x + y)^2 = 12 \quad \vee \quad (x + y)^2 = 10 \quad \Leftrightarrow \quad x + y = 2\sqrt{3} \quad \vee \quad x + y = \sqrt{10}$

$x + y = 2\sqrt{3} \quad and \quad xy = 5 \quad \Rightarrow \quad x + \frac{5}{x} - 2\sqrt{3} = 0 \quad \Leftrightarrow \quad x^2 - 2\sqrt{3}x + 5 = 0$

$d = 12 - 20 < 0$ No solution????

$x + y = \sqrt{10} \quad and \quad xy = 4 \quad \Rightarrow \quad x + \frac{4}{x} - \sqrt{10} = 0 \quad \Leftrightarrow \quad x^2 - \sqrt{10}x + 4 = 0$

$d = 10 - 16 \quad < 0 \quad No\ solution$?????

$x + y = 2\sqrt{3} + \quad or \quad x + y = \sqrt{10}$

579. Solve for x *and* y: $x^y = y^x$

One solution is of course: $x = y$, but…

We shall apply the theorem if: $a^a = b^b \quad then \quad a = b$.

$x^y = y^x \quad \Leftrightarrow y \ln x = x \ln y \quad \Leftrightarrow$

$\frac{\ln x}{x} = \frac{\ln y}{y} \quad \ln x^{\frac{1}{x}} = \ln y^{\frac{1}{y}} \quad \Leftrightarrow \quad x^{\frac{1}{x}} = y^{\frac{1}{y}} \quad \Leftrightarrow \quad \left(\frac{1}{x}\right)^{-\frac{1}{x}} = \left(\frac{1}{y}\right)^{-\frac{1}{y}}$

So according to the theorem above, we have $x = y$.

580. Solve for x *and* y: $x + xy + y = 54$

$x + xy + y = 54 \quad \Leftrightarrow \quad (x+1)(y+1) = 55 = 5 \cdot 11 \quad \Leftarrow \quad x + 1 = 5 \quad and \quad y + 1 = 11 \quad \Leftrightarrow \quad x = 4 \quad and \quad y = 10$

581. Solve for x: $x^3(x^2 - 28) = -171x$

$x^3(x^2 - 28) = -171x \quad \Leftrightarrow \quad x \neq \quad \wedge \quad x'^2(x^2 - 28) = -171 \quad \Leftrightarrow \quad x'^4 - 28x'^2 + 171 = 0$

$d = 28^2 - 4 \cdot 171 = 100; \quad x^2 = \frac{28 \pm 10}{2} \quad \Leftrightarrow \quad x^2 = 19 \quad \vee \quad x^2 = 9 \quad \Leftrightarrow \quad x = \pm\sqrt{19} \quad \vee \quad x = \pm 3$

581. Solve for x: $3^{3x} - 3^{2x} = 3^x$

$3^{3x} - 3^{2x} = 3^x \quad \Leftrightarrow \quad 3^{2x} - 3^x - 1 = 0 \quad \textit{We put} \quad y = 3^x$

$$y^2 - y - 1 = 0 \;\; ; \; d = 5 \quad y = \frac{1 \pm \sqrt{5}}{2} \quad \Rightarrow \quad 3^x = \frac{1+\sqrt{5}}{2} \quad \Rightarrow \quad x = \frac{\ln\frac{1+\sqrt{5}}{2}}{\ln 3}$$

581. Solve for x: $2^{x^2} = 1 - x^8$ **????? It has only the solution *x* = 0**

582. Solve for $\frac{x}{y}$ **:** $\frac{x^2}{y^2} + \frac{y^2}{x^2} + 4\frac{x}{y} + 4\frac{y}{x} + 6 = 0$

We put: $a = \frac{x}{y}$ and we get:

$$a^2 + \frac{1}{a^2} + 4a + 4\frac{1}{a} + 6 = 0 \quad \Leftrightarrow \quad (a + \frac{1}{a})^2 - 2 + 4(a + \frac{1}{a}) + 6 = 0$$

We put: $z = (a + \frac{1}{a})$, and we get:

$z^2 + 4z + 4 = 0 \quad (z+2)^2 = 0 \quad z = -2 \quad \Leftrightarrow \quad (a + \frac{1}{a}) = -2$

$a^2 + 2a + 1 = 0 \quad \Leftrightarrow \quad a = -1 \quad \Leftrightarrow \quad x = -y$

583. Determine *m,* such that: $m^3 + m^2 = 150$**. The solution is** $m = 5$**, since:**
$5^3 + 5^2 = 150$

583. Determine *a* and *b,* such that: $4^a - 36^b = 28$**. The solution is** $a = 3 \;\; and \;\; b = 1$**,**
since: $4^3 - 36 = 28$

584. *Solve* for *x*: $x^{\ln x} = 2$

$x^{\ln x} = 2 \quad \Leftrightarrow \quad \ln x \ln x = \ln 2 \quad \Leftrightarrow \quad (\ln x)^2 = \ln 2 \quad \Leftrightarrow \quad \ln x = \pm\sqrt{\ln 2} \quad \Leftrightarrow \quad x = e^{\pm\sqrt{\ln 2}}$

585. Determine; $x - \frac{7}{\sqrt{x}}$ **from** $x - \sqrt{x} = 7$

$x - \sqrt{x} = 7 \quad \Leftrightarrow \quad \sqrt{x} - 1 = \frac{7}{\sqrt{x}} \quad \Leftrightarrow$

$\sqrt{x} = \frac{7}{\sqrt{x}} + 1 \quad \textit{insert in} \quad x - \sqrt{x} = 7 \quad x - \frac{7}{\sqrt{x}} - 1 = 7 \quad \Leftrightarrow \quad x - \frac{7}{\sqrt{x}} = 8$

586. Solve for *x*: $\sqrt[3]{x} + \sqrt[3]{x-16} = \sqrt[3]{x-8}$ **The solution is:** $x = 8$

585. Solve for *x*: $x^3 + 6x^2 + 12x + 8 = 0$
If it has 3 roots α, β, γ we should have: $\alpha + \beta + \gamma = -6 \quad and \quad \alpha\beta\gamma = -8$.

A solution is might be: $\alpha = \beta = \gamma = -2$. We make polynomial division with $x + 2$

$x+2 \mid x^3+6x^2+12x+8 \mid x^2+4x+4$

x^3+2x^2

$4x^2+12x$

$4x^2+8x$

$4x+8$

$4x+8$

$x^2+4x+4=0 \quad \Leftrightarrow \quad (x+2)^2=0 \quad \Leftrightarrow \quad x=-2$ Double root.

586. Determine *x* and *y* such that: $9^a+4^b=9 \quad and \quad 3^a \cdot 2^b = 3$

$9^a+4^b=9 \quad and \quad 3^a \cdot 2^b=3 \quad \Leftrightarrow \quad (3^a)^2+(2^b)^2=9 \quad and \quad 3^a \cdot 2^b=3$

We put: $x=3^a \quad and \quad y=2^b$ and we have thus the equations:

$$x^2+y^2=9 \quad and \quad xy=3 \quad \Rightarrow \quad x^2+\left(\frac{3}{x}\right)^2=9 \quad \Rightarrow \quad x^4-9x^2+9=0$$

$$d=81-36=45=5\cdot 9. \quad x^2=\frac{9\pm 3\sqrt{5}}{2} \quad \Leftrightarrow \quad x=\pm\sqrt{\frac{9\pm 3\sqrt{5}}{2}}$$

$$3^a=\sqrt{\frac{9\pm 3\sqrt{5}}{2}} \quad \Rightarrow \quad | \; a=\frac{\ln\sqrt{\frac{9\pm 3\sqrt{5}}{2}}}{\ln 3} \; |$$

587. Solve for *x*: $12x^3-28x^2-3x+7=0$

Since the polynomial is not normalized (coefficient to x^3 is), we may not apply the theorem of sum and product of the roots. So we will have to guess one root. It seems that $x=\frac{1}{2}$ is a root, so we make polynomial division with $x-\frac{1}{2}$.

$x-\frac{1}{2} \mid 12x^3-28x^2-3x+7=0 \mid 12x^2-22x-14$

$12x^3-6x^2$

$-22x^2-3x$

$-22x^2+11x$

$-14x+7$

$-14x+7$

$12x^2-22x-14=0 \quad \Leftrightarrow \quad 6x^2-11x-7=0 \qquad d=121+168=289=17^2$

$$x=\frac{11\pm 17}{12} \quad \Leftrightarrow \quad x=\tfrac{7}{3} \quad or \quad x=-\tfrac{1}{2}$$

If we write the original polynomial as: $x^3-\frac{7}{3}x^2-\frac{1}{4}x+\frac{7}{12}=0$

We can see that: $\frac{1}{2}-\frac{1}{2}+\frac{7}{3}=\frac{7}{3} \quad and \quad \frac{1}{2}\cdot(-\frac{1}{2})\cdot\frac{7}{3}=-\frac{7}{12}$ as it should be.

588. Determine *a,b*: such that : $a+b=6\sqrt{ab}$

It is not specified that the equation has integer solution, as one should think.

We put: $a=x^2 \quad and \quad b=y^2$, and find;

$x^2+y^2=6xy \quad \Leftrightarrow \quad (x-y)^2=4xy$

Trying to find integer solutions to this equation fails. We therefore put $y=kx$:

$x^2+y^2=6xy \quad \Rightarrow \quad x^2+k^2x^2=6kx^2 \quad \Rightarrow \quad k^2-6k+1=0; \qquad d=36-4=32$

If the equation has integer solutions *k* should be a rational number, but it is not.

$k=\frac{6\pm4\sqrt{2}}{2}=3\pm2\sqrt{2}$ With this value for *k*, the equation will be fulfilled for any *x*.

589. evaluate the integrals: $\int x\sqrt{x^2+1}dx \quad and \quad \int x^5\sqrt{x^2+1}dx$

The first one is easy: $\int x\sqrt{x^2+1}dx=\frac{1}{3}(x^2+1)^{\frac{3}{2}}+c$

The second one requires integration by parts:

$\int f(x)g'(x)dx = f(x)g(x)-\int g(x)f'(x)dx$ or in shorthand: $\int fdg = fg-\int gdf$

$\int x^5\sqrt{x^2+1}dx=\frac{1}{3}\int x^4 d(x^2+1)^{\frac{3}{2}}=\frac{1}{3}x^4(x^2+1)^{\frac{3}{2}}-\frac{1}{3}\int (x^2+1)^{\frac{3}{2}}dx^4$

$\frac{1}{3}\int (x^2+1)^{\frac{3}{2}}dx^4=\frac{4}{3}\int x^3(x^2+1)^{\frac{3}{2}}dx=\frac{4}{3\cdot5}\int x^3 d(x^2+1)^{\frac{5}{2}}=\frac{4}{3\cdot5}x^3(x^2+1)^{\frac{5}{2}}-\frac{4}{3\cdot5}\int (x^2+1)^{\frac{5}{2}}dx^3=$

$\frac{3}{3\cdot5}\int x^2(x^2+1)^{\frac{5}{2}}dx=\frac{3}{3\cdot5\cdot2}\frac{2}{7}\int x^2 d(x^2+1)^{\frac{7}{2}}dx=$

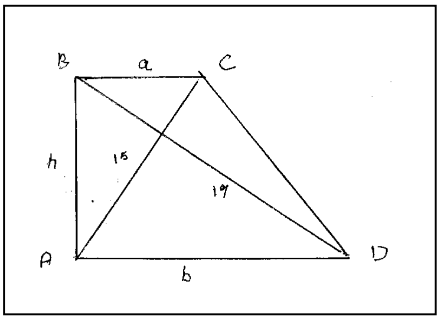
$\frac{1}{5\cdot7}x^2(x^2+1)^{\frac{7}{2}}-\frac{1}{5\cdot7}\int (x^2+1)^{\frac{7}{2}}dx^2$

$\frac{1}{5\cdot7}\int (x^2+1)^{\frac{7}{2}}dx^2=\frac{2}{5\cdot7}\int x(x^2+1)^{\frac{7}{2}}dx=\frac{1}{2\cdot5\cdot7\cdot9}(x^2+1)^{\frac{9}{2}}$

The integral is then;

$\int x^5\sqrt{x^2+1}dx=\frac{1}{3}x^4(x^2+1)^{\frac{3}{2}}-\frac{4}{3\cdot5}x^3(x^2+1)^{\frac{5}{2}}+\frac{1}{5\cdot7}x^2(x^2+1)^{\frac{7}{2}}-\frac{1}{2\cdot5\cdot7\cdot9}(x^2+1)^{\frac{9}{2}}$

390. Determine the height and the length of the parallel sides in the trapez



From the two right angle triangles we have:

$h^2+a^2=15^2 \quad and \quad h^2+b^2=19^2 \quad \Rightarrow$

$b^2-a^2=19^2-15^2=(19-15)(19+15)=4\cdot34=8\cdot17$

$(b-a)(b+a)=4\cdot34 \quad \Rightarrow$

$b-a=8 \quad and \quad b+a=17 \Rightarrow$

$b=\frac{25}{2} \quad and \quad a=\frac{9}{2}$

Then it is easy to find the height:

$$h^2 = 15^2 - a^2 = 225 - \frac{81}{4} = \frac{819}{4} \quad \Rightarrow \quad h = \tfrac{1}{2}\sqrt{819} \qquad or$$

$$h^2 = 19^2 - b^2 = 361 - \frac{625}{4} = \frac{819}{4} \quad \Rightarrow \quad h = \tfrac{1}{2}\sqrt{819}$$