Contents

301. Simplify: $(\sqrt{3}+3)^3-(\sqrt{3}-3)^3$

$$(a+b)^3 = a^3+b^3+3ab(a+b)$$
$$(\sqrt{3}+3)^3 = \sqrt{3}^3+3^3+3\sqrt{3}\cdot 3(\sqrt{3}+3) = 3^3+3^3\sqrt{3}$$
$$(\sqrt{3}-3)^3 = \sqrt{3}^3-3^3-3\sqrt{3}\cdot 3(\sqrt{3}-3) = 3^3-3^3\sqrt{3}$$
$$(\sqrt{3}+3)^3-(\sqrt{3}-3)^3 = 2\cdot 3^3+2\sqrt{3}\cdot 3^3 = 2\cdot 3^3(1+\sqrt{3})$$

302. Determine integers (*x, y*), such that $x+y+xy=54$

There is really no analytic way to solve this, so we must resort to qualified guesswork.
$x+y+xy=54 \quad \Leftrightarrow \quad x+y=54-xy$
It seems that $xy = 40$ is a reasonable assumption, 5 and 8 does not work since 5+8 =13,
But $x=4 \quad and \quad y=10$ does the trick, since: $4+10+4\cdot 10=54$

303. $x+y=2 \quad xy=3$ **Determine** x^5+y^5

This is indeed a very strange exercise, since $x+y=2 \;\; and \;\; xy=3$ has no real solution:

$$xy=3 \;\Leftrightarrow\; y=\frac{3}{x} \;\Rightarrow\; x+\frac{3}{x}=2 \;\Leftrightarrow\; x^2-2x+3=0; \quad d=4-12<0$$

Nevertheless, if we ignore this, we find;

$$(x+y)^2=4 \;\Leftrightarrow\; x^2+y^2+2xy=4 \;\Leftrightarrow\; x^2+y^2+6=4 \;\Leftrightarrow\; x^2+y^2=-2\,!$$
$$(x+y)^3=8 \;\Leftrightarrow\; x^3+y^3+3xy(x+y)=8 \;\Leftrightarrow\; x^3+y^3+9\cdot 2=8 \;\Leftrightarrow\; x^3+y^3=-10$$
$$(x^3+y^3)(x^2+y^2)=(-2)(-10) \;\Leftrightarrow\; x^5+x^3y^2+x^2y^3+y^5=20 \;\Leftrightarrow$$
$$x^5+x^3y^2+x^2y^3+y^5=20 \;\Leftrightarrow$$
$$x^5+y^5+x^2y^2(x+y)=20 \;\Leftrightarrow\; x^5+y^5+9\cdot 2=20$$
$$x^5+y^5=2$$

304. Solve for *x*; $\left(\sqrt{\dfrac{2x-1}{x-1}}\right)^2-\left(\sqrt{\dfrac{2x+1}{x+1}}\right)^2=4$

$$\left(\sqrt{\frac{2x-1}{x-1}}\right)^2-\left(\sqrt{\frac{2x+1}{x+1}}\right)^2=4 \;\Leftrightarrow\; \frac{2x-1}{(x-1)^2}-\frac{2x+1}{(x+1)^2} \;\Leftrightarrow$$
$$(2x-1)(x+1)^2-(2x+1)(x-1)^2=(x-1)^2(x+1)^2 \;\Leftrightarrow$$
$$(2x-1)(x^2+1+2x)-(2x+1)(x^2+1-2x)=(x^2-1)^2 \;\Leftrightarrow$$
$$2x^3+2x+4x^2-x^2-1-2x-(2x^3+2x-4x^2+x^2+1-2x)=x^4+1-2x^2 \;\Leftrightarrow$$

$$2x^3+2x+4x^2-x^2-1-2x-2x^3-2x+4x^2-x^2-1+2x=x^4+1-2x^2 \quad \Leftrightarrow$$
$$6x^2-2=x^4+1-2x^2 \quad \Leftrightarrow$$
$$x^4-8x^2+3=0$$
$$y=x^2 \quad \Rightarrow \quad y^2=\frac{8\pm 2\sqrt{13}}{2}=4\pm\sqrt{13}$$
$$x=\sqrt{4+\sqrt{13}} \quad \vee \quad x=\sqrt{4-\sqrt{13}}$$

305. Solve for *x*: $2^{3x}-2^x=120$

$2^{3x}-2^x=120 \quad \Leftrightarrow \quad (2^x)^3-2^x=120$

We put: $y=2^x$, and then we have: $y^3-y-120=0$

Obviously it has the solution y =5, since $125-5-120=0$

To find possible other solutions, we dive with y – 5

$$y-5\,|\,y^3-y-120\,|\,y^2+5y+24$$
$$y^3-5y^2$$
$$5y^2-y$$
$$5y^2-25y$$
$$24y-120$$
$$24y-120$$

$y^2+5y+24=0$ has no solutions, since $d=25-4\cdot 25<0$, so the only solution is: $2^x=5$

$$x=\frac{\ln 5}{\ln 2}$$

306. $3^{2x}=2^{3x}=5184$

$5184=2^6\cdot 3^4$

$3^{2x}=2^{3x}=5184 \quad \Leftrightarrow \quad 9^x=8^y=2^6\cdot 3^4=8^2\cdot 9^2$

$$9^x=8^2\cdot 9^2 \quad \Leftrightarrow \quad 9=(8^2\cdot 9^2)^{\frac{1}{x}} \quad \textit{and} \quad 8^y=8^2\cdot 9^2 \quad \Leftrightarrow \quad 8=(8^2\cdot 9^2)^{\frac{1}{y}}$$
$$8=(8^2\cdot 9^2)^{\frac{1}{y}} \quad \Leftrightarrow \quad 8=(8\cdot 9)^{\frac{2}{y}}$$
$$8\cdot 9=(8^2\cdot 9^2)^{\frac{1}{y}}(8^2\cdot 9^2)^{\frac{1}{x}} \quad \Leftrightarrow \quad 8\cdot 9=(8\cdot 9)^{\frac{2}{x}+\frac{2}{y}} \quad \Rightarrow \quad \frac{2}{x}+\frac{2}{y}=1 \quad \Leftrightarrow \quad \frac{x+y}{xy}=\frac{1}{2}$$

307. $2^a=2^b=1296$ **Determine** $\frac{a+b}{ab}$

$1296=2^4\cdot 3^4$

$$2^a = 1296 \quad \Leftrightarrow \quad 2 = (1296)^{\frac{1}{a}}$$

$$3^b = 1296 \quad \Leftrightarrow \quad 3 = (1296)^{\frac{1}{b}}$$

$$(2\cdot 3)^1 = (1296)^{\frac{1}{a}}(1296)^{\frac{1}{b}} = (1296)^{\frac{1}{a}+\frac{1}{b}} = (2^4\cdot 3^4)^{\frac{a+b}{ab}} = (2\cdot 3)^{4(\frac{a+b}{ab})} \quad \Rightarrow$$

$$4(\frac{a+b}{ab}) = 1 \quad \Leftrightarrow \quad \frac{a+b}{ab} = \frac{1}{4}$$

308. Simplify: $2^{20} - 20^2$ **(very easy)**

$2^{20} - 20^2 = (2^{10})^2 - 20^2 = (2^{10} - 20)(2^{10} + 20) = (1024 - 20)(1024 + 20) = 1004 \cdot 1044$

309. Determine positive integers *x* and *y* such that $x^3 - y^3 = xy + 61$

There is to my knowledge no analytic way to solve this problem, so we shall resort to guesswork.
It seems that *x* and *y* should be less than 6 and greater than 4.
We ca see that *x* = 6 and *y* = 5 does the trick, since $6^3 - 5^3 = 212 - 125 = 91 = 5\cdot 6 + 61$

310. Determine *a* and *b* from $a^3 + b^3 = 7 \quad and \quad a^2 + b^2 + a + b + ab = 4$

We put: $u = a + b \quad and \quad v = ab$

$(a+b)^2 = a^2 + b^2 + 2ab = 4 - ((a+b) + ab) + 2ab \quad \Leftrightarrow$

$u^2 = 4 - u - v + 2v \quad \Leftrightarrow \quad u^2 + u - v - 4 = 0$

$(a+b)^3 = a^3 + b^3 + 3ab(a+b) \quad \Leftrightarrow$

$u^3 = 7 + 3uv$

$I : u^3 - 3uv - 7 = 0$

$II : \; u^2 + u - v - 4 = 0 \quad \Leftrightarrow \quad v = u^2 + u - 4$

v inserted in I

$u^3 - 3uv - 7 = 0 \quad \Rightarrow \quad u^3 - 3u(u^2 + u - 4) - 7 = 0 \quad \Leftrightarrow$

$-2u^3 - 3u^2 + 12u - 7 = 0$

We can see that *u* = 1 is a root, since: $-2 - 3 + 12 - 7 = 0$
So we make polynomial division with; *u* – 1

$u - 1 \,|\, -2u^3 - 3u^2 + 12u - 7 \,|\, -2u^2 - 5u + 7$

$-2u^3 + 2u^2$

$-5u^2 + 12u$

$-5u^2 + 5u$

$7u - 7$

$7u - 7$

$$-2u^2-5u+7=0\,;\quad d=25+56=81$$
$$u=\frac{5\pm 9}{-4}\quad\Leftrightarrow\quad u=1\quad or\quad u=-\frac{7}{2}$$

$$v=u^2+u-4\quad\Leftrightarrow\quad v=-2\quad or\quad v=\frac{49}{4}-\frac{14}{4}-\frac{16}{4}=\frac{19}{4}$$
$$u=a+b\quad and\quad v=ab$$
$$a+b=1\quad and\quad ab=-2\quad\Rightarrow\quad a-\frac{2}{a}-1=0\quad\Leftrightarrow\quad a^2-a-2=0;\quad d=1+8=9$$
$$a=\frac{1\pm 3}{2}\quad\Leftrightarrow\quad a=2\quad or\quad a=-1\quad and\quad b=-\frac{2}{a}\quad\Leftrightarrow\quad b=-1\quad or\quad b=2$$
$$u=a+b\quad and\quad v=ab$$
$$a+b=-\frac{7}{2}\quad and\quad ab=\frac{19}{4}\quad\Rightarrow\quad a+\frac{19}{4a}+\frac{7}{2}=0\quad\Leftrightarrow\quad 4a^2+14a+19=0;\quad d=256-16\cdot 19<0$$
$$a=\frac{1\pm 3}{2}\quad\Leftrightarrow\quad a=2\quad or\quad a=-1\quad and\quad b=-\frac{2}{a}\quad\Leftrightarrow\quad b=-1\quad or\quad b=2$$

311. Solve for *x*: $x^{\log x}-100x=0$

$$x^{\log x}-100x=0\quad\Leftrightarrow\quad x^{\log x}=100x\quad\Leftrightarrow\quad \log(x^{\log x})=\log(100x)\quad\Leftrightarrow$$
$$(\log x)(\log x)=\log 100+\log x\quad\Leftrightarrow$$
$$(\log x)^2-\log x-2=0;\quad y=\log x$$
$$y^2-y-2=0;\quad d=1+8=9$$
$$y=\frac{1\pm 3}{2}\quad\Leftrightarrow\quad y=2\quad or\quad y=-1\quad\Rightarrow\quad \log x=2\quad or\quad \log x=-1$$
$$x=100\quad or\quad x=\frac{1}{10}$$

312. $100^{x-1}=99.$ **Determine** 100^{x+1}

$$\frac{100^{x+1}}{100^{x-1}}=100^2\quad\Rightarrow\quad 100^{x+1}=100^2\cdot 100^{x-1}=99\cdot 10^4$$

313. Given: $12^x=18$**. Determine** $2^{\frac{2x-1}{x-2}}$

$$12^x=18\quad\Leftrightarrow\quad 3^x\cdot 2^{2x}=2\cdot 3^2\quad\Leftrightarrow\quad 3^{x-2}\cdot 2^{2x-1}=1\quad\Rightarrow$$
$$(3^{x-2}\cdot 2^{2x-1})^{\frac{1}{x-2}}=1^{\frac{1}{x-2}}\quad\Rightarrow\quad 3\cdot 2^{\frac{2x-1}{x-2}}=1\quad\Leftrightarrow\quad 2^{\frac{2x-1}{x-2}}=\frac{1}{3}$$

314. Calculate the sum: $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+...+\frac{1}{\sqrt{8}+\sqrt{9}}$

$$\frac{\sqrt{1}-\sqrt{2}}{(\sqrt{1}+\sqrt{2})(\sqrt{1}-\sqrt{2})}+\frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}+\frac{(\sqrt{3}-\sqrt{4})}{(\sqrt{3}+\sqrt{4})(\sqrt{3}-\sqrt{4})}+...+\frac{(\sqrt{8}-\sqrt{9})}{(\sqrt{8}+\sqrt{9})(\sqrt{8}-\sqrt{9})}=$$

$$\frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{(\sqrt{3}-\sqrt{4})}{4-3}+...+\frac{(\sqrt{8}-\sqrt{9})}{8-9}=$$

$$\frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{(\sqrt{3}-\sqrt{4})}{4-3}+...+\frac{(\sqrt{8}-\sqrt{9})}{8-9}=$$

$$\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+...+\sqrt{9}-\sqrt{8}=\sqrt{9}-1=2$$

315. Which is greater: $\sqrt{5+5\sqrt{5}}$ **or 4**

We solve the inequality: $\sqrt{5+5\sqrt{5}}>4 \iff 5+5\sqrt{5}>16 \iff 5\sqrt{5}>11 \iff 125>121$

316. Given: $\frac{1}{a}+\frac{1}{b}=\frac{2}{35}$ **Determine all positive values of *a* and *b***

It is rather obvious that if $a = b = 35$, the equation is fulfilled, but there may be other solutions. To investigate this we isolate b.

$$\frac{1}{b}=\frac{2}{35}-\frac{1}{a} \iff \frac{1}{b}=\frac{2a-35}{35a} \iff b=\frac{35a}{2a-35}$$

The condition that b is an integer is that $2a – 35$ is a divisor in $35a$.
The positive divisors in 35 are 1, 5, 7, 35

$2a-35=1 \Rightarrow a=18;\ b=35\cdot 18=630:\ \frac{1}{a}+\frac{1}{b}=\frac{1}{18}+\frac{1}{630}=\frac{2}{35}$

$2a-35=5 \Rightarrow a=20;\ b=35\cdot 20:5=140:\ \frac{1}{a}+\frac{1}{b}=\frac{1}{20}+\frac{1}{140}=\frac{2}{35}$

$2a-35=7 \Rightarrow a=21;\ b=35\cdot 21:7=105:\ \frac{1}{a}+\frac{1}{b}=\frac{1}{21}+\frac{1}{105}=\frac{2}{35}$

$2a-35=35 \Rightarrow a=35;\ b=35\cdot 35:35=35:\ \frac{1}{a}+\frac{1}{b}=\frac{1}{35}+\frac{1}{35}=\frac{2}{35}$

We now consider solutions, where either a or b are negative.
The negative divisors in 35 are: -1, -5, -7, -35.

$2a-35=-1 \Rightarrow a=17;\ b=35\cdot 17:(-1)=-595:\ \frac{1}{a}+\frac{1}{b}=\frac{1}{17}-\frac{1}{595}=\frac{2}{35}$

$2a-35=-5 \Rightarrow a=15;\ b=35\cdot 15:(-5)=-105:\ \frac{1}{a}+\frac{1}{b}=\frac{1}{15}-\frac{1}{105}=\frac{2}{35}$

$2a-35=-7 \Rightarrow a=14;\ b=35\cdot 14:(-7)=-70:\ \frac{1}{a}+\frac{1}{b}=\frac{1}{14}-\frac{1}{70}=\frac{2}{35}$

317. Solve for (*a*, *b*): $\frac{1}{a}+\frac{1}{b}=\frac{1}{2004}$

Her we have one equation with 2 variables, which cannot be solved by analytic methods.
However, if the variables are positive integers, we may resort to guesswork.
Since the equation is symmetric in *x* and *y*, we may first assume that *a* = *b*.
Then it becomes easy, if we put *x* and *y* equal to the double of 2004, it works, since;

$$\frac{1}{4008}+\frac{1}{4008}=\frac{2}{4008}=\frac{1}{2004}$$

However the situation is far more complex, so we take the starting point in isolating *b* in

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2004} \quad\Leftrightarrow\quad \frac{1}{b}=\frac{1}{2004}-\frac{1}{a} \quad\Leftrightarrow\quad \frac{1}{b}=\frac{a-2004}{2004a} \quad\Leftrightarrow\quad b=\frac{2004a}{a-2004} \quad\Leftrightarrow$$

$$b=\frac{2004(a-2004)+2004^2}{a-2004} \quad\Leftrightarrow\quad b=2004+\frac{2004^2}{a-2004}$$

$$2004=2\cdot 1002=2\cdot 2\cdot 501=2\cdot 2\cdot 3\cdot 167$$

The condition that *b* is an integer is then that: $a-2004$ is a divisor in 2004^2, that is, an divisor in 2004. The divisors *k* in 2004 are: (1, 2, 2, 3, 167, 2004)

The condition that $a-2004$ is a divisor *k* in 2004 is: $a-2004=k$
This can also be formulated, where *q* now is the quotient.

$$\frac{2004}{a-2004}=q \quad\Leftrightarrow\quad \frac{2004}{q}=a-2004 \quad\Leftrightarrow\quad a=\frac{2004(q+1)}{q},$$

Since *q* is a divisor in 2004, then *a* is always an integer.

We have seen the example where $q=1$; $a=2004\cdot 2=4008$ and

$$b=\frac{2004a}{a-2004}=\frac{2004\cdot 2008}{4008-2004}=\frac{2004\cdot 2008}{2004}=2008$$

Any $a=2004+k$, where *k* is any of the divisors in 2004, or an arbitrary multiple of the divisors, will be a solution to the equation: $\frac{1}{a}+\frac{1}{b}=\frac{1}{2004}$.

We shall not make all the examples, but we shall take only one with *k* =167.

$$a=2004+167=2171; \quad b=\frac{2004a}{a-2004}=\frac{2004\cdot 2171}{2171-2004}=26052$$

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2004} \quad \Leftrightarrow \quad 2004=\frac{ab}{a+b} \quad and \quad \frac{2171\cdot 26052}{2171+26052}=2004\,!$$

But this does not include negative a. The condition for the negative solutions are, where k is a positive divisor in 2004.

$a-2004=-k \quad \Leftrightarrow \quad a=2004-k$ Or where q is the quotient.

$$\frac{2004}{a-2004}=-q \quad \Leftrightarrow \quad \frac{2004}{q}=-(a-2004) \quad \Leftrightarrow \quad a=-\frac{2004(-q+1)}{q}$$

Let us take a nontrivial example $k = 2$:

$a=2004-2=2002$ and $b=\dfrac{2004a}{a-2004}=\dfrac{2004\cdot 2002}{-2}=-2006004$

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2002}-\frac{1}{2006004}=\frac{1}{2004}$$

318. Solve for *x*: $\sin x+\sin 2x+\sin 3x=0$

We make use of the logarithmic formulas for the addition of to sine or cosine functions.

$\sin u+\sin v=2\sin\frac{u+v}{2}\cos(\frac{u-v}{2})$

We shall apply this formula on $\sin x+\sin 3x$

$\sin 3x+\sin x=2\sin\frac{3x+x}{2}\cos(\frac{3x-x}{2})=2\sin 2x\cdot\cos x$

$\sin x+\sin 2x+\sin 3x=0 \quad \Leftrightarrow \quad \sin 2x+2\sin 2x\cdot\cos x \quad \Leftrightarrow$

$\sin 2x(1+2\cos x)=0 \quad \Leftrightarrow \quad \sin 2x=0 \quad \Leftrightarrow \quad \sin 2x=0 \quad \vee \quad 1+2\cos x=0 \quad \Leftrightarrow$

$2x=\pi \quad \vee \quad 2x=2\pi \quad \vee \quad \cos=-\frac{1}{2} \quad \Leftrightarrow$

$x=\frac{\pi}{2} \quad \vee \quad x=\pi \quad \vee \quad x=\frac{2\pi}{3} \quad \vee \quad x=\frac{4\pi}{3}$

319.Determine *a* and *b*, such that: $ab+c=2020 \quad and \quad a+bc=2021$

$ab+c=2020 \quad and \quad a+bc=2021$, If we add these two equations, we find;

$ab+bc+a+c=4041 \quad \Leftrightarrow \quad b(a+c)+a+c=4041$

$(a+c)(b+1)=4041$

To find the possible values of the two factors, we resolve 4041 in prime factors. Fortunately there are not so many, since: $4041=3\cdot 3\cdot 441=3\cdot 1347$ (441 is a prime)

So if $(a+c)(b+1)=4041$, then we have the following possibilities:

$a+c=3 \quad and \quad b+1=1347 \quad b=1346$

$a+c=1347 \quad and \quad b+1=3 \quad b=2$

$a+c=9 \quad and \quad b+1=449 \quad b=448$

$a+c=449 \quad and \quad b+1=9 \quad b=8$

The first choice does not satisfy the two initial equations, but if we chose $b = 2$, we have:
$ab+c=2020 \quad \Rightarrow \quad 2a+c=2020$, and subtracting $a+c=1347$, we find: $a=2020-1347=673$
Then $c=2020-a=674$. These values fulfils both initial equations, since:

$$ab+c=2\cdot 673+674=2020 \quad and \quad a+bc=2\cdot 674+673=2021$$

320. Determine a^2 + b^2 from: $a+b=\frac{a}{b}+\frac{b}{a}$

It is obvious that: $a=b=1$, is a solution, so $a^2+b^2=2$, but it may be formally proven:

$$a+b=\frac{a}{b}+\frac{b}{a} \quad \Leftrightarrow \quad a+b=\frac{a^2+b^2}{ab} \quad \Leftrightarrow \quad ab(a+b)=a^2+b^2 \quad \Leftrightarrow$$

$$ba^2+ab^2=a^2+b^2 \quad \Leftrightarrow \quad a^2(b-1)\ +b^2(a-1)\ =0$$

Since a and b are considered non negative integers, the only solution is: $a=b=1$, so $a^2+b^2=2$.

321. Solve for *x* and *y*: $x-y=\frac{x+y}{7}=\frac{xy}{12}$

An analytic attempt leads nowhere: but an obvious guess is $x=4 \quad and \quad y=3$, since:

$$x-y=\frac{x+y}{7}=\frac{xy}{12} \quad \Leftrightarrow \quad 4-3=\frac{4+3}{7}=\frac{3\cdot 4}{12}$$

322. Determine *m* and *n* from the equation: $2^n - 2^m = 4080$

Well: $2^{11}=4048 \quad and \quad 4080-4048=32=2^5 \quad so \quad n=11 \quad and \quad m=5$

323. Determine *x* and *y* from: $2^x-2^y=1 \quad and \quad 4^x-4^y=\frac{5}{3}$

$$4^x-4^y=\frac{5}{3} \quad \Leftrightarrow \quad (2^x)^2-(2^y)^2=\frac{5}{3}$$

We put: $u=2^x \quad and \quad v=2^y$ then $2^x-2^y=1 \quad and \quad (2^x)^2-(2^y)^2=\frac{5}{3}$ gives:

$$u-v=1 \quad and \quad u^2-v^2=\frac{5}{3} \quad \Rightarrow \quad u^2-(u-1)^2=\frac{5}{3}$$

$$u^2-(u^2+1^2-2u)=\frac{5}{3} \quad \Leftrightarrow \quad 2u-1=\frac{5}{3} \quad \Leftrightarrow \quad 2u=\frac{8}{3} \quad \Leftrightarrow \quad u=\frac{4}{3} \quad \wedge \quad v=\frac{1}{3}$$

$$2^x=\frac{4}{3} \quad \Leftrightarrow \quad x=\frac{2\ln 2-\ln 3}{\ln 2} \quad \wedge \quad 2^y=\frac{1}{3} \quad \Leftrightarrow \quad x=-\frac{\ln 3}{\ln 2}$$

324. Solve the differential equation: $xydy = (x^2 + y^2)dx$

$$xydy = (x^2 + y^2)dx \quad \Leftrightarrow \quad \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \quad \Leftrightarrow \quad \frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

We put:

$$y = xz \quad \Rightarrow \quad \frac{dy}{dx} = z + x\frac{dz}{dx} \quad \Rightarrow$$

$$z + x\frac{dz}{dx} = \frac{1}{z} + z \quad \Leftrightarrow \quad x\frac{dz}{dx} = \frac{1}{z} \quad \Leftrightarrow$$

$$z\frac{dz}{dx} = \frac{1}{x} \quad \Leftrightarrow \quad zdz = \frac{1}{x}dx \quad \Leftrightarrow$$

$$\int zdz = \int \frac{1}{x}dx$$

$$\tfrac{1}{2}z^2 = \ln x + c \quad \Leftrightarrow$$

$$\tfrac{1}{2}\frac{y^2}{x^2} = \ln x + c \quad \Leftrightarrow \quad \frac{y^2}{x^2} = 2\ln x + c \quad \Leftrightarrow$$

$$y^2 = x^2(2\ln x + c) \quad \Leftrightarrow$$

$$y = \pm x\sqrt{2\ln x + c}$$

325. Solve for *x*: $x^2 + \left(\frac{x}{x+1}\right)^2 = 3$

$$x^2 + \left(\frac{x}{x+1}\right)^2 = 3 \quad \Leftrightarrow \quad x^2 + \left(\frac{x+1-1}{x+1}\right)^2 = 3$$

$$x^2 + \left(1 - \frac{1}{x+1}\right)^2 = 3 \quad \Leftrightarrow$$

$$x^2 + \frac{1}{(x+1)^2} - \frac{2}{x+1} + 1 = 3 \quad \Leftrightarrow$$

$$x^2 + \frac{1}{(x+1)^2} + \frac{-2}{x+1} + 1 = 3 \quad \Leftrightarrow$$

$$x^2+\frac{1}{(x+1)^2}-\frac{2}{x+1}+1=3 \quad \Leftrightarrow$$
$$x^2+\frac{1}{(x+1)^2}+\frac{x-1}{x+1}=3 \quad \Leftrightarrow$$
$$x^2+\frac{1}{(x+1)^2}+\frac{x-1}{x+1}+1-1=3 \quad \Leftrightarrow$$
$$x^2+\frac{1}{(x+1)^2}+\frac{2x}{x+1}-1=3 \quad \Leftrightarrow$$
$$(x+\frac{1}{x+1})^2=4 \quad \Leftrightarrow$$
$$x+\frac{1}{x+1}=2 \quad \vee \quad x+\frac{1}{x+1}=-2$$

$$x+\frac{1}{x+1}=2 \quad \Leftrightarrow \quad x^2+x+1-2(x+1)=0 \quad \Leftrightarrow \quad x^2-x-1=0 \qquad d=1+4=5$$
$$x=\frac{1\pm\sqrt{5}}{2}$$
$$x+\frac{1}{x+1}=-2 \quad \Leftrightarrow \quad x^2+x+1+2(x+1)=0 \quad \Leftrightarrow \quad x^2+3x+3=0 \qquad d=9-12<0.$$ No solution!

The solution is therefore: $x=\frac{1+\sqrt{5}}{2}$ *or* $x=\frac{1-\sqrt{5}}{2}$

326. Solve for *x* and *y*: $x^{99}+y^{99}=x^{100}$

There seem very little prospect in searching for an analytic solution, but tnat does not imply that we .may finfd a solution. We rewrite the equation as:

$$x^{99}+y^{99}=x^{100} \quad \Leftrightarrow \quad x^{99}+y^{99}=x\cdot x^{99}$$
Then it becomes obvious that $x=y=2$, since $2^{99}+2^{99}=2\cdot 2^{99}$

327. Verify that: $2\sqrt{2+\sqrt{3}}=\sqrt{2}+\sqrt{6}$

We shall show this by showing that: $(2\sqrt{2+\sqrt{3}})^2=4(2+\sqrt{3})=(\sqrt{2}+\sqrt{6})^2$ $4(2+\sqrt{3})=$
$$(\sqrt{2}+\sqrt{6})^2=2+6+2\sqrt{2}\sqrt{6}=$$
$$2+6+2\sqrt{2}\sqrt{2}\sqrt{3}=$$
$$8+4\sqrt{3}=4(2+\sqrt{3})$$

328. Determine integer values, such that: $x+y+xy=54$

There is really no analytic way to solve this equation, since there is one equation and two variables.

$$x+y+xy=54 \quad \Leftrightarrow \quad x+y=54-xy$$

$x=8 \quad and \quad y=5$ Could be a candidate, but $13=54-40=14$
$x=9 \quad and \quad y=4$ Could also be a candidate, but $13=54-36=19$
$x=10 \quad and \quad y=4$ Could also be a candidate, since $14=54-40=14$

$x=10 \quad and \quad y=4$ is a solution.

329. Simplify: $\frac{2\cdot\sqrt[3]{8}}{\sqrt[3]{2}\cdot\sqrt[3]{32}}$

$$\frac{2\cdot\sqrt[3]{8}}{\sqrt[3]{2}\cdot\sqrt[3]{32}}=\frac{2\cdot 2}{\sqrt[3]{2}\cdot\sqrt[3]{4\cdot 8}}=\frac{4}{\sqrt[3]{2}\cdot\sqrt[3]{2^2}\cdot\sqrt[3]{8}}=\frac{4}{\sqrt[3]{2^3}\cdot\sqrt[3]{8}}=\frac{4}{2\cdot 2}=1$$

330. Determine *a* + *b*, when: $a^3+b^3=2\sqrt{5} \quad and \quad a^2b+ba^2=\sqrt{5}$

$(a+b)^3=a^3+b^3+3ab(a+b) \quad and \quad a^2b+ba^2=ab(a+b)=\sqrt{5} \quad \Rightarrow$
$(a+b)^3=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}=\sqrt{5}^3 \quad \Rightarrow$
$a+b=\sqrt{5}$

331. Determine integer values (*x,y*) such that. $x+xy+y=54$

There is really no way to determine integer values for this equation to fulfil , so we will resort to qualified guesswork:
$x+xy+y=54 \quad \Leftrightarrow \quad x+y=54-xy$
It seems that $x=10 \quad and \quad y=4$ does the trick: Since: 10+4=54-40

332. Solve the differential equation : $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)-1$

$$\frac{dy}{dx}=\sin(x+y)+\cos(x+y)-1$$

We put $z=x+y \quad \Rightarrow \quad \frac{dz}{dx}=1+\frac{dy}{dx}$ and then we get:

$$\frac{dz}{dx}-1=\sin(x+y)+\cos(x+y)-1 \quad \Leftrightarrow$$

$$\frac{dz}{dx}=\sin(z)+\cos(z) \quad \Leftrightarrow \quad \frac{dz}{dx}=\sin(z)+\sin(\frac{\pi}{2}-z) \quad \Leftrightarrow$$

We then apply the formula:

$$\sin u+\sin v=2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$$

$$\frac{dz}{dx}=\sin(z)+\sin(\frac{\pi}{2}-z) \quad \Leftrightarrow \quad \frac{dz}{dx}=2\sin(\frac{\pi}{4})\cdot\cos(z-\frac{\pi}{4})=\sqrt{2}\cos(z-\frac{\pi}{4})$$

And then we have: $\cos(z-\frac{\pi}{4})=\sin(\frac{\pi}{2}-(z-\frac{\pi}{4})=-\sin(z-\frac{3\pi}{4})$

We put: $v=\frac{3\pi}{4}-z \quad \Rightarrow \quad dv=.-dz \quad and \quad -\sin(z-\frac{3\pi}{4})=\sin v$

$$\frac{dv}{dx} = \sqrt{2}\sin v \quad \Leftrightarrow \quad \frac{dv}{\sin(v)} = \sqrt{2}dx \quad \Leftrightarrow$$

$$\frac{\left(\cos^2(\frac{v}{2}) + \sin^2(\frac{v}{2})\right)dv}{2\sin(\frac{v}{2})\cos(\frac{v}{2})} = \sqrt{2}dx \quad \Leftrightarrow$$

Then we divide with $\sin(\frac{v}{2})\cos(\frac{v}{2})$

$$\frac{\left(\frac{\cos(\frac{v}{2})}{\sin(\frac{v}{2})} + \frac{\sin(\frac{v}{2})}{\cos(\frac{v}{2})}\right)dv}{2} = \sqrt{2}dx \quad \Leftrightarrow$$

$$(ctn(\frac{v}{2}) + \tan(\frac{v}{2}))dv = 2\sqrt{2}dx \quad \Leftrightarrow$$

$$\int (ctn(\frac{v}{2}) + \tan(\frac{v}{2}))dv = 2\sqrt{2}\int dx \int (ctn(\frac{v}{2}) + \tan(\frac{v}{2}))dv = 2\sqrt{2}\int dx \quad \Leftrightarrow$$

$$\tfrac{1}{2}\ln\sin\tfrac{v}{2} - \tfrac{1}{2}\ln\cos\tfrac{v}{2} = 2\sqrt{2}x + c$$

Where have applied;

$$\int \frac{\sin x}{\cos x}dx = -\int \frac{d\cos x}{\cos x}dx = -\ln\cos x \quad and \quad \int \frac{\cos x}{\sin x}dx = \int \frac{d\sin x}{\sin x}dx = \ln\sin x$$

We then substitute back; $v = \frac{3\pi}{4} - z$

$$\tfrac{1}{2}\ln\sin\tfrac{v}{2} - \tfrac{1}{2}\ln\cos\tfrac{v}{2} = 2\sqrt{2}x \quad \Leftrightarrow \quad \frac{1}{2}\ln\sin(\frac{3\pi}{8} - \frac{z}{2})' - \frac{1}{2}\ln\cos(\frac{3\pi}{8} - \frac{z}{2}) = 2\sqrt{2}x \quad \Leftrightarrow$$

$$\frac{1}{2}\ln(-\sin(\frac{z}{2} - \frac{3\pi}{8}))' - \frac{1}{2}\ln(\cos(\frac{z}{2} - \frac{3\pi}{8})) = 2\sqrt{2}x$$

Finally we substitute back: $z = x + y$

$$\frac{1}{2}\ln(-\sin(\frac{x+y}{2} - \frac{3\pi}{8}))' - \frac{1}{2}\ln(\cos(\frac{x+y}{2} - \frac{3\pi}{8})) = 2\sqrt{2}x$$

As it is obvious, that there is no way to isolate y from this equation.

333. Simplify $\sqrt[20]{13\cdot 1021\cdot 79 + 9}$

$\sqrt[20]{13\cdot 1021\cdot 79 + 9} = ?$

$13\cdot 79 = 1027 = 1024 + 3$

$1021 = 1024 - 3$

So:

$$\sqrt[20]{13\cdot1021\cdot79+9}=\sqrt[20]{(1024+3)(1024-3)+9}=\sqrt[20]{(1024^2-3^2)+9}=\sqrt[20]{1024^2}=\sqrt[20]{2^{20}}=2$$

334. simplify $\dfrac{500^{1000}}{1000^{500}}$

$$\frac{500^{1000}}{1000^{500}}=\frac{500^{2\cdot500}}{1000^{500}}=\frac{500^{500}}{1000^{500}}=\frac{500^{500}}{1000^{500}}\cdot500^{500}=\left(\frac{500}{1000}\right)^{500}\cdot500^{500}=\left(\frac{1}{2}\right)^{500}\cdot500^{500}=\left(\frac{500}{2}\right)^{500}=$$
$$250^{500}$$

335. Solve for *x*: $\dfrac{\ln\sqrt{x}}{\sqrt{\ln x}}=2$

$$\frac{\ln\sqrt{x}}{\sqrt{\ln x}}=2\quad\Leftrightarrow\quad\frac{\frac{1}{2}\ln x}{\sqrt{\ln x}}=2\quad\Leftrightarrow\quad\frac{\frac{1}{2}\sqrt{\ln x}^{\,2}}{\sqrt{\ln x}}=2\quad\Leftrightarrow$$
$$\tfrac{1}{2}\sqrt{\ln x}=2\quad\Leftrightarrow\quad\sqrt{\ln x}=4\quad\Leftrightarrow\quad\ln x=16\quad\Leftrightarrow\quad x=e^{16}$$

336. Determine posive integers *a, b, c* such that; $2^a+4^b+8^c=328$

$2^a+4^b+8^c=328\quad\Leftrightarrow\quad 2^a+2^{2b}+2^{3c}=328$

There is no way to solve one equations having three unknowns, so we resort to qualified guesswork $c=1,2,3$ $c=3$ is too large, so we try with: $c=2$ that gives $2^6=64$, then we try with the highest possible value of *b*, which is 4. And $2^8=256$ Now 256+64 = 320 and 328 – 320 = 8 = 2^3, so *a* =3.
The solution is then: *a* =3, *b* =4 and *c* =2.

337. Determine *a* , *b* and *c*, such that: $1+2^a+3^b=6^c$

It is very easy to see that: *a* = *b* =*c* is a solution, since: 1+2+3=6
We then try *c* = 2. 6^2= 36 . Then we try with *b* = 3 $3^b=3^3$ = 36 - 27 = 9, and indeed $1+2^3=9$,
So one solution is; *a* = 3, *b* = 3 , *c* = 2.

338. Find integer values of *a* and *b,* such that; $\sqrt{a}+\sqrt{b}=\sqrt{2009}$

An analytic approach is rather difficult:
But we observe that $2009=7^2\cdot41$, and 41 is a prime. So the equation may be written:
$\sqrt{a}+\sqrt{b}=7\sqrt{41}$, this implies that if two integers; $x+y=7\sqrt{41}$ then they may both be a integer multiple of $\sqrt{41}$. The possibilities are: $(x,y)=(1,6),(2,5),(3,4)$,
In the first case: $a=1\cdot41=41,\quad and\quad b=36\cdot41$, which gives:

$$\sqrt{a}+\sqrt{b}=7\sqrt{41}\quad\Leftrightarrow\quad\sqrt{41}+\sqrt{36\cdot41}=\sqrt{41}+6\sqrt{41}=7\sqrt{41}$$

The two other cases are likewise: $a=4\cdot41=165\quad and\quad b=25\cdot41=1025$, or
$a=9\cdot41=369\quad and\quad b=16\cdot41=656$ which gives:

$$\sqrt{a}+\sqrt{b}=7\sqrt{41}\quad\Leftrightarrow\quad\sqrt{9\cdot41}+\sqrt{16\cdot41}=3\sqrt{41}+4\sqrt{41}=7\sqrt{41}$$

339. Solve for *x*; $\frac{1}{x}e^{\frac{1}{x}} = e$

This is a ridiculous exercise, since it is a transcendent equation, which in general has no analytic solution.

However, it is obvious, that x = 1 is the only solution, since: $\frac{1}{1}e^{\frac{1}{1}} = e$

340. Solve for *x*. $x^5 + x^4 + x^3 + x^2 + x = -1$

At a glance it is obvious that x = -1 is a solution, since: -1 +1 -1 +1+ .1 = -1, but there may be other solutions (max 5). We do the following rewriting.

$$x^5 + x^4 + x^3 + x^2 + x + 1 = 0 \quad \Leftrightarrow \quad x^3(x^2 + x + 1) + x^2 + x + 1 = 0 \quad \Leftrightarrow$$

$$(x^2 + x + 1)(x^3 + 1) = 0 \quad \Leftrightarrow \quad x^3 = -1 \quad \vee \quad x^2 + x + 1 = 0 \quad (d = 1 - 4 < 0) \quad \Leftrightarrow$$

$$x = -1$$

341. Solve for *x*: $x^{2x^6} = 3$

$x^{2x^6} = 3$ There is (to my knowledge) no analytic approach to this equation, so we shall resort to qualified guesswork. $\sqrt[6]{3}$ could be a candidate:

$$(\sqrt[6]{3})^{2\sqrt[6]{3}^6} = (\sqrt[6]{3})^{2\cdot 3} = (\sqrt[6]{3})^6 = 3 \text{, so}$$

$$x^{2x^6} = 3 \quad \Leftrightarrow \quad x = \sqrt[6]{3}$$

342. $\frac{3^x}{4^x} = \frac{5}{2^x}$

$$\frac{3^x}{4^x} = \frac{5}{2^x} \quad \Leftrightarrow \quad 3^x = \frac{5 \cdot 4^x}{2^x} \quad \Leftrightarrow \quad 3^x = \frac{5 \cdot (2^x)^2}{2^x} \quad \Leftrightarrow \quad \left(\frac{3}{2}\right)^x = 5 \quad \Leftrightarrow \quad x = \frac{\ln 5}{\ln 3 - \ln 2}$$

343. Determine *a, b, c*, such that: $a^3 + b^3 + c^3 = (abc)^3$

Well, to my knowledge the only numbers where $a + b + c = abc$ are 1, 2, 3, since $1 + 2 + 3 = 1 \cdot 2 \cdot 3$ so the solution is obviously: $a = \sqrt[3]{1}, b = \sqrt[3]{2}$, $c = \sqrt[3]{3}$, which gives:

$$a^3 + b^3 + c^3 = \sqrt[3]{1}^3 + \sqrt[3]{2}^3 + \sqrt[3]{3}^3 = 1 + 2 + 3 = (\sqrt[3]{1} \cdot \sqrt[3]{2} \cdot \sqrt[3]{3})^3 = 1 \cdot 2 \cdot 3$$

344. Find integer solutions to: $ab - cd = 34$ and $ac - bd = 19$

$$ab - cd = 34$$
$$ac - bd = 19$$

Well, we have four unknowns and two (non linear equations), so it requires a lot of guesswork: Adding the two equations gives however:

$$ab - cd + ac - bd = 53 \quad \Leftrightarrow \quad a(b + c) - d(b + c) = 53 \quad \Leftrightarrow \quad (b + c)(a - d) = 53$$

Now since 53 is a prime is has only the divisors 1 and 53, so we chose: $b + c = 53 \quad and \; a - d = 1$

If we choose *a*, then we can find *d*, To determine how to distribute 53 on *b* and *c*. we use the firt equation, where we insert $d = a - 1$ and $c = 53 - b$.

$ab - cd = 34 \quad \Rightarrow \quad ab - (53-b)(a-1) = 34 \quad \Leftrightarrow \quad 2ab - 53a + 53 - b = 34$
Then we try with various values of *a* to determine *b*.

$a = 1$: $a = 1: \Rightarrow \quad b = 32,\ c = 19, d = 0$, Which is easily verified as a solution.
$a = 2$ wont work, because it results in a non integer *b*.
$a = 3 \quad \Rightarrow \quad 2ab - 53a + 53 - b = 34 \quad 6b - 53\cdot 3 + 53 - b = 34 \quad \Rightarrow \quad 5b = 140 \quad \Leftrightarrow \quad b = 28$
So the solution is:

$a = 3, b = 28, c = 25,\ d = 2$

345. Solve for *x*. $x^{x^3} = 729$.

It is clear that 3 is a key number to this problem. We could try with $x = \sqrt[3]{3}$. But this gives 3.

We also notice that: $729 = 3^6$. Then we could try with $x = \sqrt[3]{3}^2$, and;

$(\sqrt[3]{3}^2)^{(\sqrt[3]{3}^2)^3} = (\sqrt[3]{3}^2)^9 = 3^6 = 729$

346. Solve for *x*: $\sqrt{1 - \frac{x^3}{1000}} = (1 - \frac{2}{3})^{-1}$

$\sqrt{1 - \frac{x^3}{1000}} = (1 - \frac{2}{3})^{-1}. \quad (1 - \frac{2}{3})^{-1} = (\frac{1}{3})^{-1} = 3$

$\sqrt{1 - \frac{x^3}{1000}} = 3 \quad \Leftrightarrow \quad 1 - \frac{x^3}{1000} = 9 \quad \Leftrightarrow \quad x^3 = -8000 \quad \Leftrightarrow \quad x = -20$

347. Solve for *x*. $\sqrt[3]{2-x} + \sqrt{x-1} = 1$

A good guess is $x = 10$, and indeed; $\sqrt[3]{2-10} + \sqrt{10-1} = \sqrt[3]{-8} + \sqrt{9} = -2 + 3 = 1$

348. Find integer solutions to: $\frac{1}{x} - \frac{1}{y} = \frac{2}{35}$

$\frac{1}{x} - \frac{1}{y} = \frac{2}{35} \quad \Leftrightarrow \quad \frac{x-y}{xy} = \frac{2}{35}$

Since 5 times 7 is 35, it is straightforward to guess: $x = 7$ and $y = 5$, and we find:

$\frac{x-y}{xy} = \frac{7-5}{35} = \frac{2}{35}$

359. Find the largest integer for which $\dfrac{n^3+100}{n+10}$ is an integer

The first thought is to make polynomial division with n +1 and require that the division goes up.

$$
\begin{aligned}
&n+10 \mid n^3+100 \mid n^2-10n \\
&\quad n^3+10n^2 \\
&\quad -10n^2+100 \\
&\quad -10n^2-100n \\
&\qquad 100+100n
\end{aligned}
$$

If the remainder should be 0, n should be -1, and $\dfrac{-1+100}{-1+10}=\dfrac{99}{9}=9$

If we want to find a positive solutions, we shall first resort to guesswork:

It seems that n = 5 is a possible candidate since: $\dfrac{n^3+100}{n+10}=\dfrac{125+100}{5+10}=\dfrac{225}{15}=15$

So we have two solutions: n = -1 or n = 5.

However to find possible other solutions, we rewrite n^3+100 as

$$
\begin{aligned}
&n^3+100=n^3+1000-900 \\
&(n+10)^3=n^3+1000+3n\cdot 10(n+10) \\
&n^3=(n+10)^3-1000-3n\cdot 10(n+10) \\
&n^3+100=(n+10)((n+10)^2-30n)-900
\end{aligned}
$$

So the possible divisors n+10, can principle be all divisors in 900.

$$900=9\cdot 4\cdot 25=3^2\cdot 2^2\cdot 5^2$$

We have already found n = 5. The next candidate is n = 10, and $\dfrac{10^3+100}{10+10}=\dfrac{1100}{20}=55$

The next candidate is n = 20, and $\dfrac{20^3+100}{20+10}=\dfrac{8100}{30}=270$

The next candidate is n = 25, but n + 10 =35 is not a divisor in 900

The next candidate is n = 40, and $\dfrac{40^3+100}{40+10}=\dfrac{64100}{50}=1282$, but n + 10 = 35 is not a divisor in 900

The largest divisor in 900 is 900 = n + 10, So the largest n for which the division goes up is 890.

$$\frac{890^3+100}{900}=\frac{704969100}{900}=783299$$

350. Determine *x* and *y* from $x+y=2$ *and* $x^4+y^4=1234$

It is obvious that two positive numbers can satisfy these two equations. We proceed as follows;

$x+y=2 \quad \Rightarrow \quad (x+y)^2=4 \quad \Leftrightarrow \quad x^2+y^2+2xy=4 \quad \Leftrightarrow x^2+y^2=4-2xy$

$x+y=2 \quad \Rightarrow \quad (x+y)^3=8 \quad \Leftrightarrow \quad x^3+y^3+3xy(x+y)=8 \quad \Leftrightarrow \quad x^3+y^3=8-6xy$

$x+y=2 \quad \Rightarrow \quad (x+y)^4=16 \quad \Leftrightarrow \quad (x+y)(x+y)^3=16 \quad \Leftrightarrow (x+y)\,(x^3+y^3+6xy)=16 \quad \Leftrightarrow$
$(x+y)\,(x^3+y^3+6xy)=16 \quad \Leftrightarrow \quad x^4+y^4+xy^3+yx^3+12xy=16 \quad \Leftrightarrow$
$x^4+y^4+xy(x^2+y^2)+12xy=16 \quad \Leftrightarrow \quad 1218+xy(4-2xy)+12xy-16=0 \quad \Leftrightarrow$
$-2(xy)^2+16xy+1218=0 \quad \Leftrightarrow$
$(xy)^2-8xy-609=0; \quad d=8^2+4\cdot 609=2500=50^2$
$xy=\frac{8\mp 50}{2} \quad \Leftrightarrow \quad xy=29 \quad \vee \quad xy=-21$

The positive solution can not satisfy $x+y=2$, so we solve the two equations:
$x+y=2$ and $xy=-21$

$xy=-21 \quad \Leftrightarrow \quad y=-\frac{21}{x}$ inserted in $x+y=2$ gives

$x-\frac{21}{x}=2 \quad \Leftrightarrow \quad x^2-2x-21=0; \quad d=4+4\cdot 21=4\cdot 22$

$x=\frac{2\pm 2\sqrt{22}}{2} \quad x=1+\sqrt{22} \quad \vee \quad x=1-\sqrt{22} \quad \Rightarrow$

$y=2-x \quad \Rightarrow y=1-\sqrt{22} \quad \vee \quad y=1+\sqrt{22}$

351. Simplify: $((2-\sqrt{5})^{198}\cdot(9+4\sqrt{5})^{99}$

$(2-\sqrt{5})^2=4+5-4\sqrt{5}=9-4\sqrt{5}$, so

$((2-\sqrt{5})^{198}\cdot(9+4\sqrt{5})^{99}=((2-\sqrt{5})^2)^{99}\cdot(9+4\sqrt{5})^{99}=(9-4\sqrt{5})^{99}\cdot(9+4\sqrt{5})^{99}=$

$((9-4\sqrt{5})(9+4\sqrt{5}))^{99}=(81-80)^{99}=1^{99}=1$

352. Simplify: $2\sqrt[3]{8}+\sqrt[3]{2}\sqrt[3]{32}$

$2\sqrt[3]{8}+\sqrt[3]{2}\sqrt[3]{32}=2\cdot 2+\sqrt[3]{64}=4+\sqrt[3]{2^6}=4+4=8$

353. Solve for *x*: $7^{\log_8 x}\cdot x^{\log_8 9}=3969$

$3969=7^2\cdot 3^4$

$$7^{\log_8 x}\cdot x^{\log_8 9}=3969 \quad\Leftrightarrow\quad \log_8(7^{\log_8 x}\cdot x^{\log_8 9})=\log_8(3969) \quad\Leftrightarrow$$
$$(\log_8 x)(\log_8 7)+(\log_8 9)(\log_8 x)=\log_8(7^2\cdot 3^4) \quad\Leftrightarrow$$
$$\log_8 x(\log_8 7+\log_8 3^2)=2\log_8 7+4\log_8 3 \quad\Leftrightarrow$$
$$\log_8 x(\log_8 7+2\log_8 3)=2(\log_8 7+2\log_8 3) \quad\Leftrightarrow$$
$$\log_8 x=2 \quad\Leftrightarrow\quad x=8^2$$

354. Calculate the sum: $\frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+....$

$$S=\frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+....=\frac{3}{2}+\frac{5}{2^3}+\frac{7}{2^5}+\frac{9}{2^7}+....=$$
$$\frac{1+2}{2}+\frac{3+2}{2^3}+\frac{5+2}{2^5}+\frac{7+2}{2^7}+....=(1+\frac{1}{2})+(\frac{3}{2^3}+\frac{1}{2^2})+(\frac{5}{2^5}+\frac{1}{2^4})+(\frac{7}{2^7}+\frac{1}{2^6})+...=$$
$$(1+\frac{3}{2^3}+\frac{5}{2^5}+\frac{7}{2^7}...)+(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}...)=1+\frac{1}{2^2}(\frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+...)+\frac{1}{2}+(\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}...)=$$
$$1+\frac{1}{2^2}(\frac{3}{2}+\frac{5}{8}+\frac{7}{32}+\frac{9}{128}+...)+\frac{1}{2}+\frac{1}{2^2}(1+\frac{1}{2^2}+\frac{1}{2^3}...)=\frac{3}{2}+\frac{1}{2^2}S+\frac{1}{2^2}(\frac{1}{1-\frac{1}{2}})=$$
$$S=\frac{3}{2}+\frac{1}{2^2}>S+\frac{1}{2^2}2 \quad\Leftrightarrow\quad S=\frac{3}{2}+\frac{1}{2^2}S+\frac{1}{2} \quad\Leftrightarrow\quad S(1-\frac{1}{2^2})=2 \quad\Leftrightarrow$$
$$S=\frac{8}{3}$$

355. Determine *a* and *b* from: $a^{\ln a}=b^{\ln b}$ **and** $a-b=1$

$$a^{\ln a}=b^{\ln b} \quad\Leftrightarrow\quad \ln a(\ln a)=\ln b(\ln b) \quad\Leftrightarrow\quad (\ln a)^2=(\ln b)^2 \Leftrightarrow$$
$$\ln a=\ln b \quad or \quad \ln a=-\ln b \quad\Leftrightarrow\quad a=b \quad or \quad \ln ab=0$$

The first solution does gives $a-b=0$ but the second solution gives $ab=1$.

$$a-b=1 \quad and \quad ab=1 \quad\Rightarrow\quad a-\frac{1}{a}-1=0 \quad\Rightarrow$$
$$a^2-a-1=0;\quad d=1+4=5$$
$$a=\frac{1\pm\sqrt{5}}{2} \quad\Rightarrow\quad a=\frac{1+\sqrt{5}}{2}$$

Since a cannot be negative.

356. Solve for *x*. $2^{\log_2\frac{x+1}{x+2}}=4$

$$2^{\log_2\frac{x+1}{x+2}}=4 \quad\Leftrightarrow\quad \log_2\frac{x+1}{x+2}=4 \quad\Leftrightarrow\quad \frac{x+1}{x+2}=2^4$$
$$\frac{x+1}{x+2}=16 \quad\Leftrightarrow\quad 15x-31=0 \quad\Leftrightarrow\quad x=\frac{31}{15}$$

357. Solve for *x*; $4^x + 9^x + 25^x = 6^x + 10^x + 15^x$

$4^x + 9^x + 25^x = 6^x + 10^x + 15^x \quad \Leftrightarrow \quad (2^x)^2 + (3^x)^2 + (5^x)^2 = 2^x 3^x + 2^x 5^x + 3^x 5^x$

We put $2^x = a,\ 3^x = b,\ \ 5^x = c$ and then we have:

$a^2 + b^2 + c^2 = ab + ac + bc$

We then multiply this equation with 2.

$2a^2 + 2b^2 + 2c^2 = 2ab + 2ac + 2bc$

Which can be written as:

$(a-b)^2 + (a-c)^2 + (b-c)^2 = 0$

This is however only possible if $a = b = c \quad \Leftrightarrow \quad 2^x = 3^x = 5^x$

So the only solution is $x = 0$.

358. Solve for *x*: $(1+\frac{1}{x})^{x+1} = (1+\frac{1}{6})^6$

At a glance this seems impossible, but when you realise, that x might be a negative number, is it not.

$(1+\frac{1}{x})^{x+1} = (1+\frac{1}{6})^6 \quad \Leftrightarrow \quad (1+\frac{1}{x})^{x+1} = (\frac{7}{6})^6 \quad \Leftrightarrow$

$(1+\frac{1}{x})^{x+1} = (\frac{6}{7})^{-6} \quad \Leftrightarrow \quad (1+\frac{1}{x})^{x+1} = (1-\frac{1}{7})^{-6} \quad \Leftrightarrow \quad x = -7$

359. Determine *f*(*x*) from the equation: $3f(x) + 4xf(\frac{1}{x}) = 5x - 6$

1) $3f(x) + 4xf(\frac{1}{x}) = 5x - 6$ We multiply this equation with 3

We replace x with $\frac{1}{x}$:

2) $3f(\frac{1}{x}) + \frac{4}{x}f(x) = \frac{5}{x} - 6$ We multiply this equation with 4x

1) $9f(x) + 12xf(\frac{1}{x}) = 15x - 18$

2) $12f(\frac{1}{x}) + 16f(x) = 20 - 24x$

We then subtract 1) from 2): to give:

$7f(x) = 39x - 38 \quad \Leftrightarrow \quad f(x) = \frac{39x - 38}{7}$

360. Solve for *x*: $x^{\sqrt{x}} = \frac{1}{2}$

$x^{\sqrt{x}} = \frac{1}{2}$

To my knowledge there is no way to solve this equation analytically, so we shall resort to guesswork; $x = \frac{1}{2}$ does not work, so we try with $x = \frac{1}{4}$ and indeed:

$(\frac{1}{4})^{\sqrt{\frac{1}{4}}} = (\frac{1}{4})^{\frac{1}{2}} = \frac{1}{2}$

On the other hand:

$x^{\sqrt{x}} = \frac{1}{2} \quad \Leftrightarrow \quad x^{\sqrt{x}} = (\frac{1}{4})^{\sqrt{\frac{1}{4}}}$ so $x = \frac{1}{4}$

361. Solve for *a*: $2^{3a} + 2^a = 10$

$2^{3a} + 2^a = 10$ We put $y = 2^a$ and we get:

$y^3 + y - 10 = 0$ We see that $y = 2$ is a solution:

To find possible other solutions, we make polynomial division with *y* - 2;

$y - 2 \mid y^3 + y - 10 \mid y^2 + 2y + 5$

$y^3 - 2y^2$

$2y^2 + y$

$2y^2 - 4y$

$5y - 10$

$5y - 10$

0

$y^2 + 2y + 5 = 0 \quad d = 4 - 20 < 0$ No solution.

362. Solve the integral: $\int \cos(x - \frac{1}{x})dx$

$\int \cos(x - \frac{1}{x})dx$ This integral has no immediate solution, but we try with the substitution

$t = \frac{1}{x} \quad \Rightarrow \quad dt = -\frac{1}{x^2}dx \quad \Rightarrow \quad dx = -x^2 dt = -\frac{1}{t^2}dt$

$\int \cos(x - \frac{1}{x})dx = -\int \frac{1}{t^2}\cos(\frac{1}{t} - t)dt = -\int \frac{1}{t^2}\cos(t - \frac{1}{t})dt = -\int \frac{1}{x^2}\cos(x - \frac{1}{x})dx \quad \Rightarrow$

$$\int \cos(x-\frac{1}{x})dx + \int \frac{1}{x^2}\cos(x-\frac{1}{x})dx = 0 \quad \Leftrightarrow \quad \int (1+\frac{1}{x^2})\cos(x-\frac{1}{x}) = 0 \quad \Leftrightarrow \quad \int d\sin(x-\frac{1}{x}) = 0 \quad \Leftrightarrow$$

$$\sin(x-\frac{1}{x}) = c$$

363. Solve for *x*: $\frac{27^x - 8^x}{18^x - 12^x} = \frac{19}{6}$

$$\frac{27^x - 8^x}{18^x - 12^x} = \frac{19}{6} \quad \Leftrightarrow \quad \frac{(3^x)^3 - (2^x)^3}{(3^2 \cdot 2)^x - (2^2 \cdot 3)^x} = \frac{19}{6}$$

We put: $a = 2^x \quad and \quad b = 3^x$

$$\frac{(3^x)^3 - (2^x)^3}{(3^2 \cdot 2)^x - (2^2 \cdot 3)^x} = \frac{19}{6} \quad \Leftrightarrow \quad \frac{b^3 - a^3}{b^2 a - ba^2} = \frac{19}{6} \quad \Leftrightarrow \quad \frac{b^3 - a^3}{ab(b-a)} = \frac{19}{6}$$

We notice that:

$$(b-a)^3 = b^3 - a^3 - 3ba(b-a) \quad \Leftrightarrow$$

$$b^3 - a^3 = (b-a)^3 + 3ba(b-a) = (b-a)((b-a)^2 + 3ba)$$

$$\frac{b^3 - a^3}{ab(b-a)} = \frac{19}{6} \quad \Leftrightarrow \quad \frac{(b-a)((b-a)^2 + 3ba)}{ab(b-a)} = \frac{19}{6} \quad \Leftrightarrow \quad \frac{(b-a)^2 + 3ba}{ab} = \frac{19}{6}$$

$$\frac{b^2 + a^2 - 2ab + 3ba}{ab} = \frac{19}{6} \quad \Leftrightarrow \quad \frac{b^2 + a^2 + ab}{ab} = \frac{19}{6}$$

We divide the lhs by ab :

$$\frac{\frac{b}{a} + \frac{a}{b} + 1}{1} = \frac{19}{6} \quad \Leftrightarrow \quad \frac{b}{a} + \frac{a}{b} + 1 = \frac{19}{6}$$

We put; $y = \frac{b}{a}$ and we get:

$$y + \frac{1}{y} + 1 = \frac{19}{6} \quad \Leftrightarrow \quad 6y^2 - 13y + 6 = 0; \quad d = 169 - 144 = 25$$

$$y = \frac{13 \pm 5}{12} \quad \Leftrightarrow \quad y = \frac{3}{2} \quad \vee \quad y = \frac{2}{3} \quad \Rightarrow$$

$$(\frac{2}{3})^x = \frac{3}{2} \quad \vee \quad (\frac{2}{3})^x = \frac{2}{3} \quad \Leftrightarrow$$

$$x = -1 \quad \vee \quad x = 1$$

364. Solve for *x*: $\sqrt[x]{9}+\sqrt[x]{6}=\sqrt[x]{4}$

First I shall argue that this exercise is not formulated in accordance with mathematical terminology.
The symbol $\sqrt[n]{a}$ is defined only for integers greater than 1, and a is a non negative real number.
The symbol a^x is defined only for positive real numbers a, and x real.
Furthermore x must be negative (which excludes the symbol $\sqrt[n]{a}$) because othjerwise both terms on the lhs are bigger than the rhs. So I assume that is meant.

$$9^{\frac{1}{x}}+6^{\frac{1}{x}}=4^{\frac{1}{x}}$$

For typographical convenience we put: $y=\frac{1}{x}$: So we have: $9^y+6^y=4^y$

We divide this equation with: 9^y to give: $1+\frac{6^y}{9^y}=\frac{4^y}{9^y} \quad \Leftrightarrow \quad 1+\left(\frac{2}{3}\right)^y=\left(\left(\frac{2}{3}\right)^y\right)^2$

We put; $z=\left(\frac{2}{3}\right)^y$ and find: $z^2-z-1=0$; $\quad d=1+4 \quad \Rightarrow \quad z=\frac{1\pm\sqrt{5}}{2}$

Since z must be positive root, we discard the negative root.

$$\left(\frac{2}{3}\right)^y=\frac{1+\sqrt{5}}{2} \quad \Rightarrow \quad y=\frac{\ln(\frac{1+\sqrt{5}}{2})}{\ln 2-\ln 3}; \quad x=\frac{1}{y} \quad \Rightarrow \quad x=-\frac{\ln 3-\ln 2}{\ln(\frac{1+\sqrt{5}}{2})}$$

365. Solve for *x*: $\frac{3^{x^2}}{9^x}=27$

$$\frac{3^{x^2}}{9^x}=27 \quad \Leftrightarrow \quad \frac{3^{x^2}}{3^{2x}}=3^3 \quad \Leftrightarrow \quad 3^{x^2-2x}=3^3 \quad \Leftrightarrow$$

$$x^2-2x=3 \quad \Leftrightarrow \quad x^2-2x-3=0; \quad d=4+12=16$$

$$x=\frac{2\pm\sqrt{16}}{2}=1\pm 2 \quad \Leftrightarrow \quad x=3 \quad \vee \quad x=-1$$

366. Determine *a, b, c* such that: $(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})=2$

Well, we have one equation with three unknowns, so we will resort to guesswork.
If $a=b=c$ we can find a solution, whereas it is difficult to prove that there are other solutions.

$$a=b=c \quad \Rightarrow \quad (1+\frac{1}{a})^3=2 \quad \Rightarrow \quad (1+\frac{1}{a})=\sqrt[3]{2} \quad \Rightarrow \quad a=b=c=\frac{1}{\sqrt[3]{2}-1}$$

367. Evaluate the integral: $\int x^2\sqrt{1-x^2}dx$

We make the substitution $t=\cos x \quad \Rightarrow \quad dt=-\sin x dx$ and we find:

$\int x^2\sqrt{1-x^2}\,dx = -\int \cos^2 t\sqrt{1-\cos^2 t}\,\sin t\,dt =$
$-\int \cos^2 t \sin^2 t\,dt = -\frac{1}{4}\int 4\cos^2 t\sin^2 t\,dt$
$-\frac{1}{4}\int \sin^2 2t\,dt = -\frac{1}{8}\int(1-\cos 4t)dt =$
$-\frac{1}{8}(t-\frac{1}{4}\sin 4t)+c$

We have made use of:

$\sin 2t = 2\sin t\cos t \quad and \quad \cos 2t = 1-2\sin^2 t \;\Rightarrow$
$\cos 4t = 1-2\sin^2 2t \quad \Rightarrow = \sin^2 2t = \frac{1}{2}(1-\cos 4t)$

368. Solve for *x*: $x - x^2 - 2x^3 = \dfrac{1}{3}$

$x - x^2 - 2x^3 = \dfrac{1}{3} \quad \Leftrightarrow \quad 3x - 3x^2 - 6x^3 = 1$
We notice that: $(x-1)^3 = x^3 - 1^3 - 3x(x-1)$
x-y

369a. Solve for *x* and *y*; $x^3 + y^3 = 2xy + 8$??

$x^3 + y^3 = 2xy + 8$
There is no reason to do calculation, since the solution evidently $x = y = 2$,
since $2^3 + 2^3 = 2\cdot 2\cdot 2 + 8$

369b. Solve for *x* and *y*; $x^3 - y^3 = 2xy + 8$

$x^3 - y^3 = 2xy + 8$
$(x-y)^3 = x^3 - y^3 - 3xy(x-y) \quad \Rightarrow \quad x^3 - y^3 = (x-y)^3 + 3xy(x-y) = 2xy + 8$
We put: $a = x - y \quad and \quad b = xy$
$a^3 + 3ab = 2b + 8$
We cannot solve this third degree equation. So we guess some a ,and we see if we can find *b*, if we require integer values.

$a = 1; \quad 1 + 3b - 2b - 8 = 0 \;\Rightarrow\; b = 7$. A solution.

$a = x - y = 1 \quad and \quad b = xy = 7 \quad \Rightarrow y = \dfrac{7}{x} \quad \Rightarrow$

$x - \dfrac{7}{x} = 1 \quad \Leftrightarrow \quad x^2 - x - 7 = 0 \quad d = 1 + 28 = 29; \quad x = \dfrac{1\pm\sqrt{29}}{2} \quad y = \dfrac{14}{1\pm\sqrt{29}}$

$a = 2: \quad 8 + 6b - 2b - 8 = 0 \quad \Rightarrow \quad b = 0$. a solution

$x - y = 2 \quad and \quad xy = 0 \quad \Rightarrow \quad x = 2 \quad and \quad y = 0$

$a=-2: \quad -8-6b-2b-8=0 \quad \Rightarrow \quad -8b=16 \quad \Leftrightarrow b=-2$. a solution.

$a=x-y=-2 \quad and \quad b=xy=-2 \quad \Rightarrow$

$x-\frac{2}{x}+2=0 \quad \Leftrightarrow \quad x^2+2x-2=0 \quad d=4+8=12=3\cdot 4$

$x=\frac{-2\pm 2\sqrt{3}}{2}=-1\pm\sqrt{3} \quad \Rightarrow \quad y=\frac{1}{-1\pm\sqrt{3}}$

No solution.

370. Solve for *x*: $x^{\sqrt{x}}=\sqrt{x^x}$

It is rather easy to realize that: $x=4$ is the solution, since; $x^{\sqrt{x}}=4^{\sqrt{4}}=4^2 \quad and \quad \sqrt{x^x}=4^{4\cdot\frac{1}{2}}=4^2$

371. Solve for *x:* $\sqrt{2x-4}-\sqrt{3x+4}=-2$

Well no need to make long calculations, since it is obvious that $x = 4$ is a solution, since:

$\sqrt{2\cdot 4-4}-\sqrt{3\cdot 4+4}=\sqrt{4}-\sqrt{16}=2-4=-2$

372. Determine $\frac{x}{y}$ from $2^{x+y}=125$ and $2^{x-y}=25$

$2^{x+y}=125$ and $2^{x-y}=25 \quad \Leftrightarrow \quad 2^{x+y}=5^3$ and $2^{x-y}=5^2 \quad \Leftrightarrow$

$2=5^{\frac{3}{x+y}}$ and $2=5^{\frac{2}{x-y}} \quad \Rightarrow$

$5^{\frac{3}{x+y}}=5^{\frac{2}{x-y}} \quad \Rightarrow \quad \frac{3}{x+y}=\frac{2}{x-y} \quad \Leftrightarrow \quad 3(x-y)=2(x+y) \quad \Leftrightarrow \quad x-5y=0 \quad \Leftrightarrow \quad \frac{x}{y}=5$

373. Simplify: $\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^{2013}$

$2013=3\cdot 671$

$\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^3=\left(\sqrt[3]{5+\sqrt{2}}\right)^3+\left(\sqrt[3]{5-\sqrt{2}}\right)^3+3\sqrt[3]{5+\sqrt{2}}\sqrt[3]{5-\sqrt{2}}(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}})=$

$5+\sqrt{2}+5-\sqrt{2}+3\cdot\sqrt[3]{25-2}(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}})=$

$10+3\sqrt[3]{23}(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}})$

$y=\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)$

$y^3=10+3\sqrt[3]{23}y$

The last equation can only be solved by Cardanos formula, and it will probablygive

$y=\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)$, so it is a step back.

$$\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^{2013}=\left(\left(\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}\right)^{3}\right)^{671}=?^{671}$$

374. Solve for *x*: $\log(\ln x)=\ln(\log x)$

We recall:

$$y=\log x \quad \Leftrightarrow \quad x=10^{y} \quad \Leftrightarrow \quad \ln x = y\ln 10=\log x \ln 10 \quad \Leftrightarrow$$
$$\log x=\frac{\ln x}{\ln 10}$$

$$\log(\ln x)=\ln(\log x) \quad \Leftrightarrow \quad \frac{\ln(\ln x)}{\ln 10}=\ln(\frac{\ln x}{\ln 10})$$

We put: $y=\ln x$:

$$\frac{\ln y}{\ln 10}=\ln(\frac{y}{\ln 10}) \quad \Leftrightarrow \quad \ln y^{\frac{1}{\ln 10}}=\ln(\frac{y}{\ln 10}) \quad \Leftrightarrow$$
$$y^{\frac{1}{\ln 10}}=\frac{y}{\ln 10} \quad \Leftrightarrow \quad y^{\frac{1}{\ln 10}-1}=\frac{1}{\ln 10} \quad \Leftrightarrow$$
$$(\frac{1}{\ln 10}-1)\ln y=-\ln 10$$
$$(\frac{1}{\ln 10}-1)\ln y=-\ln 10 \quad \Leftrightarrow \quad \ln y=\frac{\ln 10}{1-\frac{1}{\ln 10}}=\frac{(\ln 10)^2}{\ln 10-1}$$
$$\ln(\ln x)=\frac{(\ln 10)^2}{\ln 10-1} \quad \Leftrightarrow \quad \ln x=\exp(\frac{(\ln 10)^2}{\ln 10-1}) \quad \Leftrightarrow$$
$$x=\exp(\exp(\frac{(\ln 10)^2}{\ln 10-1}))$$

375. Simplify: $\sqrt{3+\sqrt{8}}$

$$\sqrt{3+\sqrt{8}}=\sqrt{3+2\sqrt{2}}$$

We seek to determine *a* and *b*, such that: $(a+b\sqrt{2})^2=3+2\sqrt{2}$

$$(a+b\sqrt{2})^2=a^2+2b^2+2ab\sqrt{2}$$

We see that this is fulfilled if; $a=b=1$, so: $(1+\sqrt{2})^2=3+\sqrt{2}$ and therefore:

$$\sqrt{3+2\sqrt{2}}=\sqrt{(1+\sqrt{2})^2}=(1+\sqrt{2})$$

375. Simplify: $\frac{\sqrt{2-\sqrt{3}}}{3-\sqrt{3}}$

We seek to determine *a* and *b*, such that: $(a-b\sqrt{3})^2=2-\sqrt{3}$

$(a-b\sqrt{3})^2=a^2+3b^2-2ab\sqrt{3}$ which gives; $a=b=1$, so

$(1-\sqrt{3})^2=2-\sqrt{3}$, so $\sqrt{2-\sqrt{3}}=\sqrt{(1-\sqrt{3})^2}=(\sqrt{3}-1)$

$$\frac{\sqrt{2-\sqrt{3}}}{3-\sqrt{3}}=\frac{\sqrt{3}-1}{3-\sqrt{3}}=\frac{(\sqrt{3}-1)(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}=\frac{3\sqrt{3}+3-\sqrt{3}-3}{9-3}=\frac{\sqrt{3}}{3}$$

376. Solve for *x*: $2^{3^{4^x}}=512$

We notice that: $512=2^9$

$$2^{3^{4^x}}=2^9 \quad\Leftrightarrow\quad 3^{4^x}=9 \quad\Leftrightarrow\quad 3^{4^x}=3^2 \quad\Leftrightarrow\quad 4^x=2 \quad\Leftrightarrow\quad x=\frac{1}{2}$$

377. Evaluate the infinite fraction chain

$$x=1-\cfrac{1}{1-\cfrac{1}{1-....}}$$

We notice that

$$x=1-\cfrac{1}{1-\cfrac{1}{1-....}} \quad\Leftrightarrow\quad x=1-\frac{1}{x} \quad\Leftrightarrow$$

$$x^2-x+1=0; \quad d=1-4=-3$$

$$x=\frac{1\pm i\sqrt{3}}{2} \quad\Leftrightarrow\quad x=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

It seems utterly strange that legal operation with real numbers can actually lead to a imaginary result.
To understand it we shall look at an infinite fraction that leads to the golden ratio.

$$x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{...}}} \quad\Leftrightarrow\quad x=1+\frac{1}{x} \quad\Leftrightarrow\quad x^2-x-1=0;\ d=1+4=5$$

$$x=\frac{1+\sqrt{5}}{2}$$

The golden ratio

If we try to make a series where we cut off at the first, second,….we get
$x=2,\ x=1+\frac{1}{2},\ 1.667,\ldots$ which is quite near to the golden ratio.
However, if we to the same for the first chain of fraction, we get.

$$x=0,\ x=1-\frac{1}{0},....$$

However it is far from transparent that this leads to a complex number, since each denominator in the infinite chain fraction is different from zero.

378. Solve for *x*: $\sqrt{x-2}+3=\sqrt{4x+1}$

No need to calculate, it is obvious that the solution is $x = 2$, since $\sqrt{2-2}+3=\sqrt{4\cdot 2+1}$

379. Solve for *x*: $\sqrt{2x-4}-\sqrt{3x+4}=-2$

No need for calculation since it is obvious that the solution is $x = 4$, since:

$$\sqrt{2\cdot 4-4}-\sqrt{3\cdot 4+4}=-2$$

380. Determine *a* and *b* such that $a+ab+b=798$

$$(a+1)(b+1)=a+ab+b+1=799$$

$799=17\cdot 47$, and both 17 and 47 are primes. So:

$$a+1=17 \quad and \quad b+1=47 \qquad \Leftrightarrow \quad a=16 \quad b=46$$

381. Determine integer *a* such that: $a^3+a^2=36$

Well $3^3=27 \ \ and \ \ 3^2=9 \ \ and \ \ 27+9=36$, so the solution is $a = 3$

382. $x=\sqrt{3+\sqrt{8}}$ **Determine** $x^5-\dfrac{1}{x^5}$

We begin by simplifying $\sqrt{3+\sqrt{8}}=\sqrt{3+2\sqrt{2}}$.

$\sqrt{3+2\sqrt{2}}=(1+\sqrt{2})$, since $(1+\sqrt{2})^2=1+2+2\sqrt{2}=3+2\sqrt{2}$

$$x-\frac{1}{x}=1+\sqrt{2}-\frac{1}{1+\sqrt{2}}=\frac{(1+\sqrt{2})^2-1}{1+\sqrt{2}}=\frac{1+2+2\sqrt{2}-1}{1+\sqrt{2}}=\frac{2+2\sqrt{2}}{1+\sqrt{2}}=\frac{2(1+\sqrt{2})}{1+\sqrt{2}}=2$$

$$(x-\frac{1}{x})^2=4=x^2+\frac{1}{x^2}-2 \qquad \Rightarrow x^2+\frac{1}{x^2}=6$$

$$(x^2+\frac{1}{x^2})(x-\frac{1}{x})=2\cdot 6=x^3-\frac{1}{x^3}-x+\frac{1}{x}=x^3-\frac{1}{x^3}-(x-\frac{1}{x})=x^3-\frac{1}{x^3}-2 \quad \Rightarrow$$

$$x^3-\frac{1}{x^3}=14.$$

$$(x^3-\frac{1}{x^3})(x^2+\frac{1}{x^2})=14\cdot 6=x^5-\frac{1}{x^5}+x-\frac{1}{x}=x^5-\frac{1}{x^5}-2 \quad \Rightarrow$$

$$x^5-\frac{1}{x^5}=86$$

383. $147^x=189$ **Determine** $7^{\frac{2x-1}{x-3}}$

We notice that: $147=3\cdot 7^2 \quad and \quad 189=7\cdot 3^3$ so

$$147^x=189 \quad \Leftrightarrow \quad (3\cdot 7^2)^x=3^3\cdot 7 \quad \Rightarrow$$

$$3^{x-3}\cdot 7^{2x-1}=1 \quad \Leftrightarrow \quad 3\cdot 7^{\frac{2x-1}{x-3}}=1^{\frac{1}{x-3}} \quad \Leftrightarrow \quad 3\cdot 7^{\frac{2x-1}{x-3}}=1 \quad \Leftrightarrow$$

$$7^{\frac{2x-1}{x-3}}=\frac{1}{3}$$

384. Solve for *x*: $(\frac{1}{3})^x + (\frac{1}{5})^x = 34$

$(\frac{1}{3})^x + (\frac{1}{5})^x = 34 \quad \Leftrightarrow \quad 3^{-x} + 5^{-x} = 34$

Now; $3^2 = 9 \quad and \quad 5^2 = 25 \quad and \quad 9 + 25 = 34$,
So the solution is $x = -2$

385. Determine *x* from

$$2 - \frac{1}{\frac{1}{3} - \frac{1}{\frac{1}{4} - x}} = 5$$

$$2 - \frac{1}{\frac{1}{3} - \frac{1}{\frac{1}{4} - x}} = 5 \Leftrightarrow 2 - \frac{1}{\frac{1}{3} + \frac{1}{\frac{4x-1}{4}}} = 5 \quad \Leftrightarrow \quad 2 - \frac{1}{\frac{1}{3} + \frac{4}{4x-1}} = 5 \quad \Leftrightarrow$$

$$2 - \frac{1}{\frac{4x+11}{12x-3}} = 5 \quad \Leftrightarrow \quad 2 - \frac{12x-3}{4x+11} = 5$$

$$-3(4x+11) = 12x - 3 \quad \Leftrightarrow \quad 24x = -30 \quad \Leftrightarrow$$

$$x = -\frac{4}{5}$$

386. Given $a^2 - a - 1 = 0$ Determine a^6

$$a^2 - a - 1 = 0 \quad \Leftrightarrow \quad a = \frac{1 \pm \sqrt{5}}{2}$$

$$a^2 = a + 1 \quad \Rightarrow \quad a^4 = (a+1)^2$$

$$a^6 = a^4 a^2 = (a+1)^2 (a+1) = \text{Solve}$$

$$(\frac{1+\sqrt{5}}{2} + 1)^2 (\frac{1+\sqrt{5}}{2} + 1) = (\frac{3+\sqrt{5}}{2})^2 (\frac{3+\sqrt{5}}{2}) =$$

$$(\frac{9+5+6\sqrt{5}}{2})(\frac{3+\sqrt{5}}{2}) = 2(\frac{7+3\sqrt{5}}{2})(\frac{3+\sqrt{5}}{2}) = (7+3\sqrt{5})(\frac{3+\sqrt{5}}{2}) = +1)$$

$$\frac{21 + 7\sqrt{5} + 9\sqrt{5} + 15}{2} = \frac{36 + 16\sqrt{5}}{2} = 18 + 8\sqrt{5}$$

387. Solve for *x*: $x^x = 2^{1-x^2}$

There is (to my knowledge) no analytic way to solve this equation, but $x = 1$ is seen to be a solution, since $1^1 = 2^{1-1}$

388. Solve for (*x*,*y*) $2^{\frac{1}{x}} = 5^{\frac{1}{y}} = 100$

$2^{\frac{1}{x}} = 100 \quad \Leftrightarrow \quad 2 = 100^x$ and $5^{\frac{1}{y}} = 100 \quad 5 = 100^y$

$2 \cdot 5 = 100^x 100^y \quad \Rightarrow \quad 10 = 100^{x+y} \quad \Rightarrow \quad 10 = (10^2)^{x+y} \quad \Rightarrow \quad 10^1 = 10^{2(x+y)} \quad \Rightarrow$

$2(x+y) = 1 \quad \Rightarrow \quad x + y = \frac{1}{2}$

$2 = 100^x \quad \Rightarrow \quad \log 2 = x \log 100 \quad \Rightarrow \quad x = \frac{\log 2}{2}$

$5 = 100^y \quad \Rightarrow \quad \log 5 = y \log 100 \quad \Rightarrow \quad y = \frac{\log 5}{2}$

389. Solve for *x*: $\log_2 x + \log_3 x = 1$

We notice that:

$\log_2 x = y \quad \Leftrightarrow \quad x = 2^y \quad \textit{and} \quad \log_3 x = y \log_3 2 = \log_2 x \log_3 2$

And by the same token:

$\log_2 x = y \log_2 3 = \log_3 x \log_2 3 \quad \Rightarrow \quad \log_3 x = \frac{\log_2 x}{\log_2 3}$

$\log_2 x + \log_3 x = 1 \quad \Leftrightarrow \quad \log_2 x + \frac{\log_2 x}{\log_2 3} = 1 \quad \Leftrightarrow \quad (1 + \frac{1}{\log_2 3}) \log_2 x = 1 \quad \Leftrightarrow$

$(\frac{\log_2 3 + 1}{\log_2 3}) \log_2 x = 1 \quad \Leftrightarrow \quad (\frac{\log_2 3 + \log_2 2}{\log_2 3}) \log_2 x = 1 \quad \Leftrightarrow \quad (\frac{\log_2 6}{\log_2 3}) \log_2 x = 1 \quad \Leftrightarrow$

$\log_2 x = \frac{\log_2 3}{\log_2 6} \quad \Leftrightarrow \quad x = 2^{\frac{\log_2 3}{\log_2 6}}$

390. $7^{x-1} = 5$ **Determine** 7^{x+1}

$7^{x-1} = 5 \quad \Leftrightarrow \quad \frac{7^x}{7} = 5 \quad \Leftrightarrow \quad 7^x = 35$

$7^{x+1} = 7 \cdot 7^x = 7 \cdot 35 = 245$

391. Solve for *x*: $x^{x^5} = 100$

To my knowledge there is no analytic way to solve such an equation, so we shall resort to qualified guesswork.

One candidate might be $\sqrt[5]{10}$, and indeed: $\sqrt[5]{10}^{\sqrt[5]{10}^5} = \sqrt[5]{10}^{10} = 10^2 = 100$

392. Solve for *x*: $x^{-x^2}=\frac{1}{2}$

$x^{-x^2}=\frac{1}{2}$ These exercises are based on qualified guesswork, and an obvious candidate is $x=\sqrt{2}$,

since; $x^{-x^2}=\frac{1}{2} \quad \Leftrightarrow \quad \frac{1}{x^{x^2}}=\frac{1}{\sqrt{2}^{\sqrt{2}^2}}=\frac{1}{\sqrt{2}^2}=\frac{1}{2}$

393. $x^2-x+1=0$ **Show that** $x^{2020}+x^{1010}+1=0$

$x^2-x+1=0 \quad \Leftrightarrow \quad x^2=x-1 \quad x=1-\frac{1}{x} \quad \Leftrightarrow \quad x-1=-\frac{1}{x} \quad \Rightarrow \quad -\frac{1}{x}=x^2 \quad \Rightarrow \quad x^3=-1$

This can also be obtained much easier:

$x^2-x+1=0 \quad \Leftrightarrow \quad x^3-x^2+x=0 \quad \Leftrightarrow \quad x^3-(x-1)+x=0 \quad \Leftrightarrow \quad \Leftrightarrow \quad x^3+1=0 \quad \Leftrightarrow \quad x^3=-1$

So we could hope that 1010 was a multiply besides a remainder less than 3 (and it is of course)

$1010=336\cdot 3+2$

$x^{2020}+x^{1010}+1=0 \quad \Leftrightarrow \quad (x^{1010})^2+x^{1010}+1=0$
$x^{1010}=x^{3\cdot 336+2}=(x^3)^{336}x^2=(-1)^{336}=x^2$
$(x^{1010})^2=x^4=x^3x=-x$, So
$x^{2020}+x^{1010}+1=0 \quad \Leftrightarrow \quad -x+x^2+1=0 \quad \Leftrightarrow \quad x^2-x+1=0$

Which was what we should show.

394. Find the value of $\sqrt{2}^{\sqrt{2}^{\cdots}}$ **infinite power of** $\sqrt{2}$**.**

$x=\sqrt{2}^{\sqrt{2}^{\cdots}}$ since $\sqrt{2}^{\sqrt{2}^{\cdots}}$ is infinite we must have $x=\sqrt{2}^{x^{\cdots}}$ Which has the obvious solution $x = 2$, since: $x=\sqrt{2}^2=2$.

395. Show that the infinite sum: $\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+n}+\ldots=2$

For the series apply $1+2+3+\ldots+n=\frac{n}{2}(n+1)$

We write the formula for the nth and the (n+1)'th term.
The n'th term can be written as

$\frac{1}{\frac{n(n+1)}{2}}=2(\frac{1}{n}-\frac{1}{n+1})$ So we have

$2(\frac{1}{n}-\frac{1}{n+1})+2(\frac{1}{n+1}-\frac{1}{n+2})=2(\frac{1}{n}-\frac{1}{n+2})$

That is: It is a telescopic sum, where only the first term $\frac{1}{n}$ survives.
Since the first term is n = 1, the sum is 2.

396. Solve for *x* and *y*: $\sqrt{x}+\sqrt{y}=17 \quad and \quad x-y=17$

$x-y=17 \quad \Leftrightarrow \quad \sqrt{x}^2-\sqrt{y}^2=17 \quad \Leftrightarrow \quad (\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})=17 \quad \Leftrightarrow \quad (\sqrt{x}-\sqrt{y})=1$
$\sqrt{x}+\sqrt{y}=17$
$(\sqrt{x}-\sqrt{y})=1$
$2\sqrt{x}=18 \quad and \quad 2\sqrt{y}=16 \quad \Leftrightarrow \quad \sqrt{x}=9 \quad and \quad \sqrt{y}=8 \quad \Leftrightarrow$
$x=81 \quad and \quad y=64$

397. Simplify: $\frac{2^{\sqrt{27}}\cdot 8^{\sqrt{75}}}{4^{\sqrt{48}}}$

$\frac{2^{\sqrt{27}}\cdot 8^{\sqrt{75}}}{4^{\sqrt{48}}}=\frac{2^{3\sqrt{3}}\cdot 2^{3\cdot 5\sqrt{3}}}{2^{2\cdot 4\cdot\sqrt{3}}}=\frac{2^{3\sqrt{3}}\cdot 2^{15\sqrt{3}}}{2^{8\sqrt{3}}}=\frac{2^{18\sqrt{3}}}{2^{8\sqrt{3}}}=2^{10\sqrt{3}}=2024^{\sqrt{3}}$

397. Simplify: $\sqrt{46-12\sqrt{14}}$

We shall try to write $46-12\sqrt{14}=(a\sqrt{7}-b\sqrt{2})^2$ where a and b are integers:
$(a\sqrt{7}-b\sqrt{2})^2=7a^2+2b^2-2ab\sqrt{14}$
If we compare it to $46-12\sqrt{14}$, we see that ab = 6, so a = 2 and b = 3 or a = 2 and b = 3.
a = 2 and b = 3 gives: $4\cdot 7+2\cdot 9=46$, so: $\sqrt{46-12\sqrt{14}}=\sqrt{(2\sqrt{7}-3\sqrt{2})^2}=2\sqrt{7}-3\sqrt{2}$

398. Solve for *x*: $\sqrt{2-\sqrt{x+2}}=x$

We put: $\sqrt{x+2}=y \quad \Leftrightarrow \quad x+2=y^2 \quad \Leftrightarrow \quad x=y^2-2$
$\sqrt{2-y}=y^2-2 \quad \Leftrightarrow \quad 2-y=(y^2-2)^2$

However, it is relatively easy to guess solutions to this equation: We can se that y = 1 and y = -2 are solutions since: $2-1=(1^2-2)^2$ and $\sqrt{2-(-2)}=(-2)^2-2$.
Since: $\sqrt{x+2}=y \quad \Leftrightarrow \quad x+2=y^2 \quad \Leftrightarrow \quad x=y^2-2$ This correspond to $x=-1 \quad and \quad x=2$
But neither apply, since they come about from squaring a negative number.
So the equation has no solutions, which can also be shown more formally:

$y^4-4y^2+2+y=0 \quad \Leftrightarrow \quad y^2(y^2-4)+y+2=0$
$y^2(y^2-4)+y+2=0 \quad \Leftrightarrow \quad y^4-4y^2+2+y=0 \quad \Leftrightarrow \quad y^2(y^2-4)+y+2=0$
$y^2(y-2)(y+2)+y+2=0 \quad \Leftrightarrow \quad (y+2)(y^2(y-2)+1)=0 \quad \Leftrightarrow$
$y+2=0 \quad \vee \quad y^2(y-2)+1=0 \quad \Leftrightarrow \quad y=-2 \quad \vee \quad y^3-2y^2+1=0$

$y=1$ is a root in the last equation, so we make polynomial division with y -1.

$$y-1 \mid y^3-2y^2+1 \mid y^2-y-1$$
$$y^3-y^2$$
$$-y^2+1$$
$$-y^2+y$$
$$-y+1$$
$$-y+1$$

$y^2-y-1=0; \quad d=1+4=5 \quad \Rightarrow \quad y=\frac{1\pm\sqrt{5}}{2}$ where $\frac{1+\sqrt{5}}{2}$ is the golden ratio.

Now $y=\sqrt{x+2} \quad \Leftrightarrow \quad x=y^2-2$, so negative roots do not apply, nor do $y = 1$, so the only solution is: $y=\frac{1+\sqrt{5}}{2}$

399. Solve for *x*: $\sqrt{x}^{\log\sqrt{x}}=10^{\sqrt{10}}$

$\sqrt{x}^{\log\sqrt{x}}=10^{\sqrt{10}} \quad \Rightarrow \quad \log\sqrt{x}\log\sqrt{x}=\sqrt{10}\log 10 \quad \Leftrightarrow \quad (\log\sqrt{x})^2=\sqrt{10} \quad \Rightarrow$
$\log\sqrt{x}=\sqrt[4]{10} \quad \Rightarrow \quad \sqrt{x}=10^{\sqrt[4]{10}} \quad \Rightarrow \quad x=100^{\sqrt[4]{10}}$

400. Solve for *x*: $x=\sqrt{3\cdot\sqrt{9\cdot\sqrt{3\cdot\sqrt{9}}}}\cdot....$

This is an infinite radical expression that is repeated after two factors.
We may therefore write:

$$x=\sqrt{3\cdot\sqrt{9\cdot(\sqrt{3\cdot\sqrt{9}}}}\cdot....=\sqrt{3\cdot\sqrt{9\cdot x}}$$
$$\sqrt{3\cdot 3\sqrt{x}}=3\sqrt{\sqrt{x}}$$

So $x=3\sqrt{\sqrt{x}} \quad \Rightarrow \quad x^4=81x \quad \Rightarrow \quad x^3=81 \quad \Rightarrow \quad x^3=3^4 \quad \Rightarrow \quad x=3\sqrt[3]{3}$

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