The interpolation formulas of Newton and Lagrange

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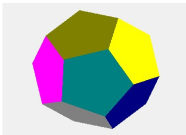

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1. The interpolation formulas of Newton and Lagrange

For n+1 mutual different point $(x_0, y_0),(x_1, y_1),(x_2, y_2),...,(x_n, y_n)$ exits one and only one polynomial of degree n: $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where $y_k = p(x_k), \quad k = 0...n$.
There can only be one polynomial with this property, since if there was another polynomial $q(x)$ having the same properties then $p(x) - q(x)$ would have the n+1 zero points $x_0, x_1,..., x_n$ but we know that a polynomial of degree n can have only n zero points.
In principle the coefficients $a_0, a_1,...a_n$ can be determined by solving the n+1 equations:

$$a_0 + a_1x_0 + a_2x_0^{\ 2} + ... + a_nx_0^{\ n} = y_0$$

....

$$a_0 + a_1x_n + a_2x_n^{\ 2} + ... + a_nx_n^{\ n} = y_n$$

However this has only theoretical interest.

Fortunately some of the great mathematicians have invented some cleaver method to establish the polynomial without finding the coefficients explicitly. The results are called interpolation formulas.

1.1 Newton's interpolation formula

Here the polynomial is written as:

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + ... + c_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

The coefficients are then being determined successively.

$$c_0 = y_0 \ , \ c_1 = \frac{y_1 - c_0}{x_1 - x_0}, \ c_2 = \frac{y_2 - (c_0 + c_1(x_2 - x_0))}{(x_2 - x_0)(x_2 - x_1)} ...$$

$$c_n = \frac{y_n - (c_0 + c_1(x_n - x_0) + ... + c_{n-1}(x_n - x_0)...(x_n - x_{n-2}))}{(x_n - x_0)(x_n - x_1)(x_n - x_{n-1})}$$

… and so on.

We have only included Newton's interpolation formula for completenes. For practical purpose it far less applicable than

1.2 Lagrange's interpolation formula

Lagrange's interpolation formula is formally written:

$$p(x) = y_0p_0(x) + y_1p_1(x) + ... + y_np_n(x)\text{, where}$$

$$p_0(x) = \frac{(x - x_1)(x - x_2)...(x - x_n)}{(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)} \qquad p_1(x) = \frac{(x - x_0)(x - x_2)...(x - x_n)}{(x_1 - x_0)(x_1 - x_2)...(x_1 - x_n)}$$

..

$$p_n(x) = \frac{(x - x_0)(x - x_2)...(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)...(x_n - x_{n-1})}$$

In this manner we obtain:

$$p_0(x_0) = 1,\ p_0(x_1) = p_0(x_2) = \ldots = p_0(x_n) = 0$$

$$p_1(x_1) = 1,\ p_1(x_0) = p_1(x_2) = \ldots = p_1(x_n) = 0$$

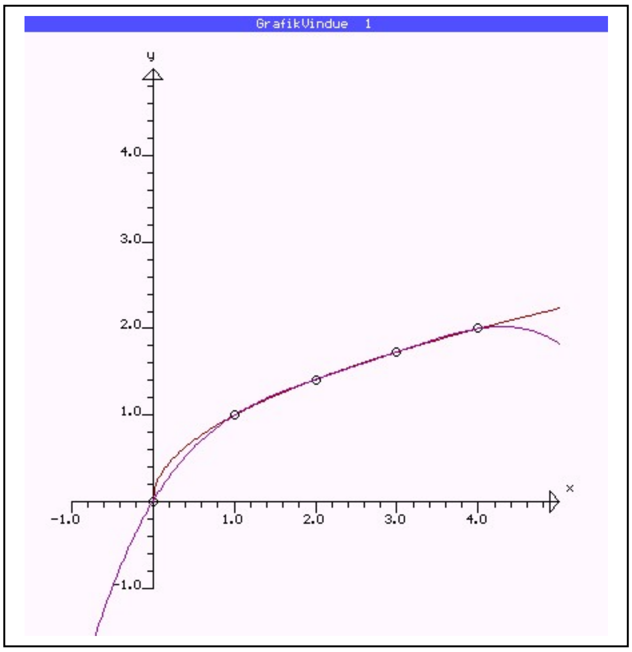
$$p_n(x_n) = 1,\ p_n(x_0) = p_n(x_2) = \ldots = p_n(x_{n-1}) = 0$$

And accordingly for the polynomial $p(x) = y_0 p_0(x) + y_1 p_1(x) + \ldots + y_n p_n(x)$

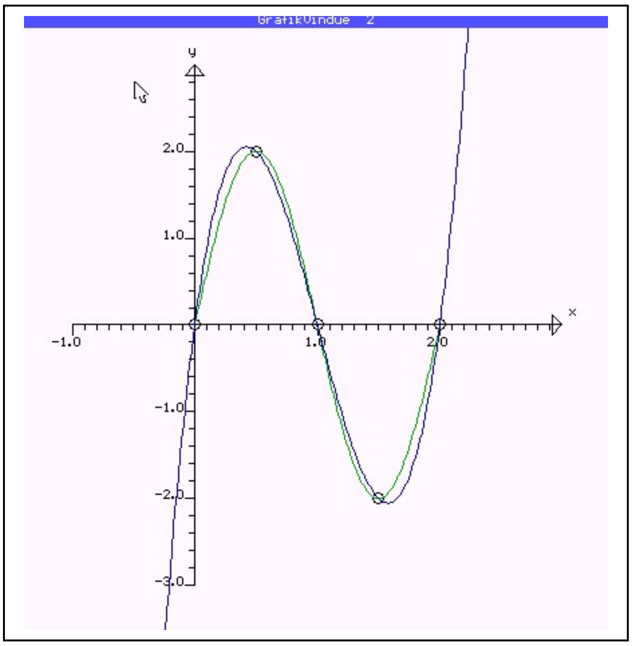
$$p(x_0) = y_0 p_0(x_0) = y_0$$
$$p(x_1) = y_1 p_1(x_1) = y_1$$
$$\ldots$$
$$p(x_n) = y_n p_n(x_n) = y_n$$

Which was the assertion, we wanted to prove.

The graph of $f(x) = \sqrt{x}$, together with the fourth order Lagrange polynomial



The graph of $f(x) = 2\sin(\pi x)$, together with the fourth order Lagrange polynomial



Although interpolation might be acceptable within the interval of the five points, we can see, that extrapolation outside the five points is meaningless. Applying a higher degree of polynomial, will not improve this – on the contrary it will get worse.

Reference: Hand written Lectures notes from professor Børge Jessen. University of Copenhagen 1965