Games

Probabilities and Strategies

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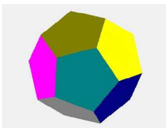

Ole Witt-Hansen 1998 (2016)

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1. Lotto

In Denmark the weekly online Lotto game, is about to guess 7 numbers out of 36 in one row. Besides the 7 drawn lotto numbers, is also drawn an additional number (an add number). One can achieve a premium, in the following manner:

1. Premium: Having all the 7 drawn numbers.
2. Premium: Having 6 correct numbers plus an add number.
3. Premium: Having 6 out of 7 correct numbers
4. Premium: Having 5 out of 7 correct numbers.
5. Premium: Having 4 out of 7correct numbers.

We shall begin by calculating the probabilities of these events, when you fill in one row.

First we remind you of the way one calculate probabilities in a finite symmetric (all elements have the same probability) probability field. An event *H* is a subset of the set of all outcomes *U.* The number of elements in *H* is denoted *n*(*H*)*.* Then the probability that the event *H* occurs is calculated by definition as:

(1.1) $$P(H) = \frac{n(H)}{n(U)} = \frac{favourable}{possible}$$

We shall also use the two fundamental formulas from combinatorial analysis.

P(*n,q*) stands for the number of permutations with *q* elements (*q*-permutations), that can be made, when taken from *n* elements:

$$P(n,q) = \frac{n!}{(n-q)!}$$

Where the factorial symbol $n! = 1\cdot 2\cdot 3 \cdots n$.

C(*n,q*) stands for the number of combinations with *q* elements (*q*- combinations) that can be make from a set of *n* elements:

$$C(n,q) = \frac{n!}{q!(n-q)!}$$

The numbers of different ways one may chose a combination of 7 numbers out of 36 is therefore:

(1.3) $$C(36,7) = \frac{36!}{7!(36-7)!} = \frac{36\cdot 35\cdot 34\cdot 33\cdot 32\cdot 31\cdot 30}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 8.347.680$$

Accordingly the probability *P*(*7*) of having the 7 correct numbers in one row is:

$$P(7) = \frac{1}{C(36,7)} = p_7 = 1{,}1979 \cdot 10^{-7}$$

When we calculate the probability of having 6 correct numbers and one correct add number, we reason in the following way. The 6 correct numbers can be chosen among the 7 numbers in *C*(*7,6*)=*7* different ways, whereas the add number may be chosen in one way only. Therefore

$$P(6 + 1\ add) = \frac{C(7,6) \cdot 1}{C(36,7)} = 7p_7 = 8.330 \cdot 10^{-7}$$

In the same manner we find, (by the multiplication principle) the probability of having 6 correct numbers. The 6 correct numbers can as before be chosen in *C*(7,6) = 7 different ways, and the "wrong" number may be chosen among 36 -7 – 1 = 28 numbers. Therefore:

$P(6) = K(7,6) \cdot 28/K(36,7) = 196 \cdot p_7 = 2.3479 \cdot 10^{-5}$

The probability of obtaining 5 correct numbers is the number of ways 5 numbers can be selected from 7 numbers, which is *C*(7,5) = 21 times the number of ways we can select 2 numbers from 36-7 =29 numbers, which is *C*(29,2)=406.

$P(5) = C(7,5) \cdot C(29,2)/C(36,7) = 8526 \cdot p_7 = 1.02 \cdot 10^{-3} = 1.02$ ‰

In the same manner we calculate the probability of having 4 correct numbers. The number of possibilities are: *C*(7,4) ·*C*(29,3)

(1.8) $P(4) = C(7,4) \cdot C(29,3)/C(36,7) = 127.890 \cdot p_7 = 0.0153 = 1{,}53\%$

From the calculated probabilities above, we see that the probability of having more than 4 correct numbers is immensely small. On the other hand there is still a fair chance of getting 4 correct numbers. This is of course a deliberate choice from the administrators of the game, since if a lotto player never wins, most will stop playing after a certain time.

A lotto coupon has 10 rows, and we shall first calculate the chance *P*(*H*) of winning on at least one of the rows.
We do it by calculating the probability of the complementary event $P(\overline{H}) = 1 - P(H)$.
Since the 10 rows are independent the probability that none has 4 correct numbers is

P(not 4 correct numbers on any of the 10 rows) = $(1-p_4)^{10} = 0{,}9847^{10} = 0{,}8557$

The chance of having at least one row with 4 correct numbers is therefore:

P(at least on coupon with 4 correct numbers) = 1 – 0.8557 = 0.1443.

If we assume that someone buys 10 rows, 5 weeks in a row, the probability of not winning is $(0.8557)^{10} = 0.4487$. So having bought a coupon 5 weeks the probability of winning is:

P(at least one coupon with 4 correct numbers in 5 weeks) = 1- $(0.8557)^{10}$ = 0.5513.

But it is worth mentioning that the premium of having 4 correct numbers is of the same magnitude or less, than the cost of one coupon.

Exercises

1. Calculate how many weeks one must play 10 rows, that makes the probability to win the 4 premium (5 correct numbers) greater than 50%.

2. A filled in row covers obviously over $K(7,4) \cdot K(29,2)$ different rows with 4 correct numbers. Find out how many different rows one has to make to be sure of getting 4 correct numbers on at least on of the rows.

2. Poker

We shall assume that the game is played with an ordinary deck of cards. We shall only occupy ourselves with the beginning of the play, where each player gets 5 cards from the deck.

What happens thereafter cannot be treated within the framework of probability theory.
We shall only be concerned with the probabilities of obtaining a hand, which corresponds to the different ranks of a poker hand.
The suits are ranking in the same manner as in bridge: Spade, hearts, diamonds and clubs.

The ranking combinations in Poker are as follows:

Royal Flush:	5 cards with highest rank in the same suit. E.g. ace, king, queen, knight, and 10 of diamonds
Straight Flush:	5 cards in sequence in the same suit..
Four equal:	4 cards in equal rang.
Full House:	3 equal rang + 2 equal rang (a pair)
Flush:	5 cards same suit.
Straight:	5 cards in sequence.
Three equal:	3 cards same rank.
Two pairs:	1 pair + 1pair (different rank).
One pair:	1 pair.
Highest card:	1 card, highest rank

We intend to calculate the probability of getting these combinations (except for the last).

The different ways to draw 5 cards from a deck of 52 cards are:

$$C(52,5) = 2.598.960$$

Royal Flush: There are 4 different *Royal Flush*, one in each suit. Hence the probability

$$P(Royal\ Flush) = \frac{4}{K(52,5)} = 6{,}156 \cdot 10^{-6}$$

Straight Flush : In each suit there are 9 different *Straight Flush*, but one of them is *Royal.*

$$P(Straight\ Flush) = \frac{4(9-1)}{C(52,5)} = \frac{32}{C(32,5)} = 4{,}925 \cdot 10^{-5}$$

Four equals*:* There are 13 possibilities of having 4 cards of equal rank. The fifth card may be chosen in 52- 4 = 48 ways.

$$P(4\ of\ the\ same\ rank) = \frac{13 \cdot 48}{C(52,5)} = \frac{624}{C(52,5)} = 2{,}401 \cdot 10^{-4}$$

Full House: There are 13 different possibilities for 3 cards of equal rank. They can be selected in *C*(4,3) ways. The pair must have a different rank, so the probability is:

$$P(Full\ House) = \frac{13 \cdot C(4,3) \cdot 12 \cdot C(4,2)}{C(52,5)} = \frac{3744}{C(52,5)} = 1{,}400 \cdot 10^{-3}$$

Flush: 5 cards in the same suit, can be selected in *C*(13,5) different ways, but we have to subtract the *Straight Flushes.*

$$P(Flush) = \frac{4 \cdot C(13,5) - 32 - 4}{C(52,5)} = \frac{5112}{C(52,5)} = 1{,}967 \cdot 10^{-3}$$

Straight: there are 13 – 4 sequences. Each card in the sequence may be chosen from 4 suits, and we must subtract the *Straight Flushes.*

$$P(Straight) = \frac{9 \cdot 4^5 - 36}{C(52,5)} = \frac{9180}{C(52,5)} = 3{,}53 \cdot 10^{-3}$$

3 ***cards of equal rank****:*There are 13 ranks, and we shall select 3 of the same rank. The remaining to cards can be selected from 52 – 4 = 48 (not 49, as it may result in 4 equal). Further we must subtract the12·*K*(4,2) pairs, since it would result in *full house.*

$$P(3\ equal) = \frac{13 \cdot C(4,3) \cdot (C(48,2) - 12 \cdot C(4,2))}{C(52,5)} = \frac{54912}{C(52,5)} = 2{,}113 \cdot 10^{-2}$$

Two pairs First we calculate the number of possibilities for two pairs. The factor ½ is caused by the fact that 1 pair (e.g. queen of hearts, queen of diamonds), will be counted both in the 13·C(4,2) and the 12·*C*(4,2) possibilities of selecting two pairs. The last card can be chosen among the 52 – 8 cards that do no belong to the two pairs.

$$P(2\ pairs) = \frac{\frac{1}{2} 13 \cdot C(4,2) \cdot 12 \cdot C(4,2)(52-8)}{C(52,5)} = \frac{123.552}{C(52,5)} = 4{,}75 \cdot 10^{-2}$$

1 *pair*: The first 3 factors in the numerator are the number of different ways to form a pair, but we have to subtract the possibilities of 2 *pairs* and *full house*..

$$P(1\ pair) = \frac{13 \cdot K(4,2) \cdot K(48,3) - 123.552 - 3755}{K(52,5)} = \frac{1221781}{K(52,5)} = 0{,}4701$$

As all poker players know, there is a fair chance of getting a pair in the first round of dealing cards, whereas all other combinations are scarce.

This ends our analysis of poker, one should notice that the sequence of the calculated possibilities are increasing in accordance with the ranking of combinations in poker game. The only surprising thing might be that a *flush* has a higher value than straight.

3. Ruin probabilities

The calculations of ruin or bankrupt possibilities are vital important in the insurance business, if it has to be driven on a sound commercial basis.
The interaction between bankrupt possibility and the size of the insurances premium is a quite complex mathematical discipline, which is launched as a special study in several universities as a degree as actuary.
Our aim is only to give some simple examples of calculating possibilities of bankruptcy of a Casino faced with an infinitely rich customer playing at the roulette.

The roulette has 37 fields, numbered 0 – 36. If you play on one field and win, the bank will return your bet 36 times, but since you have bet one unit the gain is 35 units.

I X is the stochastic variable which denote a players gain, it takes the value +35 with probability 1/37 and the value -1 with probability 36/37.

$$P(X = 35) = 1/37 \quad \text{and} \quad P(X = -1) = 36/37 .$$

If u denotes an arbitrary outcome in the universe U, then the mean value of a stochastic variable X, is denoted $E(X)$, and is calculated as:

$$E(X) = \sum_{u \in U} X(u)P(u)$$

The mean of the gain becomes:

(3.1) $$E(X) = \sum_{u \in U} X(u)P(u) = 35\frac{1}{37} + (-1)\frac{36}{37} = -\frac{1}{37}$$

If the player instead chooses to play on m fields, then the gain X has the values 36 – m with probability m/37 and the value –m with the probability (37-m)/37 = 1- m/37.
Then mean of the gain becomes:

$$E(X) = \sum_{u \in U} X(u)P(u) = (36 - m)\frac{m}{37} + (-m)(1 - \frac{m}{37}) = -\frac{m}{37}$$

The mean value of the gain per bet unit is consequently independent of how many fields you are playing on.
This establishes once and for all that there exits no system, which will enable anyone to have a gain on the roulette in the long run.
Specifically if a player plays on 18 field the gain is 36 – 18 =18 units with probability 18/37 and -18 with the probability 19/37.

Even if no player can win in the long run, coincidences (which players usually refer to as luck), may give a player possibility to win substantial amounts.

We shall the make some considerations to the possibility for a player to “break the bank”, that is, to ruin the Casino. Such a calculated probability is referred to as the “ruin possibility”

We shall assume that the bank is in possession of n units, and we wish to estimate the probability for an (infinite rich) player, who makes the same bet in every game wins n or more units, when there is no limitations on the number of games.
The ruin probability of the Casino we denote r_n.

If X_1, X_2, X_3,.. are the gains of the Casino in the individual games, then:

$$G_k = X_1 + X_2 + X_3 + .. + X_k$$

Is the accumulated gain after k games. The ruin-probability may there be defined as:

$$r_n = P(G_k \leq 0 \quad \text{for some } k)$$

The following recursion formula must hold good:

$$r_{n+1} = r_n r_1,$$

Expressing that the probability of being ruined with n+1 units is the probability of being ruined with n units times the probability of being ruined with 1 units. This because any game is independent of the outcome of the preceding games. As a consequence:

$$r_2 = r_{1+1} = r_1 r_1 = r_1^2.\ r_3 = r_{2+1} = r_2 r_1 = r_1^3$$

and consequently:

$$r_n = r_1^n$$

To establish a formula for the ruin-probability, it is most convenient first to consider the case, where the player put the same bet on 18 fields in each game. The gain of the Casino is +18 or -18 per game. For simplicity we consider the 18 bets as one unit.
We then establish the following recursion relation, which takes the first game as its starting point. The probability that the bank is ruined having n units, is equal to:

The probability that the bank wins the first game, times
the probability that it gets ruined with n+1 units, plus
The probability that the bank looses the first game, times
the probability that the bank gets ruined with n-1 units

The probability that the bank wins or looses are 19/37 and 18/37. We can therefore establish the equation:

$$r_n = \frac{19}{37} r_{n+1} + \frac{18}{37} r_{n-1} \quad \Leftrightarrow \quad r_1^n = \frac{19}{37} r_1^{n+1} + \frac{18}{37} r_1^{n-1}$$

To obtain the last equation, we have applied the relation: $r_n = r_1^n$.

It is relatively easy to determine r_1 from this equation. By division with r_1^{n-1} we get:

$$r_1 = \frac{19}{37} r_1^2 + \frac{18}{37} \Leftrightarrow 19r_1^2 - 37r_1 + 18 = 0$$

This quadratic equation is solved in the usual manner: $d = 37^2 - 4 \cdot 19 \cdot 18 = 1$

$$r_1 = \frac{37 \pm 1}{2 \cdot 19} \Leftrightarrow r_1 = \frac{36}{38} = \frac{18}{19} \vee r_1 = 1$$

We have only interest in the solution: $r_1 = 18/19$.
According to the relation $r_n = r_1^n$, we have:

$$r_n = r_1^n = (\frac{18}{19})^n$$

We are then able to answer the question of how many units must the bank have, so there is less than 0.1% chance for ruin of the bank.

$$(\frac{18}{19})^n \leq 0.001 \Leftrightarrow n \geq \frac{\ln(0.001)}{\ln(\frac{18}{19})} = 127.76$$

Remember that the players bet is 18 units, the bank can settle for a modest inventory of 2,300 units.

If a player plays in the more usual way and puts his bet on only one number, then it is (somewhat surprisingly) considerably more risky for the bank.

Again we establish a recursion relation, which takes the first game as its starting point.

The probability that the bank is ruined having n units, is equal to:

The probability that the bank wins the first game (where the bank gains 1 unit), times
the probability that it gets ruined with n+1 units, plus
The probability that the bank looses the first game (and looses 35 units), times
the probability that the bank gets ruined with n-35 units

The probability that the bank wins or looses one game is 36/37 and 1/37. We may then establish the relation.

$$r_n = \frac{36}{37} r_{n+1} + \frac{1}{37} r_{n-35} \quad \Leftrightarrow \quad r_1^n = \frac{36}{37} r_1^{n+1} + \frac{1}{37} r_1^{n-35}$$

By division by $(r_1)^{n-35}$ and rearranging the terms and putting $r_1 = x$, we obtain the equation.

$$36x^{36} - 37x^{35} + 1 = 0$$

This equation can be solved only by numerical methods, and one finds the solution: $x = 0{,}9984$.

Again, we ask the question: How many units must the bank have, so that the possibility of ruin is less than 0,001. We have to solve the inequality

$$(0{,}9974)^n \leq 0{,}001 \Leftrightarrow n \geq \frac{\ln(0.001)}{\ln(0.9984)} = 4314$$

A considerably bigger inventory than it was the case, when the player made bets on 18 fields.

If we turn the tables and calculate the ruin probability for a player who has n units and places the same bet on one number, we arrive after the same reasoning as before to the recursion equation.

$$r_n = \frac{36}{37} r_{n-1} + \frac{1}{37} r_{n+35} \quad \Leftrightarrow \quad r_1^n = \frac{36}{37} r_1^{n-1} + \frac{1}{37} r_1^{n+35}$$

Which leads to the equation:

$$x^{36} - 37x + 36 = 0$$

It is easily seen that the equation has the solution $x = 1$, and that is the only solution in the interval [0;1]. If we differentiate $f(x) = x^{36} - 37x + 36$, one finds: $f'(x) = 36x^{35} - 37$. $f'(x) = 0$ has the solution: $x = \sqrt[35]{\frac{37}{36}} > 1$. As $f'(x) < 0$ for $x < 1$, the equation can have no roots less than 1.

The probability of ruin for a player who enters a Casino (having an infinite inventory) is 1.

So again we may establish that if one continues to have the Casino as your only source of income, you will certainly be ruined, sooner or later.

2. Game Strategies

1. Presentation of some simple games

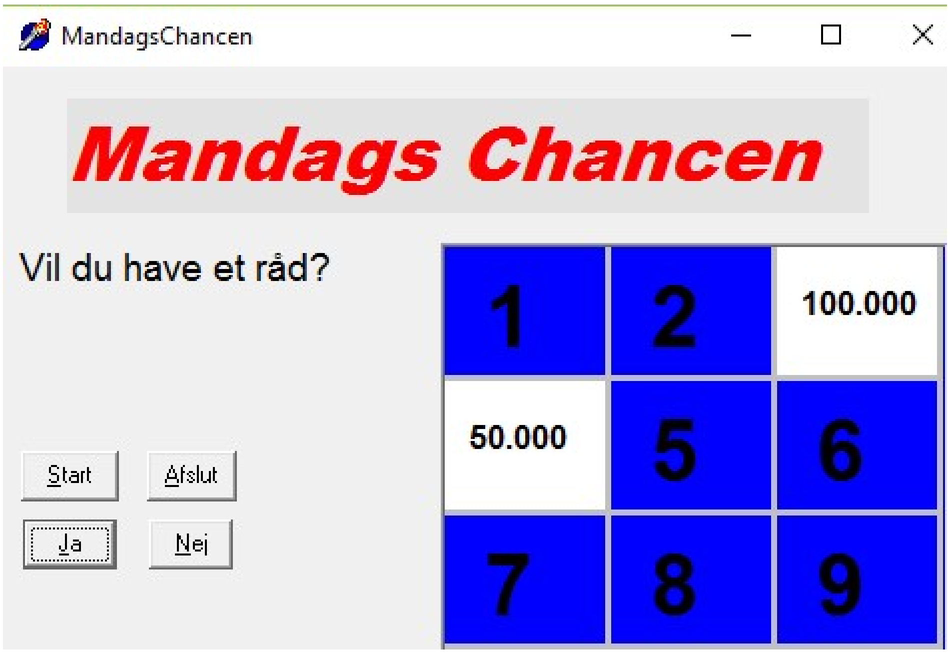
After having calculated winner probabilities and ruin probabilities, we shall now look into another field of probability theory, namely the problem of determining the optimal (the best) strategy, when playing a game. In this respect the concept of "game" is broaden.
Commonly for all such games are that they are not deterministic and it should be possible to calculate the probabilities of certain consequences of a given decision in each step of the game.

We start out describing some games, which are apt to apply the theory of *optimal strategies*.

The phrase "optimal strategy" implies that the game is conducted in a finite number of steps, and for every step, one may increase ones gain or loose ones gain entirely or partly.
Furthermore it should be possible to stop the game at any step and collect the gain.

1.1 The Monday-chance

A rather silly game in the Danish TV, which was launched about 15 years ago It is shown below as a computer simulation after two steps (The words are in Danish)



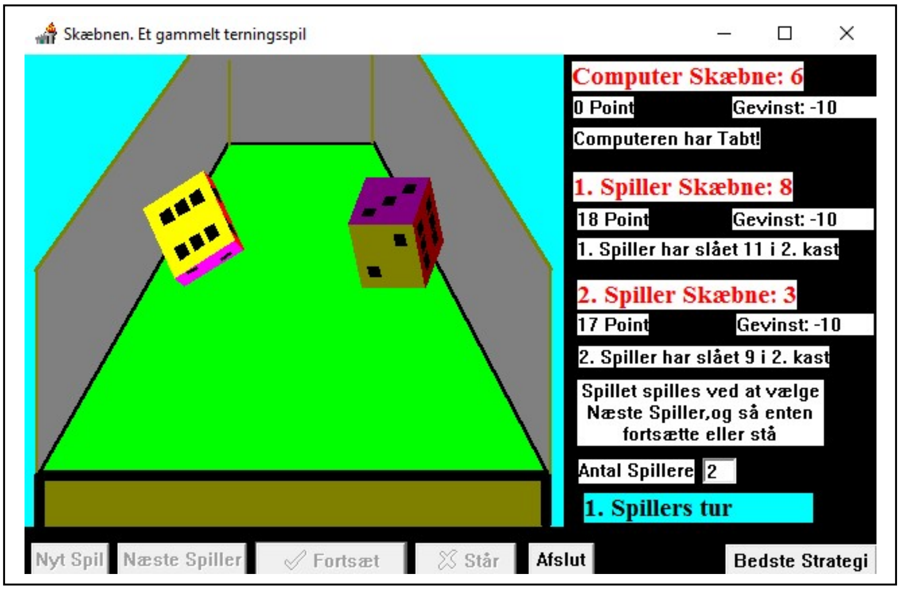
In this game, the player is presented with 9 fields, having the values (10,10,25,25,50,50,100,100,250) x 1000 crowns of which the player can chose to uncover max three of them. The player gets the amount on the last uncovered field.

It is rather obvious that you should go on if you uncover the field with 10 or 25 in the first or second attempt, and you should stop if you uncover 100 or 250, but what if you uncover 50 in the first or second try? The mathematical answer to this is a bit surprising.

1.2 "The destiny"

An old game, which allegedly comes from the orient. It is played with two dices. The game is initiated by each player making a bet and throws the two dices. The sum of the eyes on the two dices is called the players “destiny”. Then you take turns to throw as many times as you wish. The sum of eyes on the dices is added to your score.
But if you throw your destiny, you loose everything and is out of the game. You may stop any time you want. When the game ends the bets are distributed among the remaining players in proportion to their score.

Below is shown an animated computer simulation of the game. (The words are in Danish)



The issue is then, when it is statistically optional to stop. It is clear, that you should continue longer if your destiny is 2 or 12 (probability 1/36) than if it is 7 (probability 1/6)

1.3 The burglar’s pension issue

This is actually one variant of many different optimization problems.
Arthur is a burglar. Every raid gives in average K crowns. The probability that Arthur gets caught is p. If he is caught he will loose all of the income from his previous raids.
The issue is this: How many raids is it optional to conduct, before he end his career, and starts to live from his “savings”

1.4 The matches game

This game actually falls somewhat outside the scope of the other games for two reasons.
One can not stop the game at any step, and you can never loose anything from your hitherto gains. Therefore it can not be treated on equal footing with the previous games.

On the other hand it is certainly a game, which is applicable to demonstrate theoretical aspects of optimal strategies. It is probably a game which resembles stock market gambling most.
How much should you invest in the present situation, with your premonitions of the market tomorrow or in the near future.
But as it is well known, there exits no reliable mathematical theory that can predict the variations of the share prices.

The game is played with 10 matches, where one of the four sides of the matches is painted red. If a match is thrown into the air, there is a probability p = 0.25 that it lands with the red side up.

Each player has from the beginning the numbers 1..10. Then the players take turns to throw the 10 matches. The matches are thrown10 times for each player.
For each throw, the player notice the number of matches with the red side up, and he chooses to multiply this number by one of the numbers 1...10, but the number chosen must only be used once.

By the end of the game the player with the highest score has won.

As we shall see, the best strategy is actually a rather complex mathematically probability problem We may inform you (and it is easily verified) that the maximum score is 550.
The mean score, (where the number you multiply by is chosen at random) is 137.5 and the optimal mean score is 167.

1.5 Casino

Playing at the roulette also falls somewhat outside the scope of ”optimal strategies” for the simple reason that the odds are in disfavour of the player.
This is, however, not the case of Blackjack, where one can develop a strategy, which gives a weak advantage for the player.
But as mentioned before, when the odds are against you, one will always loose in the long run.
As previously established, the only possibility – in the long run – to avoid ruin is that there must be no caps for the bets, and you should have substantially more funds than the Casino, and neither is ever the case.
Any sensible mathematical theory will therefore have the result that you should stop before the first game, that is, don't enter a Casino.

This fact, however, should not prevent you of making considerations of how to play, instead of just surrender and transfer your savings to the Casino without an (intellectual) fight.

2. Optimal Strategies

If one should choose a strategy for the first 3 games mentioned above, one would probably chose one of the following 3 suggestions of a strategy.

1. You decide in advance how big your gain (or loss) must amount to be, before you stop.
   This is actually a very common “strategy”, but it is certainly not an optimal one.

2. One continues as long as one in average has the possibility of increasing the gain in the next game. You should remember that the mean of the gain also include losses, so the mean should be greater than zero.
   This is, at a glance, a very sensible strategy, and it is called the *short-term or the myope strategy*
   It is in several cases also the optimal strategy. (But not in *Monday-chance*)

3. One continues, as long as one has a average probability to increase the gain in one of the following games. This is called the long-term strategy.

Both the short-termed strategy and the long-termed strategy seem quite reasonable, and they also appear to be the optimal strategy for many (but not all) games. Should there be a conflict between short-termed and the long-termed strategy, one should choose the long-termed – naturally.
There exist, however, ”games” where neither short-termed strategy or the long-termed strategy are optimal.
A classical example is stock market speculations, where the future share prices may not be predicted by logic based on mathematics.
In such “games” the two strategies will, roughly speaking, have the result that either you should sell it all or wait until you are ruined

It can be shown that neither of the two strategies described above are optimal in all cases, but the mathematical proof is rather complex.

We are now ready to give a description of the *optimal strategy*, which has been developed by *Snell*, but I cannot present the mathematical proof, since it is extremely technical and abstract, but emphasize that it can be proven! The drawback with the Snell-strategy is, that it even for simple games can lead to rather large and incomprehensible calculations that can (as we shall see) only be done on a computer.

2.1 The optimal Snell-strategy

The Snell-strategy resembles in many ways the short-term strategy, but where the short-term strategy persistently looks on step ahead, the Snell-strategy takes it starting point at the end of the game and works its way backwards. First we invent some notations:

We assume that the game has a finite number of n steps. S_k is a stochastic variable, which denotes the total gain after k games. S_k can take one or several values with corresponding probabilities.

The new invention in the Snell-strategy is the definition of another stochastic variable G_k:

$$G_k = \max\{S_k, E(G_{k+1}|\, S_1..S_k)\}$$

Which is the largest value of S_k (the gain after having played the k’th game), and $E(G_{k+1}|\, S_1..S_k)$, which is the expected (the mean value) of maximum expected gain in the k+1’th game, under the condition $S_1..S_k$. This can be illustrated in a scheme.

$G_n = S_n$ (The gain at the end of the game)

$G_{n-1} = max\{S_{n-1}, E(G_n|\, S_1..S_{n-1})\}$

.

.

$G_k = max\{S_k, E(G_{k+1}|\, S_1..S_k)\}$

.

$G_1 = max\{S_1, E(G_2|\, S_1)\}$

The Snell-strategy tells you, that you should stop the game the first time, where S_k (the gain after the k’th game) exceeds the mean value of the expected gain in the k+1’th game.

This is called the *stop condition*. The stop condition is then:

$$S_k \geq E(G_{k+1}|\ S_1..S_k)$$

Rephrased: You should stop the game, when the accumulated gain is bigger or equal to the conditional expectation of the maximum gain in the next game.

This may sound as the short-termed strategy, but the crucial difference is that G_{k+1} is replaced by S_{k+1} (although they are the same for most games)

We shall now deal with the examples described above, and first:

2.2 The burglar's pension issue

Let us assume that Arthur's average gain from a theft raid is $K = 6000$ crowns, at that the possibility of getting picked up during the raid is 0.05. And let us assume that his previous gains after k raids are S_k, which will be confiscated if he is apprehended during a raid.

In this case one may, as shown below, realize that the short-term strategy is the optimal strategy. This is reflected in the fact that all steps in the game are equal. The mean gain from a raid is the same independently of the previous raids, apart from the fact that the collected gain grows from each raid.
X_k denotes the stochastic variable, which is equal to the gain from the k'th raid. Accordingly:

$$S_k = X_1 + X_2 + X_3 + \dots X_k$$

X_{k+1} has the value K, with probability $1\text{-}p_s = 1\text{-}P(\text{"caught"})$, meaning that the next raid is successful, and X_{k+1} has the value $-S_k$ (he looses all of his "savings"), with the probability $p_s = P(\text{"caught"})$.
Then we write down the Snell-strategy, (which in this case is the same as the short-term strategy) for the k'th step in his career.

$$G_k = max\{\ S_k\ ,\ E(S_{k+1}|\ 1..k)\ \}$$

S_k is his gain after k raids, and $E(S_{k+1}|\ 1..k)$ is the expected gain after k+1 raids.

$E(S_{k+1}|\ 1..k)$ can be calculated as the value of his previous gains plus the expected gain at the next raid, which is equal to:

$$S_k + E(X_{k+1})$$

Since $E(X_{k+1})$ is independent of the previous raids, the stop condition becomes:

$$S_k > S_k + E(X_{k+1}) \quad \Leftrightarrow \quad E(X_{k+1}) < 0$$

The mean $E(X_{k+1})$ can therefore be calculated according to the usual definition. X_{k+1} can take the values K (the average gain in one raid) with probability $1\text{-}p_s$ and the value $-\ S_k$ (looses all of his „savings") with probability p_s.

$$E(X_{k+1}) = K\cdot(1 - p_s) - S_k\cdot p_s$$

He should stop when:

$$E(X_{k+1}) < 0 \quad \Leftrightarrow \quad K\cdot(1-p_s) < S_k\cdot p_s$$

Using an example with K = 6000, and p_s = 0.05 we find:
6000·0.95 < S_k·0.05 $\Rightarrow$ S_k > 114,000, resulting in 114.00/6.000 = 0.95/0.05 = 19 raids.

But Arthur will probably neither listen to (ethical neutral) mathematical arguments nor to the advice from social workers.

One may think that this example can be equally applied in many other (equally unethically) issues, but that is not necessarily the case.

For example, how many times is it “advantageous” to travel without having bought a ticket in the subway.

The case is that you will get a ticket, if you get caught without having bought a ticket, but you can never be forced to pay for the previous times, you have travelled without a ticket (naturally).

In other words the estimate whether it is ”advantageous” to travel without a ticket does only depend only on the mean gain in the next attempt. If it is positive it is advantageous to continue, if it is negative it is more advantageous to buy a ticket. (Incidentally, it is not ethical to cheat).

We take the example that the price of a ticket is 50. The penalty for travelling without a ticket is 500. The probability of being caught we put to P_s = 0.1.
If X is the gain, travelling without a travel ticket, the mean of X is:

$$E(X) = 50\cdot P(50)-500\cdot P(-500) = 50\cdot(1-P_s)-500\cdot P_s = 50\cdot 0.9-500\cdot 0.1 = -5$$

With these suppositions (this probability of being caught), it is never profitable to travel without a valid ticket, but since a paper on probabilities and strategies in games, may not encourage to unethical behaviour, we shall abstain from doing the calculation with P(caught) = 0.05.

2.3 The "destiny"

The game is already described above. We are concerned with throwing of two dices, and to register the sum of eyes on the two dices in the stochastic variable X .
The probability distribution of X is well known, e.g. $P(X=5) = 4/36$.
The probability that $X = j$ can be written as:

$$P(X=j) = \frac{6-|7-j|}{36}$$

Let us assume that one of players *destiny* is q. If the player throws q eyes he looses everything he has gained in the previous throws, otherwise the sum of the eyes on the two dices is added to his account.
As lon as a player has not thrown his destiny the collected gain is S_n, where

$$S_n = X_1 + X_2 + X_3 + \dots + X_n.$$

And the mean value of S_n is:

$$E(S_n) = E(X_1)+ E(X_2)+E(X_3)+.... +E(X_n) = n \cdot E(X \mid X \neq q).$$

$E(X \mid X \neq q)$ is the conditional mean, where the condition is that the player does not throw his destiny.

$$E(X \mid X \neq q) = 2 \cdot P(2) + 3 \cdot P(3)+...(q-1) \cdot P(q-1) + (q+1) \cdot P(q+1)+...+12 \cdot P(12)$$

The short term strategy (and the Snell strategy) claims that you should stop, when the expected gain in the next throw is negative.
The argument can be literally copied from The Burglars Pension, since $E(S_{k+1}|S_1.. S_k)$ can be calculated as $S_k + E(X_{k+1}| S_1.. S_k)$, the previous gain plus the expected gain at the next throw.
The stop condition is therefore, as before:

$$S_k > S_k + E(X_{k+1}| S_1.. S_k) \quad \Leftrightarrow \quad E(X_{k+1}| S_1.. S_k) < 0$$

$$E(X \mid X \neq q) - S_n \cdot P(q) < 0 \quad \Leftrightarrow \quad S_n > E(X \mid X \neq q) / P(q)$$

Below $E(X \mid X \neq q)$ is calculated by a computer for $q = 2..12$

q: 2	3	4	5	6	7	8	9	10	11	12
6.94	6.83	6.67	6.44	6.17	5.83	5.89	6.00	6.17	6.39	6.67

Stop, when $S_n > E(X \mid X \neq q) / P(q)$, when the destiny is q.

q: 2	3	4	5	6	7	8	9	10	11	12
250.00	123.00	80.00	58.00	44.40	35.00	42.40	54.00	74.00	115.00	240.00

It shows that there is large variations in the stop-times, dependent on the value of the *destiny*.

2.4 The "Monday-chance"

This (utterly silly) game is, however, a bit more interesting from a theoretical point of view.
Since one may not expect that the short-term strategy will lead to the optimal strategy.
That it actually does is because of the values of the premiums chosen. This is to be understood, as if we altered in the premiums, the optimal strategy, would no longer comply with the short-term strategy. We shall however analyze the game with the Snell-strategy.

As mentioned above the game has 9 covered fields containing the numbers

(10,10,25,25,50,50,100,100,250) x 1000 crowns.

The player must at most uncover max 3 fields, and he gets the amount on the last uncovered field.

If X is the stochastic variable that denote the value on the fields, the most people would continue if $X<50$ and stop if hvis $X \geq 100$ in the first and second attempt. The only really critical decision is, if you uncover ($X=50$) in the first or second attempt.

First we notice that the sum of the fields is S=620 and that $E(X)$ = 620/9 = 68.9.

For this game we write the entire Snell-strategy.

$G_3 = X_3$ — The stochastic variable G_3 = The gain = X_3 = The value of the last uncovered field.

$G_2 = max\{ X_2 , E(G_3| X_1,X_2)\}$ — The largest of the values X_2 (value of the second uncovered field) and the conditional mean of $G_3 = X_3$.

$G_1 = max \{ X_1 , E(G_2| X_1) \}$ — Same as above

The stop criterion according to the Snell-strategy is:

$$X_k > E(G_{k+1}| X_{1..}X_k)$$

It is relatively easy to establish a formula for $E(G_3| X_1,X_2)$. When X_1 og X_2 have been uncovered, the sum of the remaining fields is $S - X_1 - X_2$. Since all fields have the same probability of being uncovered, the mean of G_3 simply the mean of the remaining numbers.

$$E(G_3| X_1,X_2) = (S - X_1 - X_2)/(9\text{-}2)$$

It is a bit more technical to establish a formular for $E(G_2| X_1)$, and also more technical to make the calculations. We shall limit ourselves to the calculation for one value of X_1, and moreover refer to the result of a computer calculation, shown below.

Assume that X_1 = 25. We let X_2 run through all its possible values (all values should be counted the number of times where are fields with this value). For every value of X_2, (e.g. 50), we must choose max from X_2 and $E(G_3| X_1,X_2) = (S - X_1 - X_2) /(9\text{-}2)$ (= (620-50-25)/7 = 77.8 in this case), and add it to a sum S_2 . $E(G_2| X_1)$ then equals $S_2/(9\text{-}1)$ (since there are 9-1 fields X_2)

```
   X2[1]=   10.00   E(G3)=   85.71   G2=   85.71
   X2[2]=   25.00   E(G3)=   83.57   G2=   83.57
   X2[3]=   50.00   E(G3)=   80.00   G2=   80.00
   X2[4]=  100.00   E(G3)=   72.86   G2=  100.00
   X2[5]=  250.00   E(G3)=   51.43   G2=  250.00

X1[1]=   10.00   E(G[2])=  107.86

   X2[1]=   10.00   E(G3)=   83.57   G2=   83.57
   X2[2]=   25.00   E(G3)=   81.43   G2=   81.43
   X2[3]=   50.00   E(G3)=   77.86   G2=   77.86
   X2[4]=  100.00   E(G3)=   70.71   G2=  100.00
   X2[5]=  250.00   E(G3)=   49.29   G2=  250.00
```

```
X1[2]=   25.00   E(G[2])=  106.79

   X2[1]=   10.00   E(G3)=   80.00   G2=   80.00
   X2[2]=   25.00   E(G3)=   77.86   G2=   77.86
   X2[3]=   50.00   E(G3)=   74.29   G2=   74.29
   X2[4]=  100.00   E(G3)=   67.14   G2=  100.00
   X2[5]=  250.00   E(G3)=   45.71   G2=  250.00

X1[3]=   50.00   E(G[2])=  105.00

   X2[1]=   10.00   E(G3)=   72.86   G2=   72.86
   X2[2]=   25.00   E(G3)=   70.71   G2=   70.71
   X2[3]=   50.00   E(G3)=   67.14   G2=   67.14
   X2[4]=  100.00   E(G3)=   60.00   G2=  100.00
   X2[5]=  250.00   E(G3)=   38.57   G2=  250.00

X1[4]=  100.00   E(G[2])=   96.43

   X2[1]=   10.00   E(G3)=   51.43   G2=   51.43
   X2[2]=   25.00   E(G3)=   49.29   G2=   49.29
   X2[3]=   50.00   E(G3)=   45.71   G2=   50.00
   X2[4]=  100.00   E(G3)=   38.57   G2=   100.00

X1[5]=  250.00   E(G[2])=   62.68
```

Looking through the results, they can be collected in one simple rule. Proceed on 50 or below, and stop on 100 or above.
Perhaps it appears a bit surprising, that you should proceed on 50.000 also prefers to stop at 50.000, but it is not the optimal strategy.

2.5 Casino and the roulette

As mentioned in the introduction to this section, there exists no strategy, which can enable anyone to win on the roulette in the long run. If you consider letting the Casino be the only source of income, you will with mathematical certainty be ruined, and probably also deeply in dept.

This, however, does not mean that you cannot win at the roulette, but if have fantasies of breaking the bank, at least two conditions must be satisfied

1. Your inventory must be larger than that of the Casino.
2. There should be no caps on the bets.

As it is well known these conditions are never satisfied.

Although you can never win in the long run, there are no reasons why you should give away your money earned by hard labour to the Casino, without an (intellectual) fight.

There exists several strategies but many of them are based on superstition or lack of knowledge of the most elementary concepts of probability theory.

There are others based on mathematical logic, and the most notorious is probably The Martingale system.

Here one plays only on red and black. This is (oppositely to playing at the roulette) a completely equal game, since you receive the double your bet, if you have placed your bet right, (with a slight disadvantage, which we shall ignore), since if zero comes out, your bet must remain until the next game, and if you win, only the value of the bet is returned.

The Matingale system is indeed very simple. If you win you should collect your gain, then stop or making the same bet again, but if you loose, you should double your bet.
For simplicity we put the bet to 1. If you have played (and lost) n times, and therefore doubled up $n - 1$ times, the accumulated bet amounts to:

$$S_n = 1 + 2 + 2^2 + \ldots 2^{n-1} = 2^n - 1$$

By the formula for the sum of the geometric series: $S_n = a_0 + a_0 q + a_0 q^2 + \ldots\ a_0 q^{n-1}$ (n-terms)

$$S_n = a_0 \frac{q^n - 1}{q - 1}$$

If you win the n'th game, you win $2 \cdot 2^{n-1} = 2^n$, so your gain, as you win, is always $2^n - (2^n - 1) = 1$

It thus seems an undeniable fact, that if you continue to double your bet, you shall always win on bet unit.
When I have entertained my students with this idea of making easy money, I always urge them to hear the rest of the story, because before you engage yourself in this attractive conception, you should make yourself clear of how many times you may loose in a row before you are ruined.

The probability that you loose 5 times in a row, is $(\frac{1}{2})^5 = 1/32$. Then you have lost $2^5 - 1$ times the value of your first bet, say 10$. amounting to 310$.

To obtain at better insurance, you may choose to be able to double up 9 times. The probability of losing 10 games is certainly small 1/1024, but if it should happen you have lost 10$x1013 =10,230$ (or about half the yearly mortgage on your house). And you will probably (at least in the last bet) hit the roof on the bets?

One could however put the following sensible question forward. If you have n chips, what is the probability of winning q chips, if you play by the Martingale system.
This question has however a more favourable answer (as long as you don't get greedy).

A direct calculation is possible, but rather comprehensive, and we shall instead use the same methods, as we applied to find the optimal strategy.

We base the calculation on the fact that playing red/black on the roulette is an equal game. Therefore the mean value of the gain must be zero, no mater how you play.

The mean value (which is zero) for a gain of q chips is equal to q times the probability $P(q)$ of winning q chips, minus n (your inventory of chips) times the probability of losing the last game, which is $1 - P(q)$.

$$q \cdot P(q) - n(1 - P(q)) = 0 \quad \Leftrightarrow \quad P(q) = \frac{n}{n+q}$$

Thus the answer is exceedingly simple.

If you enter a Casino, with the intension to double your capital, then $n = q$, and the probability of doing so is ½. So you might (from a mathematical point of view) as well settle for one game, where you place your entire capital on red.

On the other hand if you bring 10,000 $ to the Casino, and will settle for a gain of 1000 $ (to pay the hotel and other plesant accommodations), then your chance of succeeding is 10/11 or 91%.

However this perspective is far from attractive to most people who visit a Casino, rather they enter with 1000 $ and are keen to win say 10,000 $. From the formula above, it is seen that the probability that such a player will succeed is 1/11 or 9.1% .

Most people think that the large incomes to the Casino's come from the unequal odds with respect to the costumers. This is however not the case. Their main income comes is bound in the human nature of greed, which make them play in an less intelligent way.

Since playing on the roulette the odds are only slightly in favour of the Casino, everyone should stick to the rule:

Once you have won about the same amount that you have brought with you to the Casino, YOU SHOULD STOP AND LEAVE

But if you don't, your will get ruined, almost with mathematical certainty

2.6 The game with the matches

As outlined previously this game is played with 10 matches having a quadratic cross section, where one side has been coloured red. As the matches are thrown in the air, the probability is $p = 0.25$ that the red side is upward, when they land on the table.
The players take turn throwing the 10 matches, repeating it 10 times. For each throw the player notice the number of matches showing the red side, and the player must then multiply this number by one of the numbers 1..10. He may however only use each of the selected numbers once.
The result is then the score of this throw, and the aim is to obtain the highest score after the 10 throws.

To obtain the highest score is clearly a matter of strategy, based on mathematical probabilities. E.g. if you get 4 coloured matches in the second attempt, should you multiply with the highest or the next highest number left or? You may have a sensation, but this question can only be settled by an mathematical theory of optimal strategies.

The number of matches with the red side up is binomial distributed with the primary probability $p = 0.25$, and the number parameter $n = 10$

$$P(X = j) = \binom{10}{j}\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{10-j} \qquad j = 0,1,2..,10$$

Furthermore we shall apply the cumulated probabilities $P(X \le \text{j})$ and $P(X \ge \text{j})$.

Primary probabilities *P(X*=j),j = 0..10, *n* = 10 and *p* = 0.250

j =	0	1	2	3	4	5	6	7	8	9	10
P(X=j):	0.056	0.188	0.282	0.250	0.146	0.058	0.016	0.003	0.0004	0.0000	0.0000

Backward cumulated probabilities *P(X>=j)*

j =	0	1	2	3	4	5	6	7	8	9	10
P(X>=j)	1.000	0.944	0.756	0.474	0.224	0.078	0.0197	0.0035	0.0004	0.0000	0.0000

Below is shown a computer simulation of the game.

```
 1. throw. Reds = 2. Choices { 1, 2, 3, 4, 5, 6, 7, 8, 9,10,}  Factor= 4
 2. throw. Reds = 4. Choices { 1, 2, 3, 5, 6, 7, 8, 9,10,}  Factor=10
 3. throw. Reds = 2. Choices { 1, 2, 3, 5, 6, 7, 8, 9,}  Factor= 5
 4. throw. Reds = 2. Choices { 1, 2, 3, 6, 7, 8, 9,}  Factor= 3
 5. throw. Reds = 3. Choices { 1, 2, 6, 7, 8, 9,}  Factor= 8
 6. throw. Reds = 2. Choices { 1, 2, 6, 7, 9,}  Factor= 6
 7. throw. Reds = 6. Choices { 1, 2, 7, 9,}  Factor=9
 8. throw. Reds = 3. Choices { 1, 2, 7,}  Factor= 2
 9. throw. Reds = 2. Choices { 1, 7,}  Factor= 7
10. throw. Reds = 2. Choices { 1,}  Factor= 1

Your score is 176
```

The choices of the factor to multiply with are done by an optimal strategy, to be demonstrated below.

Before we launch the optimal strategy, we shall make a try with a ”intuitively” best strategy.

We shall illustrate this with an example.

Suppose that we get 4 red matches in the third throw. The numbers left to multiply with, are denoted: (y_8, y_7, y_6, y_5, y_4, y_3, y_2, y_1). This raises the question, whether we should chose the highest factor y_8, the second highest etc.

Our reason the goes the as follows: If the probability of throwing more than 4 red matches in the in the following 7 throws is greater than 0.5, we should choose the second highest factor.

We shall sharpen this argument a bit: If the probability of throwing more than 4 red matches two times or more in the remaining 7 throws is greater than 0.5, we should choose the third highest factor, and so on..

This strategy appears very plausible.

The probabilities can be achieved directly from the binomial distribution. Firstly the probability of getting at least *q* red matches in a throw.

$$P(X \geq q) = \sum_{j=q}^{n} P(X = j) \quad \text{, where } P(X = j) \text{ is shown above.}$$

The probability that this will happen at least *r* times in the remaining *n* - *k* throws is also a binomial distribution, and it can be written as:

$$P(X \geq q; \text{ at least } r \text{ times in } n - k \text{ throws }) = \sum_{j=r}^{n-k} \binom{n-k}{j} P(X \geq q)^j (1 - P(X \geq q))^{n-k-j}$$

The calculation of these 110 probabilities is most conveniently done on a computer.

To make a strategy table, one must for each throw, and for each case of the number of red matches first choose the biggest factor of the remaining factors. Then one should calculate the possibility of incrementing the number of red matches by one, in at least one of the remaining throws.

If this possibility is greater than 0.5, one should look at the second highest factor. Then you should calculate the possibility of incrementing the number of red matches by one, in at least two of the remaining throws. If this possibility is greater than 0.5, you should look at the third highest factor, and so on.

However the calculation should be initiated, by choosing the lowest factor if you get 0 red matches, and by choosing the highest factor if you get 10 red matches.

Below is shown a computer calculation of the strategy table for this strategy.
Furthermore is shown a Monte Carlo simulation of 10.000 games using this strategy. Then it is shown the occurrences of the score in the games, and finally the cumulated occurrences.

P(X =j) for n =10 and p = 0.250

j:	0	1	2	3	4	5	6	7	8	9	10
P(X =j):	0.056	0.188	0.282	0.250	0.146	0.058	0.0162	0.003	0.0004	0.000	0.000

P(X>=j) for n =10 and p = 0.250

j:	0	1	2	3	4	5	6	7	8	9	10
P(X>=j)	1.000	0.944	0.756	0.474	0.224	0.078	0.0197	0.0035	0.0004	0.000	0.000

Strategy table: Notice that the numbers do not refer to the factors themselves, since we do not know which are the remaining factors. But to the greatest, the second greatest of the remaining factors. Example: If you throw 3 red matches in the second throw, you should choose the third highest factor (7), since there are only 9 left

```
Throw no: 1   2   3   4   5   6   7   8   9  10
score
   0      1   1   1   1   1   1   1   1   1   1
   1      3   3   3   2   2   2   2   1   1   1
   2      6   5   5   4   4   3   3   2   2   1
   3      8   7   7   6   5   4   3   3   2   1
   4      9   9   8   7   6   5   4   3   2   1
   5     10   9   8   7   6   5   4   3   2   1
   6     10   9   8   7   6   5   4   3   2   1
   7     10   9   8   7   6   5   4   3   2   1
   8     10   9   8   7   6   5   4   3   2   1
   9     10   9   8   7   6   5   4   3   2   1
  10     10   9   8   7   6   5   4   3   2   1

Average score =  164.37
```

```
Occurrences of scores:

h( 75)=  1 h( 80)=  7 h( 85)=  3 h( 90)=  5 h( 95)=  6 h(100)= 34 h(105)= 38
h(110)= 82 h(115)= 68 h(120)=161 h(125)=146 h(130)=320 h(135)=280 h(140)=534
h(145)=425 h(150)=752 h(155)=577 h(160)=878 h(165)=645 h(170)=956 h(175)=651
h(180)=863 h(185)=527 h(190)=609 h(195)=351 h(200)=389 h(205)=180 h(210)=203
h(215)= 93 h(220)= 86 h(225)= 47 h(230)= 32 h(235)= 21 h(240)= 13 h(245)=  8
h(250)=  3 h(255)=  0 h(260)=  4 h(265)=  1 h(270)=  1 h(275)=  0 h(280)=  0
```

```
Frequencies of score:

F( 80)=0.0008 F( 85)=0.0011 F( 90)=0.0016 F( 95)=0.0022 F(100)=0.0056
F(105)=0.0094 F(110)=0.0176 F(115)=0.0244 F(120)=0.0405 F(125)=0.0551
F(130)=0.0871 F(135)=0.1151 F(140)=0.1685 F(145)=0.2110 F(150)=0.2862
F(155)=0.3439 F(160)=0.4317 F(165)=0.4962 F(170)=0.5918 F(175)=0.6569
F(180)=0.7432 F(185)=0.7959 F(190)=0.8568 F(195)=0.8919 F(200)=0.9308
F(205)=0.9488 F(210)=0.9691 F(215)=0.9784 F(220)=0.9870 F(225)=0.9917
F(230)=0.9949 F(235)=0.9970 F(240)=0.9983 F(245)=0.9991 F(250)=0.9994
F(255)=0.9994 F(260)=0.9998 F(265)=0.9999 F(270)=1.0000 F(275)=1.0000
```

As it is shown in the table for $P(X >= j)$, there is only 7.8% chance to throw 5 red matches, so if you do so, you should always choose the highest remaining factor.
The situation of 3 or 4 red matches is far more speculative, and only a calculation can decide what is best.

We shall now turn away from our intuitive strategy to the Snell-strategy. Using this strategy,when deciding on choosing the highest or the second highest factor, it is in fact possible to make the necessary calculations, but for the following cases the calculations simply become to comprehensive, and we resort to a slightly improved version of our intuitive strategy.
Not surprisingly it results in only a few changes in the strategy table, but it demonstrates the theoretically comforting fact, that there exists an optimal strategy that is beyond sensations.

The changes in the intuitive strategy can be illustrated with an example. Suppose that we have thrown 3 red matches, and we consider whether we should multiply with 7 or 6. If we multiply by 6 it is because we expect to throw at least 4 red matches at least once in the remaining throws.
Our scores will be $7 \cdot 3$ and $7 \cdot 4$ respectively and the score will be improved by a factor 4/3.

The improvement comes about, if we choose to multiply by 6, expecting that we shall throw 4 red matches in the remaining games. Our correction is then that we choose the second highest factor 6 if the probability of throwing 4 red matches times the relative improvement 4/3 is greater than 0.50.

2.7 Applying the Snell-strategy to the matches game

We recapitulate the Snell-strategy for the matches game.:

$G_{10} = X_{10}$	X_{10}: The number of red matches in the final throw.
$G_9 = max\{ X_9 , E(G_{10}) \}$	$E(G_{10}) = E(X_{10}) = np$=10·0.25 = 2.5 ; X_9 = 0,1,…10.
$G_k = max\{ X_k , E(G_{k+1}) \}$	$E(G_k)$ requires a lengthy calculation but it is computed as a usual mean value of a stochastic variable, that is, the max of the two values times $P(X=X_{k-1})$
$G_1 = max\{ X_1 , E(G_2) \}$	

As an example we shall compute $E(G_9)$. Since $E(G_{10})$=2.5, we shall for X_9 = 0,1,2 apply $E(G_{10})$=2.5 in the calculation.

$$E(G_9) = 2{,}5 \cdot(P(X=0)+P(X=1)+P(X=2))+ 3\cdot P(X=3)+ ... +10\cdot P(X=10) =3.06$$

The different values for $E(G_k)$ are shown in the computer calculation below. Notice that that $E(G_k)$ increase from 2.5 to 4.25, when k decreases from 10 until 2.

The strategy is the following:
Choose the largest of the remaining numbers, if $X_k > E(G_{k+1})$, that is, if the number of red matches is larger than the expected largest number. If not, you should choose the second highest factor.

To the decision of using the 3. or 4. …highest, we shall apply the intuitive strategy, with the slight modification sketched above.
Another computer calculation, quite similar to the preceding one, is presented below. You should notice the (small) changes from the first one. The new strategy is less cautious than the first strategy, but it pays off in a increase on the mean value of the score on 2.75.

P(X =j) for n =10 and p = 0.250

j:	0	1	2	3	4	5	6	7	8	9	10
P(X =j):	0.056	0.188	0.282	0.250	0.146	0.058	0.0162	0.003	0.0004	0.000	0.000

P(X>=j) for n =10 and p = 0.250

j:	0	1	2	3	4	5	6	7	8	9	10
P(X>=j)	1.000	0.944	0.756	0.474	0.224	0.078	0.0197	0.0035	0.0004	0.000	0.000

```
E(G[10])= 2.50   E(G[9])= 3.06   E(G[8])= 3.37   E(G[7])= 3.62   E(G[6])= 3.80
E(G[5])= 3.95   E(G[4])= 4.06   E(G[3])= 4.16   E(G[2])= 4.25
```

Strategy table:

throw no:	1	2	3	4	5	6	7	8	9	10
score										
0	1	1	1	1	1	1	1	1	1	1
1	2	2	1	1	1	1	1	1	1	1
2	5	4	4	3	3	3	2	2	1	1
3	8	7	6	5	5	4	3	2	2	1
4	9	8	7	7	6	5	4	3	2	1
5	10	9	8	7	6	5	4	3	2	1
6	10	9	8	7	6	5	4	3	2	1
7	10	9	8	7	6	5	4	3	2	1
8	10	9	8	7	6	5	4	3	2	1
9	10	9	8	7	6	5	4	3	2	1
10	10	9	8	7	6	5	4	3	2	1

Average score = 167.27

Occurrences of scores:

```
h( 75)=  1 h( 80)=  1 h( 85)=  0 h( 90)=  3 h( 95)=  2 h(100)=  5 h(105)=  4
h(110)=  9 h(115)= 12 h(120)= 12 h(125)= 17 h(130)= 22 h(135)= 28 h(140)= 39
h(145)= 24 h(150)= 49 h(155)= 54 h(160)= 74 h(165)= 63 h(170)=117 h(175)= 63
h(180)= 91 h(185)= 57 h(190)= 72 h(195)= 39 h(200)= 36 h(205)= 35 h(210)= 29
h(215)= 14 h(220)= 21 h(225)=  1 h(230)=  5 h(235)=  1 h(240)=  0 h(245)=  0
h(250)=  0 h(255)=  0 h(260)=  0 h(265)=  0 h(270)=  0 h(275)=  0 h(280)=  0
```

Frequency of scores:

```
F( 80)=0.0020 F( 85)=0.0020 F( 90)=0.0050 F( 95)=0.0070 F(100)=0.0120
F(105)=0.0160 F(110)=0.0250 F(115)=0.0370 F(120)=0.0490 F(125)=0.0660
F(130)=0.0880 F(135)=0.1160 F(140)=0.1550 F(145)=0.1790 F(150)=0.2280
F(155)=0.2820 F(160)=0.3560 F(165)=0.4190 F(170)=0.5360 F(175)=0.5990
F(180)=0.6900 F(185)=0.7470 F(190)=0.8190 F(195)=0.8580 F(200)=0.8940
F(205)=0.9290 F(210)=0.9580 F(215)=0.9720 F(220)=0.9930 F(225)=0.9940
F(230)=0.9990 F(235)=1.0000 F(240)=1.0000 F(245)=1.0000 F(250)=1.0000
F(255)=1.0000 F(260)=1.0000 F(265)=1.0000 F(270)=1.0000 F(275)=1.0000
```

The improvements from the first strategies do not appear very impressive, but it is theoretically that one may surpass any logically intuitively best strategy by Snell's optimal strategy.

The crucial point of such a calculatio is of course that all events are stochastically independent, having a certain probability distribution.

If one should consider, to apply a similar theory to economics or social life, it will always fail, because actions of human beings are not guided by probabilities, but based on reason or more frequently based on lack of reason.