# Economic models for of small enterprises

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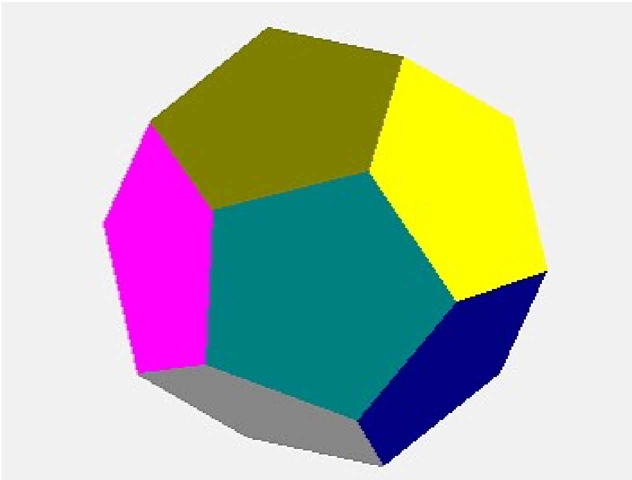

Ole Witt-Hansen 1982 (2018)

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## 1. Correlation between price and sales

The only thing of which you may be quite certain, when you want to theorize about the correlation between the price of goods and the revenue is that the sales of goods decreases with the price.

In the real world the dependence between price and sales may be very complex, and subject to various external parameters, but in a mathematical treatment, you are necessarily compelled to a correlation, which can be expressed by a simple mathematical function, and we shall therefore start out with the simplest correlation, which is a decreasing linear function.

If $p$ is the price of a commodity and the marketing is $x$, we shall express the marketing as a decreasing function of the price, where $c$ and $d$ are positive numbers.

(1.1) $$x = -cp + d$$

The sales drop to zero when $-cp + d = 0 \quad \Leftrightarrow \quad p = \frac{d}{c}$, and the maximum sales is of course, when you give away the goods, that is when $p = 0$.

In a mathematical treatment it is, however, more convenient to express the price as a function of the sales, although the causal dependency is the other way round. So we write

(1.2) $$p = -ax + b$$

Where $a$ and $b$ are positive numbers, but moreover the functional relation between $x$ and $p$ is the same as before. The turnover $y$ is the sales times the price:

(1.3) $$y = xp = -ax^2 + bx$$

Which correspond to a parabola with its peak at: $x_{\max} = \frac{b}{2a}$ having the value $y_{\max} = \frac{b^2}{4a}$.

The marginal revenue $G$ is defined as the increase in turnover, by selling one extra unit.

(1.4) $$G(x) = y(x+1) - y(x) \approx \frac{dy}{dx} \cdot 1 = -2ax + b$$

We can see that *G(x)* is growing when $x < \frac{b}{2a}$, it is zero for $x = \frac{b}{2a}$ and decreasing for $x > \frac{b}{2a}$.

### 1.1 Expense ratios

The expenses by a production may be divided into fixed expenses from the production $C_0$, which is independent of the magnitude of the production and the variable expenses $C_v$, which is a fixed amount per produced unit. The collected expenses are therefore:

(1.5) $$C = C_v x + C_0$$

The expenses per unit of goods are then:

(1.6) $$C_x = \frac{C}{x} = C_v + \frac{C_0}{x}$$

The graph for this function in a ($x$, $C_x$) coordinate system is a branch of a hyperbola, having the asymptotes $x = 0$ and $C_x = C_v$.
In the figure below, (1.5) and (1.6) are depicted in the same coordinate system.

Analogous to the marginal revenue, we shall define the marginal expenses $C_G$ as the increase in expenses caused by producing another unit of goods.

(1.7) $$C_G = C(x+1) - C(x) \approx \frac{dC}{dx}$$

In this case according to (1.5) it is equal to $C_v$
In the figure below to the left has been drawn (1.5) together with (1.6) and to the right $y$, $G(x)$ together with $p = -ax + b$ have been drawn in the same coordinate system.

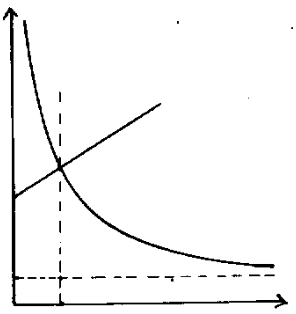
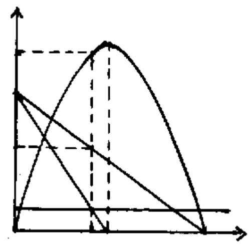


Fig. 3



Then the intersection $G(x) = C_V$ between the marginal expenses, and the marginal revenue corresponds to the optimal production

(1.8) $$G(x_0) = C_V \quad \Leftrightarrow \quad -2ax_0 + b = C_V \quad \Leftrightarrow \quad x_0 = \frac{b - C_V}{2a}$$

For $x < x_0 \Leftrightarrow G(x) > C_V$, an increase in production results in an increasing profit, whereas the opposite occur, when $x > x_0$.

The gross profit $B(x)$ by the optimal sales $x_0$ is the gross sales minus the variable costs, which is equal to the net profit $N(x)$ plus the fixed costs.

(1.9) $$B(x_0) = R(x_0) - C_V x_0 = -a\left(\frac{b - C_V}{2a}\right)^2 + b\left(\frac{b - C_V}{2a}\right) - C_V\left(\frac{b - C_V}{2a}\right) = \frac{(b - C_V)^2}{4a}$$

We may also obtain the same result by summing the profit on all units from the first to $x$.

(1.10)
$$B(x)=\int_0^{x_0}(G(x)-C_V)dx=\int_0^{x_0}(-2ax+b-C_V)dx=-a{x_0}^2+(b-C_V)x_0=$$
$$-a\left(\frac{b-C_V}{2a}\right)^2+(b-C_V)\frac{b-C_V}{2a}=\frac{(b-C_V)^2}{4a}$$

## 1.2 Example

A enterprise has experienced that the price of 9 \$ gives a turnaround of 16,000 units a day, while a price of 11.25 \$ gives a turnaround of 12,000 a day
The fixed daily costs amounts to 64,000 \$, and the variable costs are 1.50 \$ per unit.

Assuming that the sales are a linear function of price, we may according to the previous considerations determine the optimal production and price together with the collected gross and net profit at a certain price. Counting the sales in units of 1000, we may easily determine the turnaround function as the linear function through the two points (16,9) and (12, 11.25).

(1.11) $$p-9=\frac{11.25-9}{12-16}(x-16) \quad\Leftrightarrow\quad p=-\frac{9}{16}x+18$$

The optimal production is according to (1.3) $y_{\max}=\dfrac{b^2}{4a}$ with $a=\dfrac{9}{16}$ and $b=18$, which has the result: 144.000.

The marginal revenue is: $G(x)=-2ax+b=-\dfrac{9}{8}x+18$ and the marginal costs $C_V$ = 1.50.

We find thus from (1.12) $G(x)=C_V$: $-\dfrac{9}{8}x+18=1.5 \Leftrightarrow x=\dfrac{\frac{33}{2}}{\frac{9}{8}}=\dfrac{44}{3}=14\tfrac{2}{3}(1000)$ units.

And the price becomes: $p=-ax+b=-\dfrac{9}{16}\dfrac{44}{3}+18=9.75$ .

The gross profit: $B(x_0)=\dfrac{(b-C_V)^2}{4a}=\dfrac{(18-1{,}5)^2}{4\cdot\frac{9}{16}}=\dfrac{(\frac{33}{2})^2}{\frac{9}{4}}=121{,}000$

The net profit is the gross profit minus the fixed costs: 121,000 – 64,000 = 57,000.

## 2. Generalization of the theory

Trading with goods can, as it is well known, be seen from the side of the producer or that of the consumer. For that reason a somewhat different notation is often applied for the concepts involved that we have done hitherto.
As before *x* denotes the number of sold units, and *p* the price of one unit. Whereas we have previously assumed a linear relation between the two entities, we shall now generalize to an (almost) arbitrary decreasing dependence

(2.1) $$p=f(x)$$

Which we shall denote the sales function or the demand function depending on from which side you consider the trading. We shall assume that $p = f(x)$ is differentiable and that $p$ is a decreasing function of $x$, such that $f'(x) < 0$.

The gross profit: $B = B(x) = xp = xf(x)$.

(2.2) The marginal profit: $B(x+1) - B(x) \approx \frac{dB}{dx} = f(x) + xf'(x) = p(1 + \frac{x}{p} f'(x))$ .

$\frac{dB}{dx}$ is called the differential payoff, and it is less than the price, since $f'(x) < 0$.

In quite the same manner as for the linear dependence, we may express $x$ as a function of price $p$ as the inverse function of $f$: $x = f^{-1}(p)$, and hereby the gross profit: $B(p) = pf^{-1}(p)$. We then get:

(2.3) $$\frac{dB}{dp} = \frac{dB}{dx}\frac{dx}{dp} = \frac{dB}{dx}\frac{1}{f'(x)}$$

According to the differentiating rules for an inverse function.

## 2.1 Example

Let us assume that a market investigation shows that the sales or (or demand) function is given by:

$$x = \frac{40}{p^2 + 2}$$

The gross revenue is then: $B = px = \frac{40p}{p^2 + 2}$

We shall the calculate by which price $B$, is largest, and we therefore determine $B'(p)$.

(2.4) $$B'(p) = \frac{40 \cdot (p^2 + 2) - 40p2p}{(p^2 + 2)^2} = \frac{-40p^2 + 80}{(p^2 + 2)^2}$$

$$B'(p) = 0 \quad \Leftrightarrow \quad p^2 = 2 \quad \Leftrightarrow \quad p = \sqrt{2} \ (\vee \quad p = -\sqrt{2})$$

The expenditures of the company is a function of $x$: $C = C(x)$ , which is also considered to be differentiable.

The marginal costs, is then replaced by the differential costs $C(x+1) - C(x) \approx \frac{dC}{dx}$.

The costs per unit is: $C_A = \frac{C}{x}$. By differentiating we get:

(2.5) $$\frac{dC_A}{dx} = \frac{x\frac{dC}{dx} - C}{x^2}$$

$$\frac{dC_A}{dx} = 0 \quad \Leftrightarrow \quad \frac{dC}{dx} = \frac{C}{x} = C_A \tag{2.6}$$

Which shows that the differential costs per unit has a minimum, when the differential equals the costs per unit.

As it was the case in the case of a linear relation, one may determine the optimal sales and the related price by determine the intersection between the graph for the differential expenses and the differential revenue.

$$\frac{dB}{dx} = \frac{dC}{dx} \quad \Leftrightarrow \quad p(1 + \frac{x}{p} f'(x)) = C_A, \quad \text{where} \quad p = f(x)$$

## 3. Competition and monopoly

Assuming now that two enterprises $E_1$ and $E_2$ produces two different kinds of goods that compete with each other in the same market, and that this implies that the turnover of each merchandise depends on the prices of them both.

$$\begin{aligned} &E_1 : x_1 = f_1(p_1, p_2) \\ &E_2 : x_2 = f_2(p_1, p_2) \end{aligned} \tag{3.1}$$

The two functions $f_1$ and $f_2$ could for example be the result of a extrapolation of an investigation of the market.
The net profit $I$ for $E_1$ and $E_2$ is the sale times the price when subtracted from the variable costs.

$$\begin{aligned} I_1 &= B_1(x_1) - C_1(x_1) = f_1(p_1, p_2)p_1 - C_1 f_1(p_1, p_2) \\ I_2 &= B_2(x_1) - C_2(x_1) = f_2(p_1, p_2)p_2 - C_2 f_2(p_1, p_2) \end{aligned} \tag{3.2}$$

If the two manufactures of the merchandises both assume that the prices $p_1$ and $p_2$ are regulated independently of each other then the suppliers will differentiate (3.2) with respect to $p_1$ or $p_2$ to optimize the net profit, under the assumption that the prices are independent of each other.

In short, they will determine the solutions to $\frac{\partial I_1}{\partial p_1} = 0$ and $\frac{\partial I_2}{\partial p_2} = 0$ achieving the two maxima.
On the other hand, if they both accommodate their prizes to the other producers pricing, that is, if $p_1 = p(p_2) \;\; and \;\; p_2 = p(p_1)$, then the situation becomes more complex.

If finally we assume that the two producers join together, such that: $I = I_1 + I_2 = I(p_1 + p_2)$, then the max profit should be found from the two partial differential quotients , $\frac{\partial I_1}{\partial p_1} = 0$ and $\frac{\partial I_2}{\partial p_2} = 0$ being simultaneous zero. The outcome of the solution of the resulting two equations will then be the ”monopoly” prices. This we shall illustrate by a (constructed) example.

### 3.1 Example

Among the railway stations, we imagine that there is a competitive sale of chocolate bars, *A* and *B*. Let us assume that a statistic market has shown that the sale measured in 10,000 per year, may be represented by the two functions:

(3.3) $A: \quad x_1 = p_2 - p_1$

$B: \quad x_2 = 35 + p_1 - 2p_2$

Where the prices are measured in cents.
The variable costs are set to, for $E_1$: 12 cent, and for $E_2$: 20 cent.

### 3.1.1 Independent pricing

First we shall maximize the net revenue for the case where the pricings are independent of each other.

(3.4)
$$I_1 = B_1 - C_1 = p_1x_1 - 12x_1 = x_1(p_1 - 12)$$
$$= (p_2 - p_1)(p_1 - 12) = p_1p_2 - 12p_2 - p_1^2 + 12p_1$$

And by differentiating with respect to $p_1$ we get:

(3.5)
$$\frac{\partial I_1}{\partial p_1} = 0 \Leftrightarrow p_2 - 2p_1 + 12 = 0 \Leftrightarrow p_1 = \tfrac{1}{2}p_2 + 6$$

The variation of sign of $p_2 - 2p_1 + 12$ is seen to be: $p_2 - 2p_1 + 12 > 0 \Leftrightarrow p_1 < \tfrac{1}{2}p_2 + 6$, and so forth. The variation of sign is +, 0, - which means it is a max.

Accordingly we get for $E_2$:

(3.6)
$$I_2 = B_2 - C_2 = p_2x_2 - 20x_2 = x_2(p_2 - 20)$$
$$= (35 - 2p_2 + p_1)(p_2 - 20) = 35p_2 - 700 - 2p_2^2 + p_1p_2 + 40p_2 - 20p_1$$
$$\frac{\partial I_2}{\partial p_2} = 0 \Leftrightarrow p_1 + 75 - 4p_2 = 0 \Leftrightarrow p_2 = \tfrac{1}{4}p_1 + \tfrac{75}{4}.$$

The two equations must then be solved for the determination of the global max.

$$\begin{matrix} p_1 = \tfrac{1}{2}p_2 + 6 \\ p_2 = \tfrac{1}{4}p_1 + \tfrac{75}{4} \end{matrix} \Leftrightarrow \begin{matrix} p_1 - \tfrac{1}{2}p_2 = 6 \\ -p_1 + 4p_2 = 75 \end{matrix} \Leftrightarrow \tfrac{7}{2}p_2 = 81 \wedge p_1 = \tfrac{1}{2}p_2 + 6 \Leftrightarrow$$

(3.7)
$$p_1 = 17\tfrac{4}{7} \wedge p_2 = 23\tfrac{1}{7}$$

### 3.1.2 Mutual dependency of the pricings

We shall then look into what happens, when $E_1$ in his strategy to maximize his net revenue, incorporate the strategy of his competitor, and that $E_2$ does likewise.

(3.7)
$$I_1 = x_1(p_1 - 12) = (p_2 - p_1)(p_1 - 12)$$
$$I_2 = x_2(p_2 - 20) = (35 + p_1 - 2p_2)(p_2 - 20)$$

$I_1$ maximized his net revenue by choosing $p_1 = \tfrac{1}{2}p_2 + 6$ and
$I_2$ maximized his net revenue by choosing $p_2 = \tfrac{1}{4}p_1 + \tfrac{75}{4}$.

Inserting these optimal prices in the expressions for $I_1$ and $I_2$ we end up with the two expressions.

$$\begin{aligned} I_1 &= x_1(p_1 - 12) = (\tfrac{1}{4}p_1 + \tfrac{75}{4} - p_1)(p_1 - 12) \\ I_2 &= x_2(p_2 - 20) = (35 + \tfrac{1}{2}p_2 + 6 - 2p_2)(p_2 - 20) \end{aligned} \tag{3.8}$$

$$\begin{aligned} I_1 &= -\tfrac{3}{4}p_1^{\,2} + \tfrac{111}{4}p_1 - 225 \\ I_2 &= -\tfrac{3}{2}p_2^{\,2} + 71p_2 - 820 \end{aligned} \tag{3.9}$$

$$\begin{aligned} \frac{\partial I_1}{\partial p_1} = 0 \quad &\Leftrightarrow \quad -\tfrac{3}{2}p_1 + \tfrac{111}{4} = 0 \quad \Leftrightarrow \quad p_1 = 18\tfrac{1}{2} \\ \frac{\partial I_2}{\partial p_2} = 0 \quad &\Leftrightarrow \quad -3p_2 + 71 = 0 \quad \Leftrightarrow \quad p_2 = 23\tfrac{2}{3} \end{aligned} \tag{3.10}$$

The incorporation of the competitor strategy, results though in a minor increase in prices and thereby an augmented net revenue for both producers.

### 3.1.3 Forming a Monopoly. A join together of the two enterprises.

Next we shall then look into the situation, where the two enterprises join together, so that there is no longer a free competition.
The common conception is that a monopoly always will result in higher prices, as a conscious act from the producers, but let us see what a simple mathematical theory predicts on that issue.
The collected net revenue becomes:

$$\begin{aligned} I &= I_1 + I_2 = x_1(p_1 - 12) + x_2(p_2 - 20) \\ &\quad = (p_2 - p_1)(p_1 - 12) + (35 + p_1 - 2p_2)(p_2 - 20) \\ I &= I_1 + I_2 = p_1p_2 - 12p_2 + 12p_1 - p_1^{\,2} + (35p_2 - 700 + p_1p_2 - 20p_1 - 2p_2^{\,2} + 40p_2 \\ I &= I_1 + I_2 = 2p_1p_2 - p_1^{\,2} - 2p_2^{\,2} - 12p_2 - 8p_1 + 63p_2 - 700 \end{aligned} \tag{3.11}$$

To the determination of the max, we must solve the two equations:

$$\frac{\partial I}{\partial p_1} = 0 \ \text{ and } \ \frac{\partial I}{\partial p_2} = 0$$

$$\left.\begin{aligned} \frac{\partial I}{\partial p_1} = 0 \quad &\Leftrightarrow \quad 2p_2 - 2p_1 - 8 = 0 \\ \frac{\partial I}{\partial p_2} = 0 \quad &\Leftrightarrow \quad 2p_1 - 4p_2 - 63 = 0 \end{aligned}\right. \quad \Leftrightarrow \quad \begin{aligned} &2p_2 - (4p_2 + 63) - 8 = 0 \\ &p_1 = 2p_2 + 63 \end{aligned} \quad \Leftrightarrow \tag{3.12}$$

$$(p_1, p_2) = (23\tfrac{1}{2}, 27\tfrac{1}{2})$$

Which seen from the side of the producer is a nice increase in prices.

We shall finish by establishing the three cases in a tabular form.

| | | | | |
|---|---|---|---|---|
| Independent pricing | $p_1 = 17\frac{4}{7}$ | $p_2 = 23\frac{1}{7}$ | $x_1 = p_2 - p_1 = 5\frac{4}{7}$ | $x_2 = 35 + p_1 - 2p_2 = 6\frac{2}{7}$ |
| Mutual dependency | $p_1 = 18\frac{1}{2}$ | $p_2 = 23\frac{2}{3}$ | $x_1 = p_2 - p_1 = 5\frac{1}{3}$ | $x_2 = 35 + p_1 - 2p_2 = 6\frac{1}{6}$ |
| Monopoly | $p_1 = 23\frac{1}{2}$ | $p_2 = 27\frac{1}{2}$ | $x_1 = p_2 - p_1 = 4$ | $x_2 = 35 + p_1 - 2p_2 = 3\frac{1}{2}$ |

| | | |
|---|---|---|
| Independent pricing | $I_1 = x_1 p_1 - 12x_1 =$ 1521/49 $= 31\frac{2}{49}$ | $I_2 = x_2 p_2 - 20x_2 =$ 968/49$=19\frac{37}{49}$ |
| Mutual dependency | $I_1 = (p_2 - p_1)(p_1 - 12) =$ 104/3$=34\frac{2}{3}$ | $I_2 = (35 + p_1 - 2p_2)(p_2 - 20)$ $= 22\frac{11}{18}$ |
| Monopoly | $I = (p_2 - p_1)(p_1 - 12) + (35 + p_1 - 2p_2)(p_2 - 20)$ $=$105/4$=26\frac{1}{4}$ | |

From the table one may draw his own conclusion, bearing in mind that it is based on a mathematical model.
One should notice however, when forming a monopoly there is a decrease in both the turnaround and the net revenue, in contradiction to the common concept of monopolizing.

Ref: Irene Kristensen: Nogle anvendelser af matematik i økonomi. Århus universitet 1974