5. Differential equation models

When we talk about a mathematical ”model”, we refer to a mathematical expression or differential equation, which may describe some empirical data, that are not based or derived from first principles, that is, laws of nature.
One may establish several different models, in contrast to a theory, which cannot be duplicated or improved within its area of validity.

5.1 The course of an influenza epidemic.

The propagation of an epidemic is in general a very complex process, often caused by accidental events. The deadly SARS, propagating in various countries worst in China and in vest Canada was caused by a salesman dealing with poultry, who spend the night in an international hotel, and after which the disease spread to various countries through air port terminals etc.
Inherently it is not possible to establish a mathematical model for such a series of accidental events.

A far better example, are the flu epidemics, which occasionally hit Europe, USA and the rest of the world
Here it is possible to establish a (crude) model of how the epidemic will propagate and finally fade out. The model is most adequate for cities, having a large population.
This is so because the flu is propagated by drop infection, and to do so it is required that many people are in fairly close contact with each other. We shall make some designations.

$H(t)$ = ”Healthy”: The number in the population, which has not been infected yet at time t.

$S(t)$ = ”Sick: The number in the population at time t, which has been infected, and still can transfer the infection to others.

$R(t)$ = ”Recovered”. The number in the population at time t, which has been infected, but has now recovered.

If N is the size of the population, then in all cases: $H(t)+ S(t) + R(t) = N$

Since the model is based on probability arguments, we shall invent the corresponding fractions.

$$h(t) = \frac{H(t)}{N}$$: The fraction of “healthy” persons.

$$s(t) = \frac{S(t)}{N}$$: The fraction of ”ill” persons.

$$r(t) = \frac{R(t)}{N}$$: The fraction of ”recovered” persons.

At all times we must have: $h(t)+ s(t) + r(t) = 1$

As always differentiation with respect to time indicates a velocity or a rate with which a quantity change.

If there is at least one that is ”ill”, then $h(t)$ will be a decreasing function of time. The speed with which it decreases will (under the most general assumptions) be proportional to the probability that a healthy encounters an ill. So we have the equation (where a is a constant)

$$(5.1) \qquad \frac{dh}{dt} = -a \cdot h(t)s(t)$$

Since a person that has become ill, after some time gets healthy (or dies), then the velocity with which they recover must be proportional with the fraction of the ill. (b is a constant)

$$(5.2) \qquad \frac{dr}{dt} = bs(t)$$

To establish a differential equation for $s(t)$, we shall apply the normalization condition:

$$h(t) + s(t) + r(t) = 1,$$

Which we differentiate:

$$\frac{dh}{dt} + \frac{ds}{dt} + \frac{dr}{dt} = 0 \quad \Leftrightarrow \quad \frac{ds}{dt} = -\frac{dh}{dt} - \frac{dr}{dt}$$

Inserting the already established expressions for $\frac{dh}{dt}$ *and* $\frac{dr}{dt}$

$$(5.3) \qquad \frac{ds}{dt} = a \cdot h(t)s(t) - bs(t)$$

We may also argue for the equation (5.3), since a person, which is no longer healthy has become ill. So the fraction of the ill increases with that rate $a \cdot h(t)s(t)$, but decreases at the same time proportional with $s(t)$, since an ill eventually recovers (or dies) after a certain period.

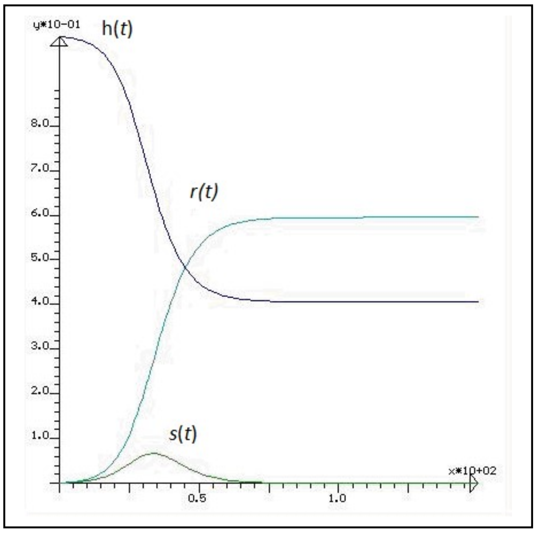
The three coupled differential equations (5.1) – (5.3) have no known analytic solution, and we must resort to a numerical solution performed by a computer.

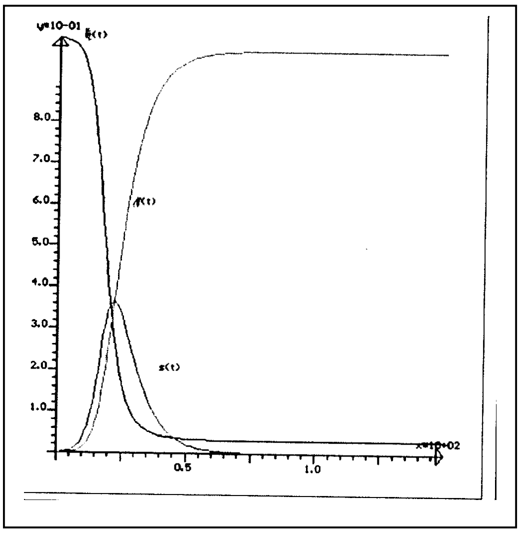
(The graphs below is made by my own DOS-based program, written in Turbo 7.0 from Borland in 1995)

The determination of the constants a and b can for example be taken from empirical data.
On the first graph we have rather arbitrarily (to illustrate the capacity of the model) chosen a=0.5 and b=0.33. The choice of b=0.33 can be argued, because an infected person can only infect others in the time of incubation, which is set to 3 days, so that a third of them can not infect others after one day.

On the second graph, we have again chosen a=0.5, but assumed that the time of incubation is 7 days b=0.141 (=1/7). In both cases we see that the number of infected is increased, passing a max until it begins to fall with almost the same pace.

Notice that the model assumes that the epidemic can develop freely, and that there are no medical counter measure such as vaccination or isolation of the infected, as it was the case both with the SARS and the Ebola epidemics.
In the first case the maximum infected becomes is 8%, whereas in the second case it is 38%. The time of incubation (where an infected person can infect others) is of vital importance.
In the first case 58% become infected in all, whereas in the second case it is 96%!
If we adjust *a*, one gets other results of course.





The model may be applied to give an estimate of the duration of an epidemic if one, from the start of the epidemic, has gathered sufficient data to estimate *a* and *b*.