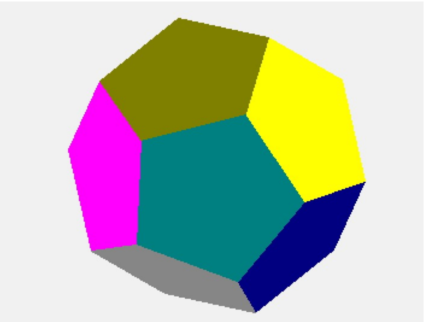
Finite and Infinite Series



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1. Sequences of numbers

A number sequence is an infinite sequence of numbers, where each element is separated from the next with a comma.

(1.1) $a_1, a_2, a_3, \ldots, a_n, \ldots$

To be able to write the individual element, one usually has a formula for the *n*'th element e.g.

$$a_n = 2^n \quad or \quad b_n = \frac{n-1}{n} \quad or \quad c_n = \left(1+\frac{1}{n}\right)^n$$

The first 3 elements of each number sequence become

$$(a): \quad 2, 4, 8, \ldots \qquad (b): \quad 0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \ldots \qquad (c): \quad 2, \tfrac{9}{4}, \tfrac{64}{27}$$

A number sequence is said to *converge*, to be *convergent*, to have a finite *limit*, if the *n*'th element approaches, or rather, becomes infinite close to a certain number.

For the three sequences above, it is evident that the first (a) is not convergent, since the *n*'th element goes to infinity.
The second sequence, however, has the limit 1, which can be seen, by the rearrangement.

$$b_n = \frac{n-1}{n} = 1 - \frac{1}{n} \quad .$$

Since $\frac{1}{n} \to 0$ (approaches 0, has the limit 0), when n approaches infinity ($n \to \infty$), it is clear that the sequence has the limit 1.

That the third sequence has the limit e (the base of the natural logarithm), however is not entirely trivial, but it can be deduced since we know that the function $f(x) = \ln x$, is differentiable: $f'(x) = (\ln x)' = 1/x$, and therefore $f'(1) = (\ln 1)' = 1$.

We remind of the definition of the derivative of a real function $y = f(x)$

(1.2) $$\frac{f(x+h) - f(x)}{h} \to f'(x) \quad for \quad h \to 0$$

We therefore take the logarithm of the *n*'th element of the sequence (c). (Notice that $ln(1)=0$, and that h is replaced with $1/n$)

$$\ln(1+\tfrac{1}{n})^n = n\ln(1+\tfrac{1}{n}) = \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \frac{\ln(1+\frac{1}{n}) - \ln(1)}{\frac{1}{n}} \to \ln'(1) = 1 \quad for \quad \frac{1}{n} \to 0 \quad (i.e.\ for\ n \to \infty)$$

Since $\ln e = 1$, it then follows that if $\ln(1+\frac{1}{n})^n$ has the limit 1 *for* $n \to \infty$, then $(1+\frac{1}{n})^n$ must have the limit *e*, and consequently: $\left(1+\frac{1}{n}\right)^n \to e \quad for \quad n \to \infty$

The theoretical definition of the limit *a,* of a number sequence a_n is this:

(1.2) For every (arbitrary small) number ε, there exist an integral number *n*, so that $|a_n - a| < \varepsilon$

Vi further introduces two mathematical operators $\forall$ and $\exists$.

$\forall$: (called: All-quantifier) It reads:
"For any…the following apply…" or "For all…. the following apply …"

$\exists$: (called: Exist-quantifier) and it reads:
"There exists….so the following apply …" or "There is at least one….so ….apply …"

With the help of quantifiers the definition of limit may be written in a more compact form

(1.3) $$\forall \varepsilon > 0 \;\; \exists\, n : |a_n - a| < \varepsilon$$

2. Series of numbers

A series is a sum of numbers. Series are always written with the *summation symbol* $\sum$, where $\sum$ is the Greek capital letter *sigma*.
The meaning of the *summation symbol* is best illustrated by showing some examples.

(2.1) $$\sum_{n=1}^{5} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \quad , \quad \sum_{k=1}^{10} k^2 - k = 0 + 2 + 6 + 12 + \ldots + 90$$

In the examples above *n* and *k* are indices. The sum is evaluated (in the first example), first by setting $n = 1$ (called the lower limit) and evaluate the expression under the summation symbol, then adding a plus sign, and set $n = 2$ and evaluate…, and so on until the upper limit $n = 5$ is reached.

Using another index does not change anything. Indices are often chosen among the letters: *i, j, k, l n, m.* The upper and lower limits can be chosen as any integral numbers.

2.1 Infinite series

An infinite series is a series of numbers, where the upper limit is infinity.
In advance it is far from obvious that it is possible to assign a definite number to an infinite sum. We shall however, by demonstrating some examples, show that it is actually possible to do so in a mathematically consistent way. An infinite series can for example be written:

(2.2) $$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

We denote by S_n the sum of the first n terms in the sum. $S_n = \sum_{k=1}^{n} a_k$

If the sequence $S_1, S_2, S_3, ..., S_n, ...$ has a limit for $n \to \infty$, the series is said to be *convergent* with a sum being that limit. If the limit does not exist the series is said to be *divergent*.

2.2 Formula for the sum of an algebraic series

An algebraic series is a series, where the difference between each term and the preceding term is constant. Below is shown two simple examples.

1 + 2 +3 + 4 + …+ 100 (difference = 1) $0 + \frac{1}{2} + 1 + \frac{3}{2} + 2 + ...$ (difference = ½)

The task is to establish a general formula for the sum of an algebraic series.
If the terms are denoted by *a* (with an index), and the difference is denoted *d*, then any algebraic series having *n* terms, can be formally written.

(2.3) $S_n(1) = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_1 + (n\text{-}1)d)$

If we write the series in the reverse order.

$$S_n(n) = a_n + (a_n - d) + (a_n - 2d) + ... + (a_n - (n\text{-}1)d)$$

And add the two (they are the same) series term by term from one end to the other, we see that

$$S_n(1) = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_1 + (n\text{-}1)d)$$

$$S_n(n) = a_n + (a_n - d) + (a_n - 2d) + ... + (a_n - (n\text{-}1)d)$$

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) ... + (a_1 + a_n)$$

Together *n* terms, which are all equal to each other. Then it follows that $2S_n = n\,(a_1 + a_n)$, or as the formula for an algebraic series is usually written:

(2.4) $$S_n = \frac{n}{2}(a_1 + a_n)$$

a_1 is the first term, and a_n is the last. The number of terms is *n*.
The formula can for example be applied to a quick calculation of the sum of integers from 1 to a 1000.

$$S_{100} = \frac{1000}{2}(1 + 1000) = 500500$$

An anecdote about the most famous mathematician J. F. Gauss, when he was nine years old, goes as follows.
His severe teacher had given the class the exercise to calculate the sum of integers from 1 to 100 (correctly), within a lesson. In less than 5 minutes Gauss came up with the correct answer. The teacher was furious, and claimed that Gauss must have been cheating. But when he rushed down to look at Gauss' calculation he became pale with embarrassment, because on Gauss' note block, there were only 3 lines:

1 + 2 + 3+…+100
100+99+98+… +1

101+101+101+…+101 The sum is ½ ·100· 101 = 5050

2.3 Formula for the sum of a geometric series

A geometric series is a series, where the quotient between each term and the preceding term is constant.
We now type this property in a general manner: The first term is denoted *a*, the quotient *q*, and the sum of the first *n* terms is S_n.

(2.5) $S_n = a_1 + a_2 + a_3 + ... + a_n$ $S_n = a + aq + aq^2 + ... + aq^{n-1}$

To make a formula for the sum S_n, we multiply S_n by *q*, and subsequently subtract S_n from qS_n

$$qS_n = aq + aq^2 + aq^3 + ... + aq^{n-1} + aq^n$$
$$S_n = a + aq + aq^2 + aq^3 + ... + aq^{n-2} + aq^{n-1}$$

$$qS_n - S_n = aq + aq^2 + aq^3 + ... + aq^{n-1} + aq^n - (a + aq + aq^2 + aq^3 + ... + aq^{n-2} + aq^n)$$

Doing the subtraction of all the terms, $aq,\ aq^2\ aq^3.....aq^{n-1}$ will cancel each other, and we find:

$$qS_n - S_n = aq^n - a \quad \Leftrightarrow \quad S_n(q - 1) = a(q^n - 1)$$

So the formula for S_n becomes:

(2.6) $S_n = a\dfrac{q^n - 1}{q - 1}$ or equivalently $S_n = a\dfrac{1 - q^n}{1 - q}$ when $q < 1$

Notice that *n* is the number of terms in the series.

The expression for S_n applies for $q \neq 1$. For $q = 1$, $S_n = na$ (naturally)

An infinite geometric series is convergent for $|q| < 1$, and otherwise divergent. This follows from:

$q^n \to 0 \quad for \quad n \to \infty$, when $|q|<1$ (since $a^x \to 0 \quad for \quad x \to \infty \ when \ 0 < a < 1$)

Applying this to (2.6), we have the formula for an infinite geometric series.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-q^n}{1-q} = a\frac{1-\lim_{n\to\infty} q^n}{1-q} = a\frac{1}{1-q}$$

The formula for the sum of an infinite geometric series, where: -1 < q < 1.

(2.7) $$S = a\frac{1}{1-q}$$

(2.8) Examples

1. Consecutive bisection of a line segment with length 1.

The infinite sum of the segments is:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots=\sum_{n=1}^{\infty}\frac{1}{2^n}=\tfrac{1}{2}\frac{1}{1-\frac{1}{2}}=1$$

Infinite decimal fractions:

$$0.3333333\ldots=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\ldots..=\sum_{k=1}^{\infty}\frac{3}{10^k}=\frac{3}{10}\cdot\frac{1}{1-\frac{1}{10}}=\frac{1}{3}$$

It should be emphasized, that the theory of infinite sums of numbers being a finite number is absolutely rigorous and consistent. And the theory resolves many of the famous (philosophical) paradoxes, which (especially) in ancient Greece that has been put forward e.g. Achilles and the turtle.

An infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ is said to be *absolute* convergent if the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \ldots \text{ is convergent.}$$

If a series is absolute convergent, it is also convergent, whereas the opposite is not always the case.

For example is the harmonic series: $\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\ldots$ divergent, but the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}\ldots \text{ is convergent.}$$

3. Criteria for convergence

As mentioned above an expression of the form:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

is called an infinite series. The sum:

(3.1) $$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

Is called the nth section of the infinite series, and the sequence $(s_n) = s_1, s_2, s_3, \ldots$ is called the *section sequence*.

If the section sequence is convergent and converges to s, then the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

Is said to be convergent with sum s.

(3.2) $$\sum_{n=1}^{\infty} a_n = s$$

A series which is not convergent is called divergent. A sum of the form:

$$u_{n,p} = \sum_{k=n+1}^{n+p} a_k = a_{n+1} + ... + a_{n+p}$$

Is called a numbered section of the series.
It happens that the terms have other indices than k or n. e.g. $\sum_{n=1}^{\infty} a_{2n+5}$, but the section s_n, is still the sum of the first n terms.

Formally the criteria for convergence of a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ..., \text{ where } s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n$$

If the sum is unknown (as it usually is) this may be formulated in a slightly different manner:

(4.1) The series $\sum_{k=1}^{\infty} a_k$ is *convergent* if and only if:

$$\forall \varepsilon > 0 \; \exists N : \forall n \geq N \; \forall p > 0 : |u_{n,p}| \leq \varepsilon, \quad \textit{where} \quad u_{n,p} = s_{n+p} - s_n$$

This follows from the rewriting: $|u_{n,p}| = |s_{n+p} - s_n| = |(s_{n+p} - s) - (s - s_n)| < |s_{n+p} - s| + |s_n - s|$
Since it is possible to choose an n such that right hand two terms are less than ε, which insures convergence, it follows that $|u_{n,p}| < \varepsilon$ will insure convergence ,otherwise the series is divergent.

From this we can immediately see, that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent:

Since for $\varepsilon = \frac{1}{4}$ it is not possible to find any N, such that the convergence criterion is satisfied.
Namely we have:

$$u_{N,N} = \sum_{k=N+1}^{2N} \frac{1}{k} \geq N \frac{1}{2N} = \frac{1}{2}$$

From the criterion of convergence, it also follows trivially that a necessary (but not sufficient, as the harmonic series demonstrates) condition for convergence of the series $\sum a_n$ is that $a_n \to 0$.

3.1 The comparison criteria

Some series are known to be convergent, for example the geometric series:

(3.2) $$S = \sum_{n=0}^{\infty} aq^n = a\frac{1}{1-q} \quad \text{or as it usually written:} \quad S = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

By comparison with the geometric series, the series $\sum_{n=1}^{\infty} a_n$ is (absolute) convergent if

$$\frac{|a_{n+1}|}{|a_n|} < 1$$

Since then the n'th term in the sum decreases faster than that of the geometric series. More formally:

If we put $x = \max\{\frac{|a_{n+1}|}{|a_n|}\} < 1$ then we find: $|a_{n+1}| < x|a_n|$ and

$|a_2| < x|a_1|, |a_3| < x|a_2| < x^2|a_1|,$ and so on, we have the inequality.

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} |a_n| < \sum_{n=1}^{\infty} |a_1| x^{n-1}$$

Since the last equation is geometric series with $|x| < 1$, it follows that the series

$\sum_{n=1}^{\infty} a_n$ is absolute convergent.

The geometric series is quite apt to decide, whether a series is convergent, but sometimes you need a series that converges slower to make the comparison.
A more general formulation of the comparison criterion is the following

Let $\sum b_n$ be a convergent series, having positive terms, and let $\sum a_n$ be a series with the condition $|a_n| \le b_n$ for all n, then $\sum a_n$ is convergent.
From the generalized condition for convergence, we may for any $\varepsilon > 0$ choose N, such that:

$$\sum_{k=n}^{n+q} b_k < \varepsilon \quad for \;\; n > N, \;\; q > 0$$

And we then have:

(3.3) $$\left|\sum_{k=n}^{n+q} a_k\right| < \varepsilon \;\; \le \sum_{k=n}^{n+q} |a_k| \le \sum_{k=n}^{n+q} b_k < \varepsilon \quad for \;\; n > N, \;\; q > 0$$

We notice trivially that if $s = \sum a_n$ and $t = \sum b_n$ are both convergent series then $\sum (a_n + b_n)$ is convergent with the sum $s + t$.

If we look at the section series, then we may choose n such that for p>0 :

$$\left|\sum_{k=n}^{n+q}(a_k + b_k)\right| = \left|\sum_{k=n}^{n+q} a_k + \sum_{k=n}^{n+q} b_k\right| < \left|\sum_{k=n}^{n+q} a_k\right| + \left|\sum_{k=n}^{n+q} b_k\right| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

We shall now look at an extended comparison condition, with the terms in a series b_n, where $b_n > 0$ decreases at least as fast than that of a convergent series a_n.

$$\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n}$$

If we choose $c \in R$, such that $b_N \le ca_N$, then $b_n \le ca_n$ for all $n > N$. This can be seen by induction, since:

$$\frac{b_{n+1}}{b_n} \le \frac{a_{n+1}}{a_n} \quad \Leftrightarrow \quad b_{n+1} \le \frac{a_{n+1}}{a_n} b_n \le \frac{a_{n+1}}{a_n} ca_n = ca_{n+1}$$

3.2 The general comparison criterion.

The most direct criterion is comparison with the geometric series, but that criterion may also be used indirectly.

1. The root criterion:

Convergence, if lim sup $\sqrt[n]{a_n} < 1$ and Divergence, if lim sup $\sqrt[n]{a_n} \ge 1$

2. The quotient criterion:

Convergence, if lim sup $\left(\frac{a_{n+1}}{a_n}\right) < 1$ and divergence, if lim sup $\left(\frac{a_{n+1}}{a_n}\right) > 1$

Proof of the root criterion:

From lim sup $\sqrt[n]{a_n} < 1$ follows the existence of a N and a $k < 1$, such that $\sqrt[n]{a_n} < k$ for $n > N$. It follows then that: $a_n < k^n$, and the convergence follows from the quotient criterion.

Proof of the quotient criterion:

From lim sup $\left(\frac{a_{n+1}}{a_n}\right) < 1$ follows in a similar way the existence of a N and a $k < 1$, such that:

$\left(\frac{a_{n+1}}{a_n}\right) < k = \frac{k^{n+1}}{k^n}$ for $n > N$. So here we are talking about an indirect comparison with a geometric series.

For a better appliance of the comparison criterion, we must make use of series which converge slower than geometric series. The convergence of such series may often be shown by the so called Integral criterion, which shall be discussed below.

Example Consider the *hypergeometric* series:

$$F(a,b,c;x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)1!}\frac{x^2}{2!} + \ldots$$

The ratio of successive terms is:

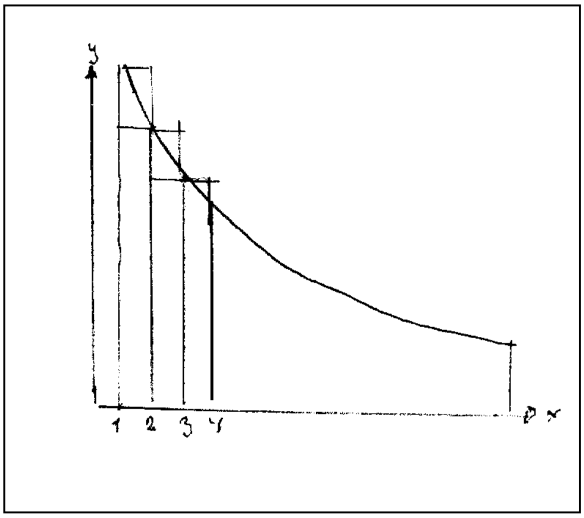
$$\frac{a_{n+1}}{a_n} = \frac{(a+n)(b+n)}{(c+n)(n+1)}x$$
$$= \frac{(1+a/n)(1+b/n)}{(1+c/n)(1+1/n)}x$$
$$= (1 + \frac{a+b-c-1}{n} + O(n^{-2}))x$$

Thus the series converge if $|x| < 1$ or if $|x| = 1$ and a + b - c < 0

3.3 The integral criterion

Let $f:[1,\infty] \to R$ be a decreasing positive function with the limiting value 0 at infinity.

Then the series $\sum_{n=1}^{n} f(n)$ and the sequence $\int_1^n f(x)dx$ are both convergent or both divergent



As you can see from the figure below:

$$\sum_{k=2}^{n+1} f(n) \le \int_1^{n+1} f(x)dx \le \sum_{k=1}^{n} f(n) = s_n$$

$$s_{n+1} - s_1 \le \int_1^{n+1} f(x)dx \le s_n$$

Both the sequence $\int_1^{n+1} f(x)dx$ and s_n are ascending, and if s_n is convergent, so is the integral and vice versa.

We shall then look into the convergence of the series: $\sum_{k=1}^{n} \frac{1}{k^\alpha}$, which are related to the function:

$f(x) = \frac{1}{x^z}$ $z > 1$, which is the argument to the Zeta-function: $\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \ldots$

If we apply the integral condition for the series $1 + \frac{1}{2^z} + \frac{1}{3^z} + \ldots + \frac{1}{n^z}$, we have

$$\int_1^n \frac{1}{x^z}dx = \left[\frac{x^{-z+1}}{1-z}\right]_1^n = \frac{1}{1-z}\left(n^{1-z} - 1\right).$$

The integral series is convergent for z > 1 and otherwise divergent. (Since $n^{-z} \to 0 \; for \; z > 0$).

So the series: $\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + ...$ is convergent for $z > 1$ otherwise divergent.

The integral condition may also be applied to the slowly convergent series: $x^{-1}(\ln x)^{-z}$ since the integral:

$$\int_2^n x^{-1}(\ln x)^{-z}\,dx = \left[\frac{(\ln x)^{-z+1}}{-z+1}\right]_2^n = \frac{(\ln n)^{-z+1}}{-z+1} - \frac{(\ln 2)^{-z+1}}{-z+1}$$

converges for $z > 1$, and so does the Zeta function.

3.4 Transformation of series

Example

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + ...$$

Integrating by terms

$$\int_0^x f(x)dx = x + x^2 + x^3 + ... = \frac{1}{1-x}$$

And then differentiating

$$f(x) = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

Example

$$f(x) = \frac{1}{1\cdot 2} + \frac{x}{2\cdot 3} + \frac{x^2}{3\cdot 4} + \frac{x^3}{4\cdot 5} + ...$$
$$x^2 f(x) = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \frac{x^5}{4\cdot 5} + ...$$

Differentiating the last equation twice:

$$(x^2 f(x))'' = 1 + x^2 + x^3 + ... = \frac{1}{1-x}$$

Two integrations give:

$$\int_0^x \frac{1}{1-x}dx = -\ln(1-x)$$

$$-\int_0^x \ln(1-x)dx = -\left[\ln(1-x)x\right]_0^x + \int_0^x xd\ln(1-x) = -x\ln(1-x) + \int_0^x xd\ln(1-x)$$

$$\int_0^x xd\ln(1-x) = -\int_0^x \frac{x}{1-x}dx = \int_0^x \frac{1-t}{t}dt = \left[\ln t - t\right]_0^x$$

$$t = 1-x \quad \Rightarrow \quad x = 1-t; \;\; dt = -dx$$

$$= \left[\ln(1-x) - (1-x)\right]_0^x = \ln(1-x) - (1-x) + 1$$

$$-\int_0^x \ln(1-x)dx = -x\ln(1-x) + \ln(1-x) - x = (1-x)\ln(1-x) + x$$

$$x^2 f(x) = (1-x)\ln(1-x) + x \;\Rightarrow\; f(x) = \frac{1}{x} + \frac{(1-x)\ln(1-x)}{x^2}$$

Example

$$S = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots$$

Define

$$f(x) = \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!} + \ldots \quad \text{Then } S = f(1) = 1$$

$$f'(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots = xe^x$$

$$f(x) = \int_0^x xe^x dx = \int_0^x xde^x = xe^x - \int_0^x e^x dx = xe^x - e^x + 1$$

$$S = f(1) = 1$$