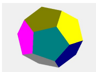
# Linear programming
# By examples



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## 1. Linear programmering

Let there be given a linear function of two variables.

$$f(x_1, x_2) = a_1x_1 + a_2x_2,$$

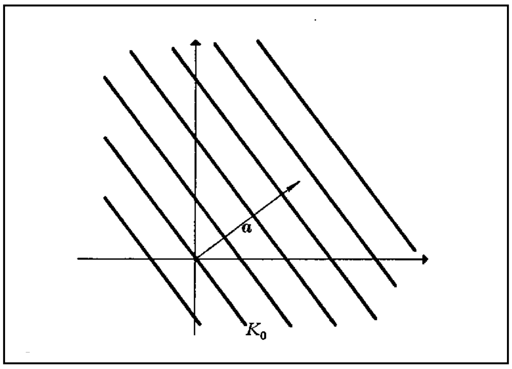
where $(a_1, a_2) \neq (0,0)$

A function $f(\bar{x}) = f(x_1, x_2, \ldots x_n)$ is considered linear if the following two conditions are fulfilled for all $\bar{x}, \bar{y}, \lambda$, where $\bar{x}$ is the set of numbers $(x_1, x_2, x_3, \ldots, x_n)$ and $\lambda \in R$

$$f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y}) \qquad \text{and} \qquad f(\lambda\bar{x}) = \lambda f(\bar{x})$$

We shall then investigate whether $f(x_1, x_2) = a_1x_1 + a_2x_2$ has a maximum value and/or a minimum value in a given set of points in the plane *M*.

The equation $a_1x_1 + a_2x_2 = c$ forms a line in the plane, having the normal vector $\vec{a} = (a_1, a_2)$
For different values of *c* the equation displays a bundle of parallel lines, also called the level lines. They are lines where the function has the same constant value *c*. They are all perpendicular to the normal vector $\vec{a} = (a_1, a_2)$. As Seen from the figure below.




It seems clear that $f(x_1, x_2)$ is growing in the direction of the normal vector, since that is the direction of increasing level lines.
If $P(x_1, x_2)$ is a point in *M*, then the projection of a point on the normal vector $\vec{a}$ is:

$$\vec{OP} \cdot \vec{a} / |\vec{a}| = (a_1x_1 + a_2x_2) / |\vec{a}| = c / |\vec{a}|$$

From this you can see that the linear function reaches its max/min, when the projection of $P(x_1, x_2)$ on the normal vector has its max/min.

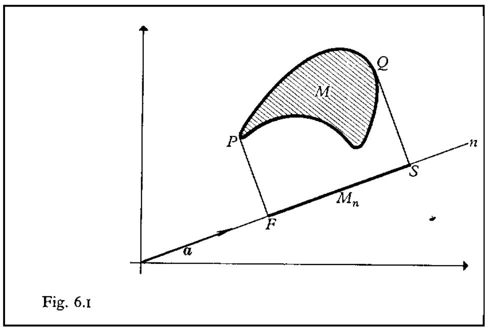


Fig. 6.1

If the set of points *M* is enclosed in a curve, it is clear that possible max and min must be situated on the border of the enclosing curve, and if the enclosing curve is a polygon, then the max and min must be situated in the corners of the polygon.
If on other hand *M* is enclosed in a differentiable curve, as shown in the figure to the left, then we find that max and min is to be found where the tangent to the curve is parallel to the level lines.

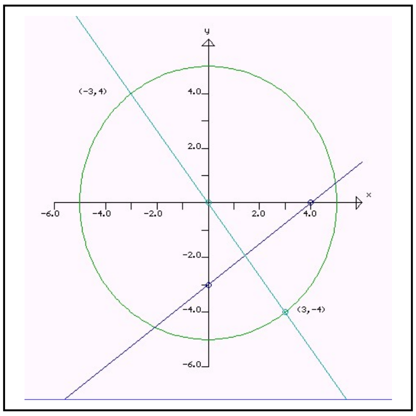
If $\vec{a} = (a_1, a_2)$ is a normal vector to the level lines, then the cross-vector $\hat{a} = (-a_2, a_1)$ is parallel to the level lines. If the enclosing curve is given by the parametric

$$\vec{f}(t) = (x(t), y(t)) \text{ then the tangent vector is } \vec{f}'(t) = (x'(t), y'(t)).$$

The condition that a level line is a tangent to the curve that encloses *M* is consequently that the cross vector $\hat{a} = (-a_2, a_1)$ is parallel to the tangent vector $\vec{f}'(t) = (x'(t), y'(t))$.
This can be expressed by the equation:

$$(x'(t), y'(t)) / |\vec{f}'(t)| = \pm(-a_2, a_1)/|\hat{a}| \quad \text{or} \quad \vec{f}'(t) \times \hat{a} = 0$$

We shall then look at a simple example, where we shall determine max and min for the linear function $f(x, y) = 3x - 4y$ in the set of points $x^2 + y^2 \le 25$.




A parametric for the circle is (5cost, 5sint), and the differential quotient is $(-5\sin t, 5\cos t)$.
A unit vector which is parallel to the level lines is $(-a_2, a_1)/|\hat{a}| = (4,3)/5$, and a unit vector in the direction of the tangent is $(-\sin t, \cos t)$. It is sufficient to solve the equations.

$$\cos t = \pm\tfrac{3}{5}$$

Resulting in: $(\cos t, \sin t) = (\tfrac{3}{5}\tfrac{4}{5})$, $(\cos t, \sin t) = (-\tfrac{3}{5}, -\tfrac{4}{5})$,

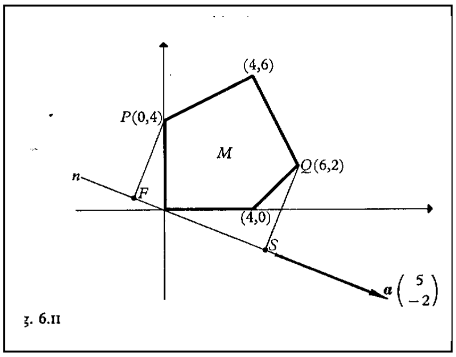
Which matches max and min for the level lines.

## 1.1 Geometrical example

We then wish to determine max and min for the function $f(x, y) = 5x - 2y$ in the set *M*.

$$\begin{aligned} M = \{(x, y) \mid x \ge 0 \wedge \\ y \ge 0 \wedge \\ 2x - 4y + 16 \ge 0 \wedge \\ -x + y + 4 \ge 0 \wedge \\ -4x - 2y + 28 \ge 0\} \end{aligned}$$

The inequalities maps into 5 semi-planes. The intersection of these semi-planes is a pentagon. The five expressions are positive in the semi-planes, in which the normal to the lines point into. The five lines have the normal vectors: (1,0), (0,1), (2,-4), (-4,-2). (-1,1)




This can be confirmed by looking at the figure to the left. In the figure we have drawn the sections of the five lines, which correspond to the five inequalities, which form the pentagon. Notice that the pentagon has two edges on the coordinate axis.
To find the rest of the intersection points in the first quadrant, we must solve the two sets of equations:

$$\begin{aligned} 2x - 4y + 16 = 0 \\ -4x - 2y + 28 = 0 \end{aligned} \quad \text{and} \quad \begin{aligned} -4x - 2y + 28 = 0 \\ -x + y + 4 = 0 \end{aligned}$$

The first set of equations is most easily solved by multiplying the upper equation by 2 and adding the equations, which gives $y = 6$, so the intersection point is (4,6). In the same manner the second set of equations is solved by multiplying the lower equation by 2 and adding the equations, giving $(x, y) = (6,2)$.

We already know that the max and min for the linear function are to be found in the corners of the pentagon, since the corners must the largest and the least projection on the normal vector to the level lines.
From the figure above it is seen that *Q*(6,2) has the largest projection on the normal and that *P*(0,4) has the least projection. To verify this, we calculate the value of $f(x,y)=5x-2y$ in all five points. $f(0,0)=0; f(0,4)=-8,\ f(4,6)=8;\ f(6,2)=26; f(4,0)=20$

If you don't have a geometrical representation, you must rather evaluate the function in all the corners of the polygon, which also usually is less time consuming than to draw the polygon.

If there are more than two variables involved, things become much more complicated. The level lines are still (hyper) planes, but finding the intersection of the surfaces that create the set of points that defines the set *M* in question, is not always easy, and is (almost) always done by computer programs. The method is called the simplex method, and it will not be treated further in this note.

## 1.1 Least collected sail route

In the two ports *A* and *B* are 8 and 7 ships respectively. These ships must sail to the ports *P*, *Q* and *R,* such that 4 ships must go to *P*, 5 ships must go to *Q*, and 6 ships must go to *R*.
The distances in nautical miles (sm) they must sail to reach the receiving ports are given in the table below.

| | *P* | *Q* | *R* |
|---|---|---|---|
| *A* | 190 | 300 | 250 |
| *B* | 250 | 400 | 300 |

We shall then try to decide how many ships must sail from *A* to *P,*Q, *R*, and how many ships must sail from *B* to *P,*Q, *R*, such that the total distance covered becomes least possible.

We shall assume that $x$ ships sail from *A* to *P*, and $y$ ships sail from *A* to *Q*.
Since there are 8 ships in *A* then $8-x-y$ ships must sail from *A* to *R*.
*P* receives 4 ships, and therefore $4-x$ ships must sail from *B* to *P*
In a similar manner we realize that $5-y$ ships must sail from *B* to *Q*, and $6-(8-x-y)=x+y-2$ ships must sails from *B* to *R*. We have collected these considerations in the table below.

| | to *P* | to Q | to R | total |
|---|---|---|---|---|
| from *A* | $x$ | $y$ | $8-x-y$ | 8 |
| from *B* | $4-x$ | $5-y$ | $x+y-2$ | 7 |
| total | 4 | 5 | 6 | 15 |

The overall sailed distance is thus.

$$190x+300y+250(8-x-y)+250(4-x)+400(5-y)+300(x+y-2)=4400-10(x+5y)$$

To minimize this distance, we can see that we must find the max for the function:

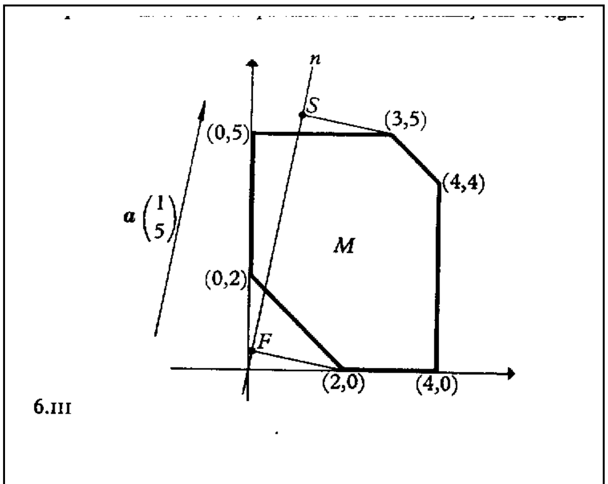
$$f(x,y)=(x+5y)$$

The numbers in the table may not be negative, so the solution (*x, y*) must be found in an area constrained by the inequalities.

$$M = \{(x,y) \mid x \geq 0 \wedge y \geq 0 \wedge 8 - x - y \geq 0 \wedge 4 - x \geq 0 \wedge 5 - y \geq 0 \wedge x + y - 2 \geq 0\}$$

We begin with a geometrical interpretation.
First we draw the axis parallel lines. $x = 0,\ y = 0,\ x = 4,\ y = 5$ .

The line $x + y - 2 = 0$ intersects the axis in (0,2) and (2,0). And we only lack to determine the intersection between $8 - x - y$ and the two lines $y$ =5 and $x$ = 4, which gives (3,5) and (4,4).
The polygon *M*, which represents the area where the inequalities are fulfilled, is shown below.
Also the normal vector to the level lines $x + 5y = c$ is drawn.




From the figure we can see that the projection on the normal vector is largest in (3,5) and least in (2,0)
We control the result by evaluating $f(x,y) = x + 5y$ in the five corners (0,2), (0,5), (3,5), (4,4) , (4,0) and (2,0) of the pentagon.

$$f(0{,}2) = 10,\ f(0{,}5) = 25,\ f(3{,}5) = 28,$$
$$f(4{,}4) = 24,\ f(4{,}0) = 4;\ f(2{,}0) = 2$$

Which confirms what we found geometrically, namely max in (3,5), and min in (2,0). Inserting the solution (3,5) in the table over distances, we find:

| | to *P* | to *Q* | to *R* |
|---|---|---|---|
| from *A* | 3 | 5 | 0 |
| from *B* | 1 | 0 | 6 |

## 1.2 Optimal distribution of goods

The two factories *A* and *B* supply 3 consumers *P, Q* and *R* with a certain product. The production that *A* and *B* may yield, and the 3 consumers need are shown in the first two tables below.
The third table shows the cost for delivering a product from *A* or *B* to the consumers *P, Q* and *R*.

| *A* | *B* |
|---|---|
| 8000 | 7000 |

| *P* | *Q* | *R* |
|---|---|---|
| 4000 | 5000 | 6000 |

| | *P* | *Q* | *R* |
|---|---|---|---|
| *A* | 19 | 30 | 25 |
| *B* | 25 | 40 | 30 |

We now set ourselves that task to determine the number of goods, which must be delivered from *A* and *B* to *P, Q, R*, such that the total cost becomes the least possible.
Let us assume that *A* delivers *x* units to *P* and *y* units to *Q*. It then follows that *A* must deliver $8000 - x - y$ units to *R*. As *P* shall receive 4000 units, then *B* must ship $4000 - x$ units to *P*.
Since *Q* shall receive 5000 units, then *B* must ship $5000 - y$ to *Q*.

*A* must ship 8000 – *x* - *y* to *R*, and *B* must ship 7000 – (4000 – *x*) – (5000 – *y*) = *x* + *y* – 2000 units to *R*. This is illustrated in the table below

| | to *P* | to Q | to R | total |
|---|---|---|---|---|
| from *A* | *x* | *y* | 8 – *x* - *y* | 8 |
| from *B* | 4 - *x* | 5 - *y* | *x* + *y* -2 | 7 |
| total | 4 | 5 | 6 | 15 |

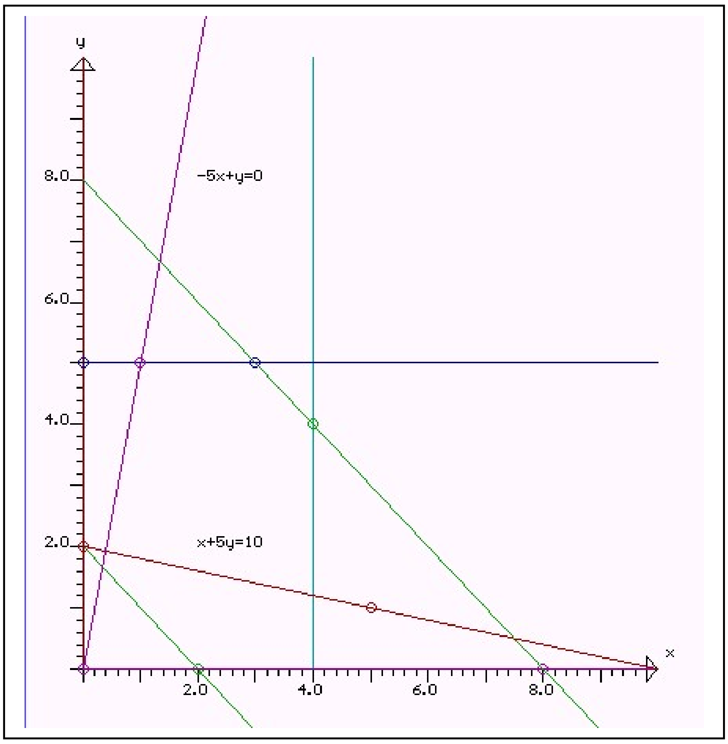
Since all the numbers in the table must be non negative, the following inequalities must be valid.

$$x \geq 0;\ y \geq 0\,;\ 8000 - x - y \geq 0\,;\ \ 4000 - x \geq 0\,;\ 5000 - y \geq 0\,;\ x + y - 2000 \geq 0$$

And the total costs becomes.

$$19x + 30y + 25(8000 - x - y) + 25(4000 - x) + 40(5000 - y) + 30(x + y - 2000) = -x - 5y + 44000$$

If the accumulated cost $-x - 5y + 44000$ should be at least as possible, then $x + 5y$ must be as large as possible.
The level lines are: $x + 5y = c$, and as we have seen earlier that it is the case when the projection on the normal to the level lines is largest.
Below is shown a graphical representation of the lines corresponding to the linear inequalities. The numbers are counted in thousands.




The following 6 lines are drawn.

$$x = 0$$
$$y = 0$$
$$8 - x - y = 0$$
$$4 - x = 0$$
$$5 - y = 0$$
$$x + y - 2 = 0$$

The intersection of the 6 positive semi-planes is a polygon having the corners, which are easily found as: (4,0), (4,4), (3,5), (0,5), (0,2) , (2,0).
Furthermore is shown a level line $x + 5y = 10$ and the normal to this line $\vec{n} = (1,5)$.
From the geometrical drawing one can see that (3,5) has the largest projection on the normal.

To confirm this geometrical observation, we calculate the value of $f(x, y) = x + 5y$ in each of the six corners.

$f(4,0)=4$ $f(4,4)=24$ , $f(3,5)=28$, $f(0,5)=25$ , $f(0,2)=10$, $f(2,0)=2$ .

As we see: $f(x,y)=x+5y$ obtains its largest value 28 in (3,5), as we also found geometrically. The least value for the total cost is hereby: 44 – 3 – 25 = 36. (thousands). When (3,5) is insertee in the table, we have:

| | to *P* | to Q | to R | total |
|---|---|---|---|---|
| from *A* | 3 | 5 | 0 | 8 |
| from *B* | 1 | 0 | 6 | 7 |
| total | 4 | 5 | 6 | 15 |

## 1.3 Optimization of production costs

A factory which manufactures cars and tractors is organized in four departments, each with a specific function.
A: Metal stampning, B: Assembly of engines, C; Assembly of cars, D: Assembly of tractors.
To the production of a car is required 1/25000 of A's monthly capacity, 1/34000 of B's monthly capacity and 1/22500 of C's monthly capacity.
For the tractors the corresponding fractions are: 1/35000, 1/17000 and 1/15000.
Counting in fractions of thousands, we apply for the cars the fractions: 1/25, 1/34 and 1/22.5 and for the tractors: 1/35, 1/17, 1/15.

Let *x* and *y* be the number of produced cars and tractors respectively.
Since the sum of fractions that a department is charged with making cars and tractors must be less or equal to one, we may establish the following inequalities:

A: $\frac{x}{25}+\frac{y}{35}\le 1$ B: $\frac{x}{34}+\frac{y}{17}\le 1$ C: $\frac{x}{22{,}5}+\frac{y}{15}\le 1$

Each of the inequalities represents a semi-plane. Together with $x\ge 0$, $y\ge 0$ the intersection of these five semi-planes form a polygon.
Assuming that the company earns 30.000 per assembled car and 25.000 per assembled tractor, we wish due to the conditions stated above to determine *x* and *y* to maximize company profit.
The profit is thus given by the linear function $f(x,y)=30x+25y$

The line $\frac{x}{25}+\frac{y}{35}=1$ is fixed by the two points (25,0) and (0,35),

The line $\frac{x}{34}+\frac{y}{17}=1$ is fixed by (0,17) and (34,0) ,

The line $\frac{x}{22{,}5}+\frac{y}{15}\le 1$ is fixed by (0,15) and (22.5,0)

We shall draw these lines in a coordinate system.

The profit is given by the function $f(x,y)=30x+25y$. Looking at the line level $f(x,y)=150$, then this line is fixed by the two points (0,6) and (5,0).
The normal vector to the level lines is (30,25), or $\vec{n}=(6,5)$. Since the maximum value of $f(x,y)=30x+25y$ is given by the largest projection on the normal vector, we also draw a line through (0,0) along the normal vector.

If all lines are drawn in a coordinate system we obtain the figure shown below.

The blue line near the starting point of the coordinate system is the level line, and the light blue line is the normal to the level lines.
We wish to find the maximum projection of the points in the polygon on the normal line.
The only intersection point between the lines that confine the polygon is (20,7). From the figure one sees that this point also have the max projection on the normal line.
The maximum gain by the production is thus obtained, when the number of cars produced is 20 and the number of tractors produced is 7. The total income then amounts to 775. (775000)