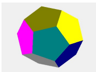
Analytic functions of a complex variable



Ole Witt-Hansen 2018 (1966)

Contents

1. Analytic (holomorphic) functions

A function of a complex variable *z*,

$$w = f(z)$$

is a mapping from the complex numbers into the complex numbers ($C \to C$). It is called an *analytic,* or when it is differentiable also a holomorphic function.

If *z* and *w* are complex numbers, the function may also be written as: $u + iv = f(x+iy)$.

Differentiability and continuity are defined as for real functions.

The function $f(z)$ is *continuous* in z_0 if:

$$f(z) \to f(z_0) \quad for \quad z \to z_0$$

The function $f(z)$ is *differentiable* in z_0 , if the fraction: $\dfrac{f(z)-f(z_0)}{z-z_0}$ has a limit for $z \to z_0$

If the limit exists, it is called the *differential quotient* of $f(z)$ *in* z_0. Written more formally:

(1.1) $$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$

As it is the case for real functions, the differentiability may be expressed in various ways.
If we put the increment $h = z - z_0$, then two equivalent ways of writing are:

(1.2) $$\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$$

(1.3) $$f(z_0+h) = f(z_0) + f'(z_0)h + \varepsilon(h)h \text{ , where } \varepsilon(h) \to 0 \ \ for \ \ h \to 0$$

Where the last equation is seen to be equivalent to: $\dfrac{f(z_0+h)-f(z_0)}{h} = f'(z_0) + \varepsilon(h)$

The differential quotient $f'(z_0)$ is also written as $\dfrac{df(z)}{dz}$, that is, the quotient between two *differentials*, since the *differential* of a function f is defined as:

(1.4) $$df(z) = f'(z_0)dz$$

Formally, we have just multiplied the differential quotient with *dz.* We shall not enter a discussion, whether differentials are infinitesimal quantities or something else, but we shall carefree use differentials as ordinary variables (as it has been done for more than 200 years).

The calculation rules for differential quotients follow the same rules as for real functions of one variable, and the rules are proven in the same manner.

- $(f(z) \pm g(z))' = f'(z) \pm g'(z)$
- $(f(z) \cdot g(z))' = f'(z) \cdot g(z) + f(z) \cdot g'(z)$
- $(\frac{f(z)}{g(z)})' = \frac{f'(z) \cdot g(z) - f(z) \cdot g'(z)}{g(z)^2}$

For the composite function: $g \circ f(z) = g(f(z))$ applies the same rule as hitherto:

- $(g \circ f)'(z) = g'(f(z))f'(z)$

For the inverse function f^{-1} (if it exists) to a differentiable function f, we have as before:

(1.5) $$w = f(z) \quad \Leftrightarrow \quad f^{-1}(w) = z$$

The inverse function f^{-1} has the differential quotient:

(1.6) $$(f^{-1})'(w) = \frac{1}{f'(z)} \qquad where \quad w = f(z)$$

The calculation rules for the composite and the inverse functions, may also be verified, if we carefree use differentials, since if $w = f(z) \quad and \quad y = g(w)$, we obtain purely algebraic:

(1.7) $$\frac{dg}{dz} = \frac{dg}{dw}\frac{dw}{dz}$$

This rule is denoted the *chain rule*, as it is the case for real functions.

Correspondingly for the inverse function:

(1.8) $$z' = (f^{-1})'(w) = \frac{dz}{dw} = \frac{1}{\frac{dw}{dz}} = \frac{1}{f'(z)} = \frac{1}{f'(f^{-1}(w))}$$

2. Cauchy – Riemann's differential equations.

An analytic function may also be written by its real and imaginary part.

(2.1) $$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

(2.2) $$df(z) = f'(z)dz = f'(z)(dx + idy) = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})dx + (\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})dy$$

Since the coefficients to *dx* and *dy* in (2.2), must be the same on both sides, we must have:

$$(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x})=f'(z) \quad and \quad \frac{1}{i}(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y})=f'(z) \quad \Rightarrow$$

(2.3) $$(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x})=\frac{1}{i}(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y})$$

Collecting the real and the imaginary part we obtain Cauchy-Riemann's differential equations.

(2.4) $$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

The equations may also be written as:

(2.5) $$\frac{df}{dz}=\frac{\partial f}{\partial x}=i\frac{\partial f}{\partial y}$$

The last formulation is very important, since it implies (as we shall demonstrate below) that every analytic function has an integral *F*, such that *F*'(*z*) = *f*(*z*).

For a real function of two variables $f(x,y)$, we have for the differential of *f*:

(2.6) $$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

From the analysis we know:

(2.7) $$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

Notice that f is an integral to df in (2.6).

One may show (not surprisingly) that (2.7) is a necessary and sufficient condition that a function $f(x,y)$ has an integral $F(x,y)$.

Namely if $dg=f_x(x,y)dx+f_y(x,y)dy$ is a differential form, *dg* is said to be a total differential, (and thus has an integral), if and only if (according to (2.7):

(2.8) $$\frac{\partial f_x}{\partial y}=\frac{\partial f_y}{\partial x}$$

But that's exactly the case for any analytic function, because of Cauchy Riemann's differential equations, since they may, according to (2.5) ,be written as:

$$\frac{\partial f}{\partial x}=i\frac{\partial f}{\partial y},$$

From which we conclude that

- Every analytic function has an integral.

For a vector function $f = (f_x, f_y)$, which has an integral *F*, that is, $(f_x, f_y) = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$, it applies that the integral of *f* along a curve $s(t) = (x(t), y(t))$ is independent of the curve it follows between the two end points (1) and (2), since:

$$\int_{(1)}^{(2)} \vec{f} \cdot d\vec{s} = \int_{(1)}^{(2)} (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}) \cdot d\vec{s} = \int_{(1)}^{(2)} (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}) \cdot (x'(t), y'(t))dt =$$

(2.9)

$$\int_{(1)}^{(2)} (\frac{\partial F}{\partial x} x'(t) + \frac{\partial F}{\partial y} \cdot, y'(t))dt = \left[F(x(t), y(t))\right]_{(1)}^{(2)} = F(x_2, y_2) - F(x_1, y_1)$$

The integral along a curve depends only on of the value of the integral *F* in the endpoints, which also applies for the integral of a real function of one variable.

$$\int_a^b f(x)dx = \left[F(t)\right]_a^b = F(b) - F(a)$$

This theorem is also well known in physics e.g. for a conservative force *F,* which may be derived from a potential *U*: $(F_x, F_y) = (-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y})$**.** The work done by the force *W*= $\int_{(1)}^{(2)} \vec{F} \cdot d\vec{s}$, is independent of the path chosen between the end points (1) and (2) of the curve.

From this follows immediately that for a vector function, which has an integral, the integral along any closed curve always equals zero, since the integral is $F(x_1, y_1) - F(x_1, y_1) = 0$.

3. The integral theorems of Cauchy

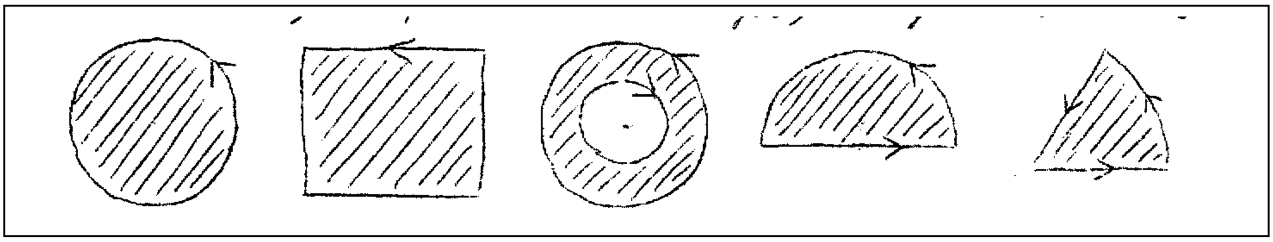
Cauchy's first integral thworem:

Since an analytic function $f(z) = u(x, y) + iv(x, y)$ may equivalently be represented by a vector function $(u(x, y), v(x, y))$, it applies for all analytic functions that the integral along a closed curve is independent of the path chosen, and it is always equal to zero.

(3.1) $$\int_{(1)}^{(2)} f(z)dz = F(z_2) - F(z_1) \qquad \oint f(z)dz = 0$$

We shall state this theorem in a somewhat different way, since we introduce the concept of a *simple figure*, as a compact set, limited by one or several closed curves.
The curves that limit a simple figure *F* is called the border (or edge) of the figure and is denoted ∂F .

We shall not specify the concept of a simple figure further, but just refer to the figures shown below.
The edges of the figures are oriented in the positive direction for curves which enclose the figure, and the edges are oriented in the negative direction for curves which are enclosed by the figure. (In which case the figure, must be enclosed by both an outer and an inner border).



Since the integral of an analytic function along a closed curve is zero, the follows:

Cauchy's first integral theorem.

For every simple figure F in $A \subseteq C$ and for every analytic function $f : A \to C$ applies:

(3.2) $$\int_{\partial F} f(z)dz = 0$$

We shall then proceed to apply the integral theorem (3.2) to the function: $f(z) = z^n$.

We consider a circular disc: $F = \{z \mid |z| \le r\}$, which as its border has the circle: $C = \{z \mid |z| = r\}$, having positive orientation and we shall evaluate the integral along the border for any integral number n. We then show that:

(3.3) $$\int_C z^n dz = \begin{cases} 0 & for \quad n \ne -1 \\ 2\pi i & for \quad n = -1 \end{cases}$$

Using the parametric for the circle: $z = re^{it} \quad 0 \le t \le 2\pi$, we find:

(3.4) $$\int_C z^n dz = \int_0^{2\pi} r^n e^{\mathrm{int}} rie^{it} dt = r^{n+1} i \int_0^{2\pi} e^{i(n+1)t} dt = \begin{cases} 0 & for \quad n \ne -1 \\ 2\pi i & for \quad n = -1 \end{cases}$$

3.1 Doing integrals, applying Cauchy's first integral theorem

As an example, we shall for real y evaluate the integral.

$$I(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{ixy} dx = \frac{1}{\sqrt{2\pi}} \lim_{a \to \infty} \int_{-a}^{a} e^{-\frac{1}{2}x^2} e^{ixy} dx$$

About the normal distribution function $e^{-\frac{1}{2}x^2}$ we know (which is also proven in analysis by using polar coordinates, when evaluating the integral of $e^{-\frac{1}{2}(x^2+y^2)}$) that:

$$I(0) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$$

We then consider the function: $f(z) = e^{-\frac{1}{2}z^2} = e^{-\frac{1}{2}(x+iy)^2} = e^{-\frac{1}{2}x^2 + \frac{1}{2}y^2 - ixy}$, and we shall look at the rectangle, having the corners $\pm a, \pm a + iy$, and the border, which we denote by *K*.
To be able to treat the two cases $y > 0$ and $y < 0$ in an equal manner, we shall choose the positive orientation on *K* if $y > 0$, and the negative orientation if $y < 0$.

According to Cauchy's integral theorem: $\int_K f(z)dz = 0$, and we then have:

$$\int_{-a}^{a} e^{-\frac{1}{2}x^2} dx - \int_{-a}^{a} e^{-\frac{1}{2}x^2 + \frac{1}{2}y^2 - ixy} dx + \int_{K_1} f(z)dz + \int_{K_2} f(z)dz = 0$$

Where K_1 and K_2 correspond to the vertical sides, with an orientation, the same as the orientation of *K.* On these sides we have, however:

$$|f(z)| \le e^{-\frac{1}{2}a^2} e^{\frac{1}{2}y^2}$$

For the lengths we have: $\lambda(K_1) = \lambda(K_2) = |y|$. And consequently:

$$\left| \int_{K_1} f(z)dz \right| \le e^{-\frac{1}{2}a^2} e^{\frac{1}{2}y^2} |y| \qquad and \qquad \left| \int_{K_2} f(z)dz \right| \le e^{-\frac{1}{2}a^2} e^{\frac{1}{2}y^2} |y|$$

From which we infer that the two integrals:

$$\int_{K_1} f(z)dz \ \ and \ \ \int_{K_2} f(z)dz$$

go to zero, when *a* goes to infinity. We therefore have:

$$\lim_{a\to\infty} \tfrac{1}{\sqrt{2\pi}} \left(\int_{-a}^{a} e^{-\frac{1}{2}x^2} dx - \int_{-a}^{a} e^{-\frac{1}{2}x^2+\frac{1}{2}y^2-ixy} + \int_{K_1} f(z)dz + \int_{K_2} f(z)dz = 0 \right) \Rightarrow$$

$$1 - \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2+\frac{1}{2}y^2-ixy} dx = 0 \qquad \Leftrightarrow \qquad \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2+\frac{1}{2}y^2-ixy} dx = 1$$

So we get:

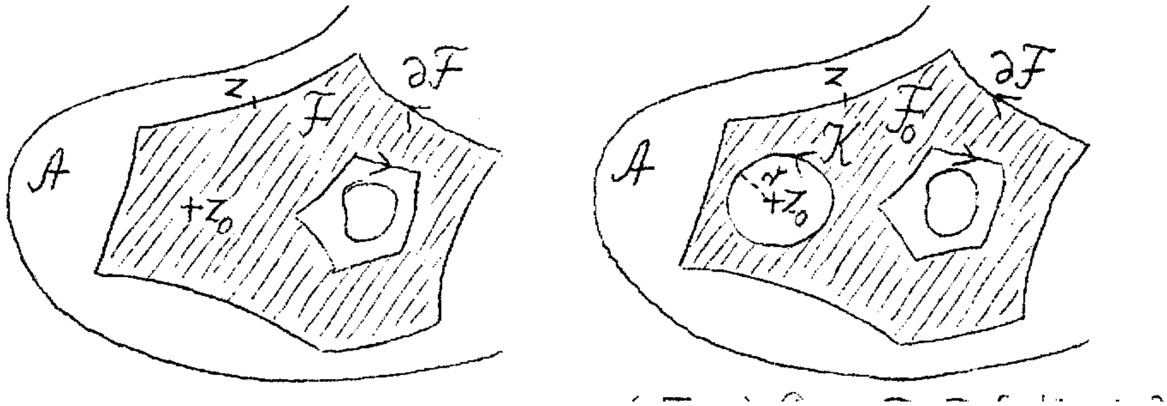
(3.5) $$I(y) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 + \frac{1}{2}y^2 - ixy} dx = e^{-\frac{1}{2}y^2}$$

3.2 Cauchy's second integral theorem.

Let A be an open compact set in C.
For every analytic function: $f : A \to C$, and for every simple figure in A, the value of f in every inner point z_0 in F is determined only from the values of the border of f, which is stated in Cauchy's famous integral formula.

(3.6) $$f(z_0) = \frac{1}{2\pi i} \int_{\partial F} \frac{f(z)}{z - z_0} dz$$

Proof: In the set $A \setminus \{z_0\}$, we consider the function: $g(z) = \dfrac{f(z)}{z - z_0}$, which obviously is analytic.



If r is a positive number less than $\operatorname{dist}(\partial F, z_0)$, then $F_0 = F \setminus \{z \mid |z - z_0| < r\}$ is also a simple figure in $A \setminus \{z_0\}$, and we have $\partial F_0 = \partial F - K$, where K denotes the circle $\{z \mid |z - z_0| = r\}$, having positive orientation. Using Cauchy's first integral theorem, we may then write:

(3.7) $$\int_{\partial F} g(z)dz - \int_K g(z)dz = 0 \quad \Leftrightarrow \quad \int_{\partial F} \frac{f(z)}{z - z_0} dz = \int_K \frac{f(z)}{z - z_0} dz$$

When r is small, then the values of $f(z)$ on K are approximately equal to $f(z_0)$.
We shall therefore rewrite the last integral.

$$\int_K \frac{f(z)}{z - z_0} dz = \int_K \frac{f(z_0)}{z - z_0} dz + \int_K \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$\int_K \frac{f(z_0)}{z - z_0} dz = f(z_0) \int_K \frac{1}{z - z_0} dz = f(z_0) 2\pi i \quad \text{According to (3.4)}$$

For the second integral we make the estimate:

$$\left|\int_K \frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \sup_{z\in K}\left|\frac{f(z)-f(z_0)}{z-z_0}\right|\cdot \lambda(K) =$$

$$\frac{1}{r}\sup_{z\in K}\left|f(z)-f(z_0)\right| 2\pi r = 2\pi \sup_{z\in K}\left|f(z)-f(z_0)\right|$$

Holding things together, we have:

(3.8) $$\left|\frac{1}{2\pi i}\int_K \frac{f(z)}{z-z_0}dz - f(z_0)\right| \le \sup_{z\in K}\left|f(z)-f(z_0)\right|$$

The left side of (3.8) is independent of r, while the right side goes to zero, when r goes to zero. But then the left side must be zero, and the proof is completed.

Cauchy's second theorem:

For every analytic function $f: A \to C$, and for an arbitrary simple figure F in A, the value of f in any inner point z_0 in F is determined from the values of f on the border of F by Cauchy's formula.

(3.8) $$f(z_0) = \frac{1}{2\pi i}\int_{\partial F} \frac{f(z)}{z-z_0}dz .$$

We shall rewrite (3.8) a little, by changing the integration variable from z to ζ, and replacing z_0 to z.

(3.9) $$f(z) = \frac{1}{2\pi i}\int_{\partial F} \frac{f(\zeta)}{\zeta - z}d\zeta$$

If we differentiate this formula with respect to the fixed z, we get:

$$f'(z) = \frac{1}{2\pi i}\int_{\partial F} \frac{f(\zeta)}{(\zeta - z)^2}d\zeta$$

And by differentiating n times:

(3.10) $$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial F} \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

The formula shows the (surprising) result that if an analytic function is differentiable, it is also differentiable an infinite number of times.