The Legendre and the associated differential equation

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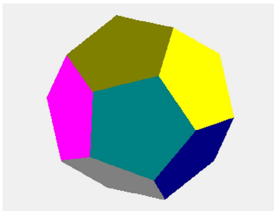

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1. The Legendre equation

The Legendre equation comes frequently up in physics, and dealing with the properties of the solutions is a substantial part of most standard books in mathematical physics.
However the Legendre polynomials are rather complex mathematics, and most often only the results are presented, but not the detailed derivation. The Legendre differential equation reads:

$$(1.1) \qquad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

There are two linear independent classes of solutions to the equation: $P_n(x)$ and $Q_n(x)$, we shall here only occupy ourselves with $P_n(x)$. $P_n(x)$ are called Legendre polynomials.
The most compact way to state the solution to the Legendre equation is the Rodrigues formula:

$$(1.2) \qquad P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

Unfortunately it is not possible to verify Rodrigues formula directly by insertion.
One has either to use the series expansion solution to the Legendre equation together with applying the binomial formula on Rodrigues equation or some mathematical tricks, as we shall see below.

Especially in connection with physics *n* is often replaced with *l*. In connection with the series expansion, it is in fact necessary.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$$

2. Series expansion solution to the equation

Assume a series expansion solution:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_nx^n + \ldots$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + na_nx^{n-1} + \ldots$$
$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots$$

$$(1-x^2)y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots$$
$$-(2a_2x^2 + 6a_3x^3 + 12a_4x^4 + \ldots + n(n-1)a_nx^n + \ldots$$
$$-2xy' = -2xa_1 - 4a_2x^2 - 6a_3x^3 - 8a_4x^4 + \ldots + -2na_nx^n + \ldots$$
$$n(n+1)y = n(n+1)a_0 + n(n+1)a_1x + n(n+1)a_2x^2 + n(n+1)a_3x^3 + n(n+1)a_4x^4 + \ldots + n(n+1)a_nx^n + \ldots$$

We put the three terms in the Differential equation together, collecting the coefficients of the various terms having the power of *x*. Since the collected series is identically zero, each of the coefficients to a certain power of *x* must necessarily be zero. It is rather messy to complete this task, but below we have established a schema for the powers from 0 to 3 and for *n*.

	const	x	x^2	x^3	x^n
y''	$2a_2$	$6a_3$	$12a_4$	$20a_5$	$(n+2)(n+1)a_{n+2}$
$-x^2y''$			$-2a_2$	$-6a_3$	$-n(n-1)a_n$
$-2xy'$		$-2a_1$	$-4a_2$	$-6a_3$	$-2na_n$
$l(l+1)y$	$l(l+1)a_0$	$l(l+1)a_1$	$l(l+1)a_2$	$l(l+1)a_3$	$l(l+1)a_n$

For the first four terms we get:

const: $2a_2 + l(l+1)a_0 = 0 \;\Rightarrow\; a_2 = -\dfrac{l(l+1)}{2}a_0$

x: $\quad 6a_3 - 2a_1 + l(l+1)a_1 = 0 \;\Rightarrow\; a_3 = -\dfrac{(l-1)(l+2)}{6}a_1$

x^2: $\quad 12a_4 - 2a_2 - 4a_2 + l(l+1)a_2 \;\Rightarrow\; a_4 = -\dfrac{(l-2)(l+3)}{12}a_2 = -\dfrac{l(l+1)(l-2)(l+3)}{4!}a_0$

x^3: $\quad 20a_5 - 12a_3 + l(l+1)a_3 \;\Rightarrow\; a_5 = -\dfrac{(l+4)(l-3)}{20}a_3$

$$= -\frac{(l-1)(l-3)(l+2)(l+4)}{5!}a_1$$

$$x^n: \quad \begin{aligned} &(n+2)(n+1)a_{n+2} + (l^2 + l - n^2 - n)a_n = 0 \;\Leftrightarrow \\ &(n+2)(n+1)a_{n+2} + (l-n)(l+n+1)a_n = 0 \end{aligned}$$

(2.1) $$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n$$

The solution then separates in two series, one for even power of *x*, and one for odd power of *x*, corresponding to the solutions $P_n(x)$.

$$y = a_0\left(1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \ldots\right) +$$
$$a_1\left(x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 - \ldots\right)$$

It is important to notice that for integer *l*, the solutions to Lagrange's differential equation are polynomials. This follows from the recursion relation:

$$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)} a_n$$

For $n = l$, $a_{n+2}=0$, and from the recursion relation it follows that $a_{n+4} = a_{n+6} = a_{n+8} = \ldots = 0$.

The first 4 Legendre polynomials: $P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$ are

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= \frac{1}{2}\frac{d}{dx}(x^2-1) = x \\
P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \tfrac{1}{2}(3x^2-1) \\
P_3(x) &= \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \tfrac{1}{2}(5x^3-3x)
\end{aligned}$$

The Legendre polynomials are especially in physics written with $x = \cos\theta$ instead of x.
The Legendre polynomial normalization and orthogonal properties are therefore defined in the interval [-1,1]. The formula is (from which we shall abstain from making the proof, because…)

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$$

You may for example easily verify:

$$\begin{aligned}
&\int_{-1}^{1} P_0(x)^2 dx = \int_{-1}^{1} 1dx = [x]_{-1}^{1} = 2 \\
&\int_{-1}^{1} P_1(x)^2 dx = 1 \quad \Leftrightarrow \quad \int_{-1}^{1} x^2 dx = \left[\tfrac{1}{3}x^3\right]_{-1}^{1} = \tfrac{2}{3} \\
&\int_{-1}^{1} P_1(x)P_2(x)dx = \int_{-1}^{1} \tfrac{1}{2}(3x^3-x)dx = \tfrac{1}{2}\left[\tfrac{3}{4}x^4 - \tfrac{1}{2}x^2\right]_{-1}^{1} = 0
\end{aligned}$$

As it is seen from the definition equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

the Legendre polynomials have the eigenvalues $n(n+1)$. Since the Legendre polynomials are an infinite series of orthogonal functions, "any" function may be expanded on them.

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \qquad \text{where} \qquad c_n = \frac{2n+1}{2}\int_{-1}^{+1} P_n(x)f(x)dx$$

3. Rodrigues formula

(3.1) $$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

We claim that the Rodrigues formula above is a solution to the Legendre differential equation.
It is, however, not possible to verify this directly without differentiation to the bottom for a fixed n

There are other ways, however. One is to use the binomial formula to expand $(x^2-1)^n$ into powers of x^2 and carry out n times of differentiation, but it is rather extensive.
Another way, (but also rather circumstantial) is to take as a starting point the function:

$$v(x) = (x^2-1)^n \quad \Rightarrow \quad (x^2-1)\frac{dv}{dx} = (x^2-1)n2x(x^2-1)^{n-1} = 2nx(x^2-1)^n = 2nxv$$

So

(3.2) $$(x^2-1)\frac{dv}{dx} = 2nxv$$

Then we differentiate this identity $n+1$ times to show that $\frac{d^n v}{dx^n}$ is in fact a solution to the Legendre differential equation, as it ought to be according to Rodrigues formula.

3.1 Differentiating a product n-times

We shall first look for a general formula, differentiating a product of two functions (fg) n-times.

The k-times differentiated function $\frac{d^k f}{dx^k}$, we write for simplicity as $f^{(k)}$ for $k>2$, and $f^{(0)} = f$

To find the formula we begin with the product rule.

$$(fg)' = f'g + fg' = \sum_{k=0}^{1}\binom{1}{k} f^{(1-k)}g^{(k)}$$

$$(fg)'' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg'' = \sum_{k=0}^{2}\binom{2}{k} f^{(2-k)}g^{(k)}$$

$$(fg)^3 = f^{(3)}g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg^{(3)}$$

$$= f^{(3)}g + 3f''g' + 3f'g'' + fg^{(3)} = \sum_{k=0}^{3}\binom{3}{k} f^{(3-k)}g^{(k)}$$

Leading to

(3.2) $$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}g^{(k)}$$

This formula may of course be verified by induction, and it goes as follows.
It is only necessary to look at two terms in sequence from (3.2).

$$\ldots + \binom{n}{k} f^{(n-k)}g^{(k)} + \binom{n}{k+1} f^{(n-k-1)}g^{(k+1)} + \ldots$$

And when we differentiate w eget:

$$\binom{n}{k} f^{(n-k+1)} g^{(k)} + \binom{n}{k} f^{(n-k)} g^{(k+1)} + \binom{n}{k+1} f^{(n-k)} g^{(k+1)} + \binom{n}{k+1} f^{(n-k+1)} g^{(k+2)}$$

The products of the two functions are the same in the second and third term, so we hold their binomial coefficients together.

$$\begin{aligned}\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\ &= \frac{n!}{k!(n-k-1)!}\left(\frac{1}{n-k} + \frac{1}{k+1}\right) = \frac{n!}{k!(n-k-1)!}\left(\frac{k+1+n-k}{(n-k)(k+1)}\right) \\ &= \frac{n!}{k!(n-k-1)!}\left(\frac{n+1}{(n-k)(k+1)}\right) = \frac{(n+1)!}{(k+1)!(n-k)!}\end{aligned}$$

As it should be, namely: $\binom{n+1}{k+1} f^{(n-k)} g^{(k+1)}$

3. Proof of Rodrigues formula

We return to the proof of Rodrigues formula, where $v = (x^2 - 1)^n$ by differentiating the equation:

$$(x^2 - 1)\frac{dv}{dx} = 2nxv \qquad n+1 \text{ times.}$$

We adapt $f = (x^2 - 1)$ and $g = \frac{dv}{dx}$, and n -> n+1 in the formula (3.2) so:

(3.3) $$(fg)^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)}$$

Since $f(x) = (x^2 - 1)$ we notice that:

$$f'(x) = 2x,\ f''(x) = 2,\ f^{(3)}(x) = 0 \text{ , so } f^{(n+1-k)} = 0 \text{ when } n + 1 - k > 2 \Leftrightarrow k < n \text{ -1,}$$

so the lower limit for k is n -1. We then get 3 terms.

$$
\begin{aligned}
(fg)^{(n+1)} &= \sum_{k=n-1}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)} \\
&= \frac{n(n+1)}{2} f^{(2)} g^{(n-1)} + (n+1) f' g^{(n)} + f g^{(n+1)} \\
\text{(3.4)} \qquad &= \frac{n(n+1)}{2} 2 \frac{d^{n-1}}{dx^{n-1}} \frac{dv}{dx} + (n+1) 2x \frac{d^n}{dx^n} \frac{dv}{dx} + (x^2-1) \frac{d^{n+1}}{dx^{n+1}} \frac{dv}{dx} \\
&= n(n+1) \frac{d^n v}{dx^n} + (n+1) 2x \frac{d^{n+1} v}{dx^{n+1}} + (x^2-1) \frac{d^{n+2} v}{dx^{n+2}} \\
&= (x^2-1) \frac{d^2}{dx^2}\left(\frac{d^n v}{dx^n}\right) + 2x(n+1) \frac{d}{dx}\left(\frac{d^n v}{dx^n}\right) + n(n+1) \frac{d^n v}{dx^n}
\end{aligned}
$$

Then we turn to the right hand term $2nxv$ in $(x^2-1)\frac{dv}{dx} = 2nxv$, using $f = 2nx$ and $g = v$ in the formula: $(fg)^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)}$.

Since $f(x) = 2nx$ we see that when $n+1-k > 1$ then $f^{(n+1-k)} = 0$, so the lower limit of k must be n.

$$
\begin{aligned}
(fg)^{(n+1)} &= \sum_{k=n}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)} = \\
\text{(3.5)} \qquad & 2n(n+1) f' g^{(n)} + f g^{(n+1)} \\
&= 2n(n+1) \frac{d^n v}{dx^n} + 2nx \frac{d}{dx} \frac{d^n v}{dx^n}
\end{aligned}
$$

So we have that

$$(x^2-1)\frac{dv}{dx} = 2nxv$$

implies

$$(x^2-1) \frac{d^2}{dx^2}\left(\frac{d^n v}{dx^n}\right) + (n+1) 2x \frac{d}{dx}\left(\frac{d^n v}{dx^n}\right) + n(n+1) \frac{d^n v}{dx^n} = 2n(n+1) \frac{d^n v}{dx^n} + 2nx \frac{d}{dx} \frac{d^n v}{dx^n}$$

$$\text{(3.6)} \qquad (x^2-1) \frac{d^2}{dx^2}\left(\frac{d^n v}{dx^n}\right) + 2x \frac{d}{dx}\left(\frac{d^n v}{dx^n}\right) - n(n+1) \frac{d^n v}{dx^n} = 0$$

If we put $y = \frac{d^n v}{dx^n}$ we get:

$$\text{(3.7)} \qquad (x^2-1) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - n(n+1) y = 0$$

So we see that $y = \frac{d^n v}{dx^n}$ is indeed a solution to Lagrange equation, and since

$$\text{(3.8)} \qquad v(x) = (x^2-1)^n \quad \text{and} \quad P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

Then it follows that the Rodrigues polynomial $P_n(x)$ is indeed a solution to Lagrange's differential equation.

4. Associated Legendre polynomials:

The Legendre polynomials:

(4.1) $$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

Are solutions to the Legendre differential equation:

(4.2) $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

The associated Legendre differential equation is:

(4.3) $$(1-x^2)y'' - 2xy' + \left(l(l+1) - \frac{m^2}{1-x^2}\right)y = 0$$

We shall prove that the equation (4.2) has the solution:

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x)$$

However, establishing this is far from simple. In the literature it is mostly stated as a fact and therefore I have chosen to fill in the details. We shall begin by trying with a solution of the form:

(4.4) $$y = (1-x^2)^n u \qquad \text{where we for simplicity have put } n = \tfrac{m}{2},$$

First we do some preliminary calculations

$$y' = n(1-x^2)^{n-1}(-2x)u + (1-x^2)^n u' =$$
$$-2nx(1-x^2)^{n-1}u + (1-x^2)^n u'$$

$$y'' = -2n(1-x^2)^{n-1}u + 4n(n-1)(1-x^2)^{n-2}x^2u - 2xn(1-x^2)^{n-1}u' - 2xn(1-x^2)^{n-1}u' + (1-x^2)^n u''$$

$$y'' = -2n(1-x^2)^{n-1}u + 4n(n-1)(1-x^2)^{n-2}x^2u - 4xn(1-x^2)^{n-1}u' + (1-x^2)^n u''$$

Insert in the associated Lagrange equation:

$$(1-x^2)\left(-2n(1-x^2)^{n-1}u+4nx^2(n-1)(1-x^2)^{n-2}u-4xn(1-x^2)^{n-1}u'+(1-x^2)^n u''\right)-$$
$$2x(-2nx(1-x^2)^{n-1}u+(1-x^2)^n u')+\left(l(l+1)-\frac{m^2}{1-x^2}\right)(1-x^2)^n u=0 \quad \Leftrightarrow$$

$$\left(-2n(1-x^2)^n u+4nx^2(n-1)(1-x^2)^{n-1}u-4xn(1-x^2)^n u'+(1-x^2)^{n+1}u''\right)$$
$$+4x^2n(1-x^2)^{n-1}u-2x(1-x^2)^n u'+\left(l(l+1)(1-x^2)^n-m^2(1-x^2)^{n-1}\right)u=0 \quad \Leftrightarrow$$

$$-2n(1-x^2)^n u+4nx^2(n-1)(1-x^2)^{n-1}u-4xn(1-x^2)^n u'+(1-x^2)^{n+1}u''$$
$$+4x^2n(1-x^2)^{n-1}u-2x(1-x^2)^n u'+\left(l(l+1)(1-x^2)^n-4n^2(1-x^2)^{n-1}\right)u=0$$

Division by $(1-x^2)^{n-1}$:

$$-2n(1-x^2)u+4nx^2(n-1)u-4xn(1-x^2)u'+(1-x^2)^2u''$$
$$+4x^2nu-2x(1-x^2)u'+\left(l(l+1)(1-x^2)-4n^2\right)u=0 \quad \Leftrightarrow$$

$$-2n(1-x^2)u+4nx^2(n-1)u-4xn(1-x^2)u'+(1-x^2)^2u''$$
$$+4x^2nu-2x(1-x^2)u'+\left(l(l+1)(1-x^2)-4n^2\right)u=0 \quad \Leftrightarrow$$

$$(1-x^2)^2u''-4xn(1-x^2)u'-2x(1-x^2)u'-2n(1-x^2)u+4nx^2(n-1)u$$
$$+4x^2nu+\left(l(l+1)(1-x^2)-4n^2\right)u=0 \quad \Leftrightarrow$$

$$(1-x^2)^2u''-2x(1-x^2)(2n+1)u'+(-2n(1-x^2)+4nx^2(n-1)$$
$$+4x^2n-4n^2)u+\left(l(l+1)(1-x^2)\right)u=0 \quad \Leftrightarrow$$

$$(1-x^2)^2u''-2x(1-x^2)(2n+1)u'+(-2n(1-x^2)+4n^2x^2$$
$$-4n^2)u+\left(l(l+1)(1-x^2)\right)u=0 \quad \Leftrightarrow$$

$$(1-x^2)^2u''-2x(1-x^2)(2n+1)u'+(-2n(1-x^2)+4n^2x^2$$
$$-4n^2)u+\left(l(l+1)(1-x^2)\right)u=0 \quad \Leftrightarrow$$

$$(1-x^2)^2u''-2x(1-x^2)(2n+1)u'-(1-x^2)(2n+4n^2)u+l(l+1)(1-x^2)u=0 \quad \Leftrightarrow$$

$$(1-x^2)^2u''-2x(1-x^2)(2n+1)u'-(1-x^2)(2n+4n^2)u+l(l+1)(1-x^2)u=0$$

Divide by $(1-x^2)$ and replace by $2n$ by m.

(4.5)	$(1-x^2)u''-2x(m+1)u'+\left(l(l+1)-(m(m+1)\right)u=0$

For $m = 0$, we recognize (4.5) as the Legendre equation

(4.6) $$(1-x^2)u''-2xu'+l(l+1)u=0$$

which has the solution $u = P_l(x)$. If we differentiate the equation (4.5) we get:

$$(1-x^2)(u')''-2x(u')'-2(m+1)u'-2x(m+1)(u')'+(l(l+1)-m(m+1))u'=0 \quad \Leftrightarrow$$

$$(1-x^2)(u')''-2x(m+2)(u')'+(l(l+1)-(m+1)(m+2))u'=0$$

But this just (4.5) with u' in place of u, and $m+1$ instead of m. But this is the differential equation of $u'=\frac{d}{dx}P_l(x)$ with m =1 instead of m = 0.

By differentiating another time, we find that $u''=\frac{d^2}{dx^2}P_l(x)$ is a solution with m = 2.

So returning to our tentative solution: $y=(1-x^2)^{\frac{m}{2}}u$, we can see that:

(4.7) $$y=P_l^m(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x)$$

Are the solutions to the associated Legendre equation.

The expressions (4.7) are therefore called the associated Legendre functions.
For m = 0 the associated Legendre functions are the same as the Legendre polynomials.
Below is listed the first six Legendre functions.

$$P_0^0(x)=1 \qquad P_1^1(x)=-\sqrt{1-x^2}$$
$$P_1^0(x)=x \qquad P_2^1(x)=-3x\sqrt{1-x^2}$$
$$P_2^0(x)=\tfrac{1}{2}(3x^2-1) \qquad P_2^2(x)=3(1-x^2)$$

As it is the case for the Legendre polynomials, the associated Legendre functions are orthonormalized functions, since we have the relation, (which we do not prove, because…)

$$\int_{-1}^{+1}P_k^m(x)P_l^m(x)dx=\frac{(l+m)!}{(l-m)!}\delta_{k\,l}$$