Elementary Mathematics

Trigonometric functions

This is an article from my home page: www.olewitthansen.dk

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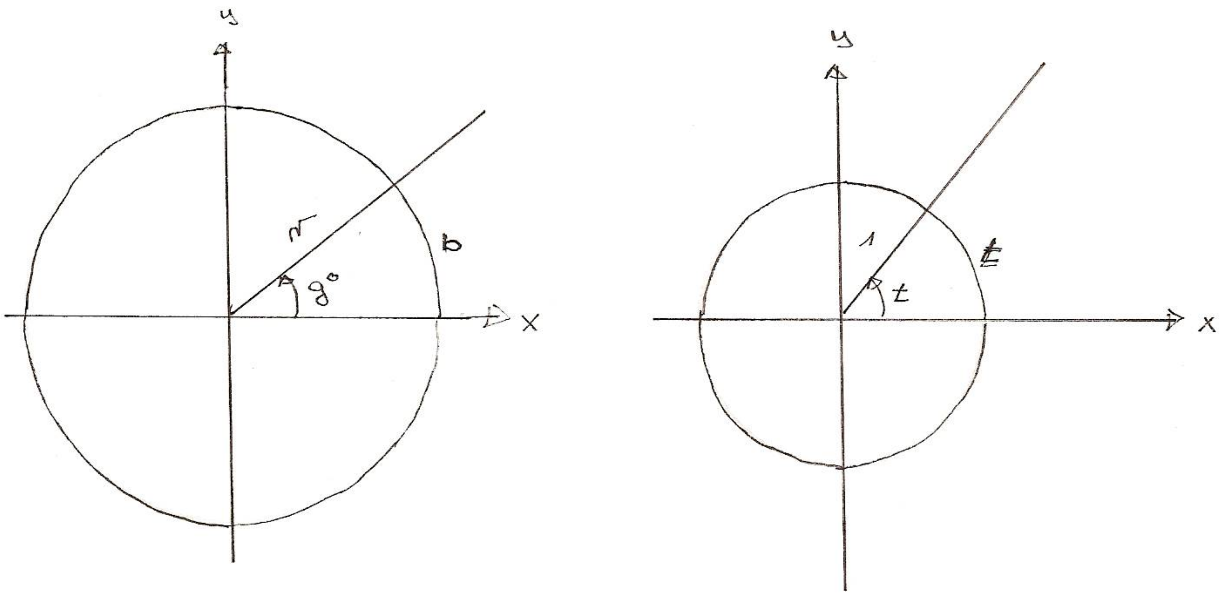
1. Degree and radian

The trigonometric functions *sine, cosine* and *tangent* may be considered as functions of a generalized angular measurement.
One reason for this is that 1^0, is not a length, but the size of an arc measure on a circle of arbitrary radius.
For this and several other reasons is invented another measurement of an angle, which is called *the radian number*.



In the first figure we have drawn a circle with radius *r* having centre in the vertex of an angle, and the right leg of the angle coincides with the *x*-axis. The angle covers the arc *b* on the circle.

By the radian number for an angle, we understand the length of the arc that is covered by the angle of a circle having centre at the vertex of the angle, measured in units of the radius.
The radian number is then defined as:

$$t = \frac{b}{r}.$$

If specially the circle is a unit circle, as show on the figure to the right, then the radius is 1, and then $t = b$, which allows us to formulate an alternative definition of the radian number.

By the radian number for an angle, we understand the length of the arc that the angle covers on a unit circle having its centre at the vertex of the angle.

The radian number is a length that is independent of which circle you measure it.
The ratio between the circumference of a circle and the diameter is constant equal to π independent of the radius and therefore the ratio between the arc (which is a fraction of the perimeter) and the radius also constant.
If you want to convert from degrees to radian, you should just notice that the circumference of the circle is $2\pi r$, which on a unit circle becomes 2π and this corresponds to 360^0. We thus have:

$$2\pi \ \ radian \ = \ 360^0 \qquad \Leftrightarrow \qquad \pi \ \ radian \ = \ 180^0$$

$$1 \ \ radian \ = \ \frac{180^0}{\pi} \qquad \text{and} \qquad 1^0 \ = \ \frac{\pi}{180} radian$$

$$t \ \ radian \ = \ \frac{180^0}{\pi} t \qquad \text{and} \qquad g^0 \ = \ \frac{\pi}{180} g \ \ radian$$

We see that we convert from degrees to radians by multiplying by $\frac{\pi}{180}$ and vice versa.

$1\ radian \approx 57{,}30^0$ and $1^0 = 0{,}0175\ radian$

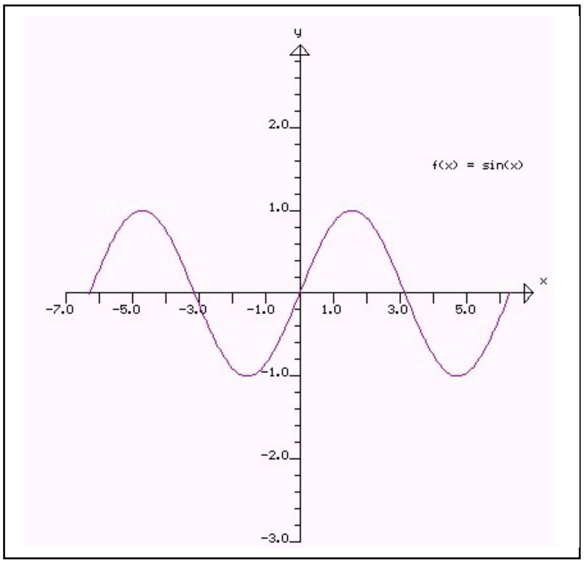
There are some radian numbers which are useful to know, and they can be easily be derived from

$$2\pi = 360^0, \quad \pi = 180^0, \quad \frac{\pi}{2} = 90^0, \quad \frac{\pi}{3} = 60^0, \quad \frac{\pi}{6} = 30^0$$

In a coordinate system the radian number can be extended to include all real numbers.

2. sin *x*, cos *x* and tan *x*

Speaking of sin(x), cos(x) and *tan*(x) as function, then x is always the radian number, and not the degree of an angle. Below is sketched the graphs for the three functions in the interval [-2π, 2π].



In the figure to the left is shown the graph

$$f(x) = \sin(x)$$

Sine has the definition set *Dm*(sin)= *R*, and the value set *Vm*(sin) = [-1, 1].
x and $x+ 2\pi$, give the same value for sin(x) and cos(x), and we say that sine and cosine are periodic with the period 2π (=360^0)

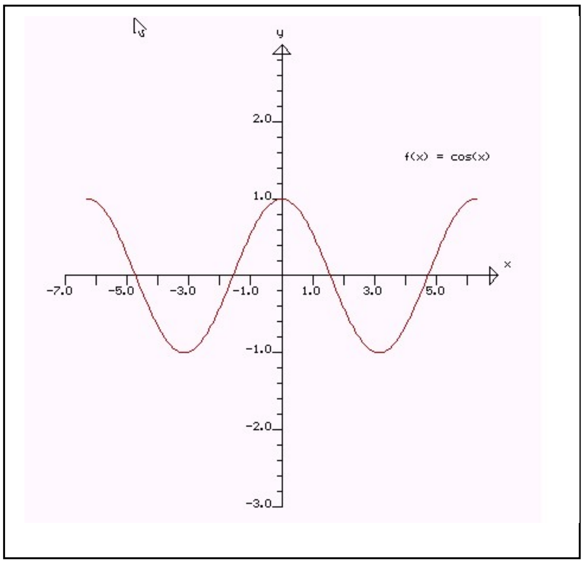
$$\sin(x + 2\pi) = \sin(x)$$
$$\cos(x + 2\pi) = \cos(x)$$

sin(x) intersects the y-axis in 0, and the intersection with the x-axis is given by:

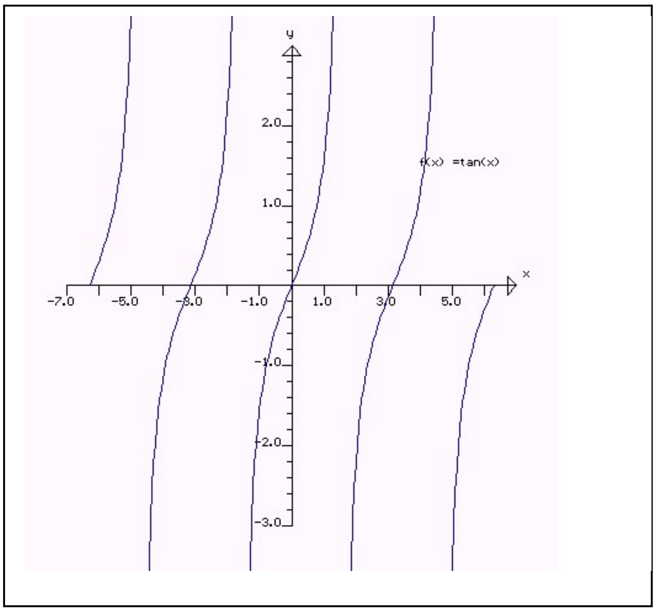
$$\sin(x) = 0 \ \Leftrightarrow x = p \cdot \pi \ ; \ \ p \in Z$$

$f(x) = \cos x$ has the same definition set as sin x. *Dm*(cos)= *R*, and *Vm*(cos) = [-1, 1]. Further $f(0) = \cos(0) = 1$, and: $f(x) = 0 \Leftrightarrow \cos(x) = 0 \Leftrightarrow$

$$x = \tfrac{\pi}{2} + p\pi; \ \ p \in Z$$



From the relation $\cos(x - \tfrac{\pi}{2}) = \sin x$, we can see that the graph for sine is the same graph as for cosine displaced $\tfrac{\pi}{2}$ along the x-axis.



The graph for $f(x) = tan\ x = \dfrac{\sin x}{\cos x}$

Is very different from the graphs for sine and cosine, since *tan x* is not defined for the zero point for cos *x*. So

$$Dm(tan):\ x = \tfrac{\pi}{2} + p\pi;\ \ p \in Z$$

tan x is periodic with the period π, since

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin(x)}{-\cos(x)} = \tan(x)$$

The value set for *tan x* is *R*

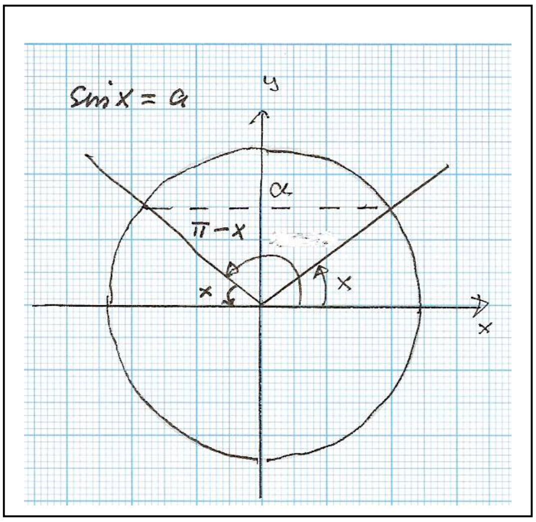
3. Trigonometric equations

A trigonometric equation is an equation which contains either sin *x*, cos *x* or *tan x*, (but not *x*).
We shall start by solving the three basic equations:

$$\sin x = a\,;\ \ a \in [-1,1] \qquad \cos x = a\,;\ \ a \in [-1,1] \qquad \tan x = a; \quad a \in R$$

To solve an equation means to investigate, whether there are solutions, and if applicable find all solutions. Older pocket computers gave only one or two solution, so it makes (still) sense to do it by hand. To visualize the set of solutions it is advantageous (if not necessary) to illustrate the solutions in a coordinate system with a unit circle.

Below we solve the three equations using a numerical example, but the method is general.



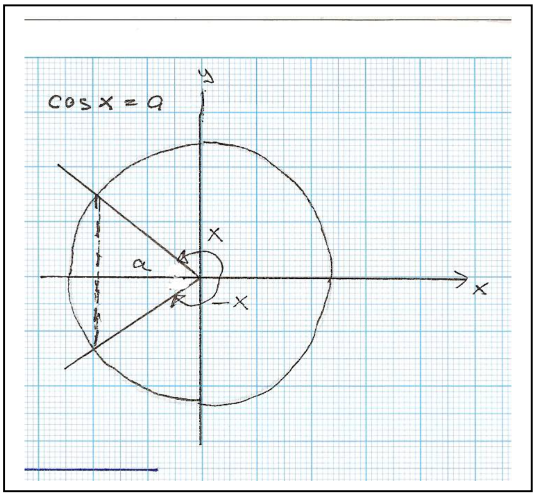
First we look at the equation: sin x = 0.62

We find one solution (in radian) on a pocket calculator (I used tables in high school) x_0 =0.6687.
We allocate 0.62 on the *y*-axis in a coordinate system, provided with a unit circle. There are then two direction points which have this sine.
The one is x_1 =0.6687, ($\frac{\pi}{2} \approx 1{,}57$), but as it can be seen from the figure, the equation has also the solution
$x_2 = \pi - 0.6687$.
We then find the complete solution by adding an integral number of 2π to the solutions.

$$x = 0.6687 + p2\pi \quad \vee \quad x = \pi - 0.6687 + p2\pi \ \ ;\ \ p \in Z$$

More generally you may find the complete solution to the equation sin x = a, as:

$$x = x_0 + p2\pi \quad \vee \quad x = \pi - x_0 + p2\pi\,,$$ where x_0 is an arbitrary solution.



In almost the same manner, we shall solve the equation

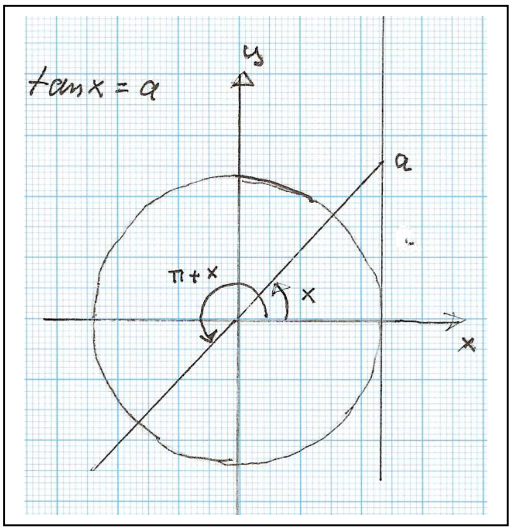
$$\cos x = -0.75$$

-0.75 is allocated on the x-axis, and we find two direction points, which have this cosine. We look it up and find one solution: x_0 = 2.419, corresponding to the angle x in the figure.
The other direction point corresponds to $-x_0$ = -2.419.
And we then find the complete solution by adding an integral number of 2π to the solutions.

$$\cos x = -0.75 \quad \Leftrightarrow \quad x = \pm 2{,}419 + p2\pi \quad ; p \in Z$$

And in general:

$$\cos x = a \quad \Leftrightarrow \quad x = \pm x_0 + p2\pi \quad ; \quad p \in Z\,,$$ where x_0 is an arbitrary solution.



We shall then solve the equation:

$$\tan x = 1.2$$

We allocate a =1.2 on the tangent in (1,0)
If we draw the line from a on the tangent to the origin, we find two direction points, which have $\tan x$ =1.2.
When we look it up, we find x_0 = 0.8761. To determine the complete solution we just have to ad an integral number of π, since tangent is periodic with period π.

$$x = 0.8661 + p\pi\,;\quad p \in Z$$

In general:

$$\tan x = a \quad \Leftrightarrow \quad x = x_0 + p\pi\,;\quad p \in Z$$

Eksempel
Some trigonometric equations may be transformed into the trigonometric base equations: We look at the equation:

$$6\sin^2 x + \sin x - 1 = 0 \qquad x \in [0, 2\pi]$$

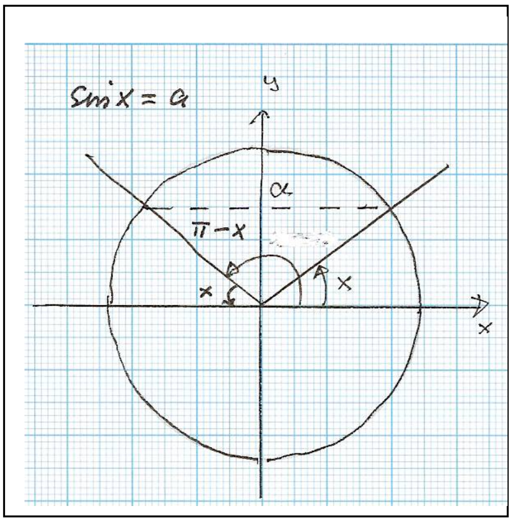
It is a quadratic equation in sin x, and we determine the determinant: $d = 1 + 24 = 25$. The solutions are then:

$$\sin x = \frac{-1 \pm 5}{12} \Leftrightarrow \sin x = -\frac{1}{2} \;\vee\; \sin x = \frac{1}{3}$$

We look up the two solutions: $x = \frac{7}{6}\pi$ and x = 0.3398. The other two solutions in the interval [0, 2π], are then found to be: $\pi - \frac{7}{6}\pi + 2\pi = \frac{11}{6}\pi$ and x = π – 0.3398 =2,8018. In the interval [0, 2π] the equation therefore has the solutions:

$$x = \frac{7}{6}\pi \;\vee\; x = \frac{11}{6}\pi \vee\; x = 0{,}3398 \vee\; x = 2{,}8018$$

4. Trigonometric inequalities



Trigonometric inequalities may be a bit tricky for the reason, that they are periodic with period 2π or π, and sine and cosine are not monotonic in the intervals of the period. As it is the case of equations, inequalities cannot be solve without a graphic representation, and this is even more outspoken than it is the case for equations.
We shall choose the same numerical examples as above, but now applied to inequalities.

$$\sin x < 0.62\,, \qquad x \in [0,2\pi]$$

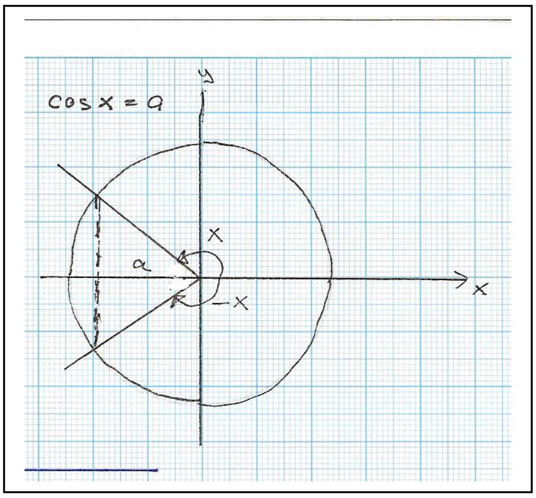
To solve the inequality, we shall first solve the equation $\sin x = 0.62$, where we find the solution: $x_0 = 0.6687$.
We then allocate 0.62 on the y-axis, and draw a horizontal line to find the two directional points having $\sin x = 0.62$
We recognize the solution $x_1 = 0.6687$ from a previous example, and as before we have the second solution: $x_2 = \pi - 0.6687 = 2.47$.
From the drawing (and only from the drawing), we see, that in the interval $x \in [0,2\pi]$, we have:

$$\sin x < 0.62 \quad \Leftrightarrow \quad 0 < x < 0.6687 \;\vee\; 2.47 < x < 2\pi$$

If we wish to have the complete solution, we only need to add an integral number of 2π.

$$p2\pi < x < 0{,}6687 + p2\pi \vee\; 2{,}47 + p2\pi < x < 2\pi + p2\pi$$

Which we may synthesize to: $2{,}47 + (p-1)2\pi < x < 0{,}6687 + p2\pi$



In almost the same manner, we may solve the inequality

$$\cos x > -0.75 \quad , \; x \in [0,2\pi]$$

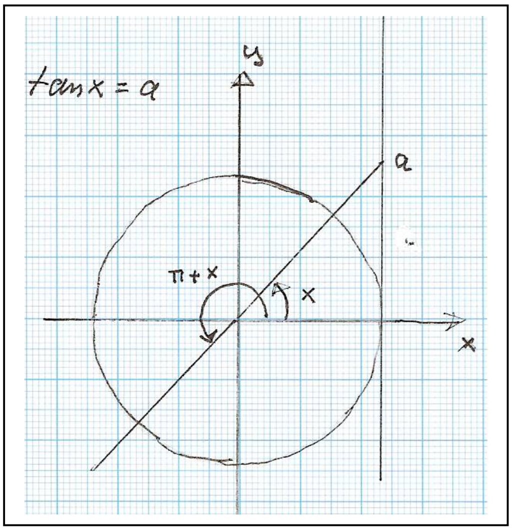
-0.75 is allocated on the x-axis, and by drawing a vertical line through this point, we find two directional points on the unit circle which have that cosine.
If we look it up, we find: $x_0 = 2.419$, corresponding to the angle x shown in the figure. The other directional point corresponds to the angle $-x_0 = -2.419$ which has the directional angle $2\pi - 2.419 = 3.864$ in $[0,2\pi]$.
From the *drawing* we can see that $\cos x > -0.75$, in the interval $[0,2\pi]$, if and only if:

$$3.864 < x < 2\pi \;\vee\; 0 \le x < 2.419$$

If we want the complete solution, we just have to add en integral number of 2π.

$$3{,}864 + p2\pi < x < 2\pi + p2\pi \;\vee\; p2\pi \le x < 2{,}419 + p2\pi$$

Which may be summarized to $3{,}864 + p2\pi < x < 2{,}419 + (p+1)2\pi$



Solve the inequality:

$$\tan x < 1.2 \quad , \; x \in [0,2\pi]$$

As in the previous examples, we begin by solving the equation: $\tan x = 1.2$
We allocate a =1.2 on the tangent in the point E (1,0).
If we draw the line from a on the tangent to the origin, we find two direction points, which have *tan x* =1.2.
When we look it up, we find: $x_0 = 0.8761$.
Since tangent is periodic with the period π, we find the other solution $x = \pi + 0{,}8661$ in $[0,2\pi]$
From the drawing we then se that in the interval $[0,2\pi]$:

$$\tan x < 1.2 \, ,$$

When:

$$0 \le x < 0{,}8761 \;\vee\; \tfrac{\pi}{2} < x < \pi + 0{,}8661$$

If we want the complete solution, we just have to add en integral number of π,

$$p\pi \le x < 0{,}8761 + p\pi \;\vee\; \tfrac{\pi}{2} + p\pi < x < \pi + 0{,}8661 + p\pi$$

Eksempel
Trigonometric inequalities of the second degree: Wi shall look at the inequality:

$$6\sin^2 x + \sin x - 1 < 0 \quad x \in [0,2\pi]$$

If we put $z = \sin x$, we shall initially determine when $6z^2 + z - 1 < 0$.

The equation $6z^2 + z - 1 = 0$, has the roots $z - \frac{1}{2} \vee z = \frac{1}{3}$.

We have earlier seen that the quadratic polynomial is negative between the roots: $-\frac{1}{2} < z < \frac{1}{3}$
We have reduced the quadratic inequality to a double inequality in sin x.

$$-\frac{1}{2} < \sin x < \frac{1}{3} \;\Leftrightarrow\; \sin x > -\frac{1}{2} \;\wedge\; \sin x < \frac{1}{3}$$

These two inequalities are then solved for $x \in [0,2\pi]$.

$$\left(\frac{11}{6}\pi < x < 2\pi \;\vee\; 0 < x < \frac{7}{6}\pi\right) \;\wedge\; \left(0 < x < 0{,}3398 \;\vee\; \pi - 0{,}3398 < x < 2\pi\right)$$

Summarized to: $\frac{11}{6}\pi < x < 2\pi \;\vee\; 0 < x < 0{,}3398$

5. Harmonic functions

From sine and cosine one may form new functions that are linear in sine and cosine e.g. 3cos x or -5 sin x.
In general

$$f(x) = A\cos x \qquad \text{or} \qquad f(x) = A\sin x$$

where A is a nonzero real number.

Since cosine varies between -1 and 1, and since all functional values are multiplied by A then:

$f(x) = A\cos x$, will vary between $-A$ and A.

We could also consider functions $f(x) = \cos(2x)$ or $f(x) = \sin(½x)$, or more generally:

$f(x) = \cos(kx)$ or $f(x) = \sin(kx)$

where k is a nonzero real number.

$\cos(kx)$ and $\sin(kx)$ are also periodic, but with another period than sine and cosine.
The period T can be determined as the increment in x that gives the phase kx an increment of 2π.

$$k(x + T) = kx + 2\pi \quad \Leftrightarrow \quad kT = 2\pi \quad \Leftrightarrow \quad T = \frac{2\pi}{k}$$

So, $\sin(2x)$ has the period π, and $\sin(½x)$ has the period 4π.

We may also consider a displacement of these functions along the x-axis.:

$f(x) = \cos(x+2)$ or $f(x) = \sin(x-3)$

$\cos(x+2)$ is a displacement of $\cos(x)$ -2 along the x- axis, and $\sin(x-3)$ is a displacement of $\sin(x)$ $+3$ along the x- axis.

If we combine the three things, we may write a general trigonometric function:

$$f(x) = A\cos(kx + \varphi)$$

In physics the constant A is denoted the *amplitude*, k is the angular velocity (usually written as ω instead of k). $kx + \varphi$ is called the phase and φ the initial phase.

In physics the function represents a oscillation, where x is the time and we therefore replace x with t

$$f(t) = A\cos(\omega t + \varphi)$$

The period in the oscillation is given by: $T = \dfrac{2\pi}{\omega}$

Every function which can be expressed as above has the form of a sine or a cosine function.
They are called harmonic functions, and an expansion on these function is called harmonic analysis.

We only need to write the expression for $f(t)$ for the cosine function since $\cos(x - \frac{\pi}{2}) = \sin(x)$, so a sine can always be written as a cosine by adding a phase $-\pi/2$.

We could also look at a linear combination of $\cos(kx)$ and $\sin(kx)$. What will the graph look like?

$$f(x) = a\cos(kx) + b\sin(kx)$$

where a, b and k are real numbers. It shows up, that such a linear combination can always be written as a cosine shaped function: $f(x) = A\cos(kx + \varphi)$

If a and b are perceived as the coordinates of a point (a,b), then it is always possible to determine a number A and an angle φ, such that:

$$a = A\cos\varphi \quad \text{and} \quad b = A\sin\varphi$$

This becomes obvious if you consider a right angled triangle with catheti a and b and hypotenuse A. It then follows:

$$\tan\varphi = \frac{b}{a} \quad \text{and} \quad a^2 + b^2 = A^2\cos^2\varphi + A^2\sin^2\varphi = A^2(\cos^2\varphi + \sin^2\varphi) = A^2 \;\Rightarrow A = \sqrt{a^2 + b^2}$$

$f(x)$ can therefore be written as:

$$f(x) = A\cos\varphi\cos(kx) + A\sin\varphi\sin(kx) = A(\cos\varphi\cos(kx) + \sin\varphi\sin(kx))$$

In the section on geometric vectors, we derive the so called addition formulas. One of them is:

$$\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

If we apply this formula to the expression for $f(x)$, then the expression can be written as:

$$f(x) = A(\cos\varphi\cos(kx) + \sin\varphi\sin(kx)) = A\cos(kx - \varphi)$$

If we want exactly the same expression as above, we can just replace φ with – φ.
Below we have drawn (with a computer program from 1995) the graphs for

$f(x) = \cos 2x$, $g(x) = \sin \frac{1}{2}x$, $h(x) = -2\cos(2x) + 3\sin(2x)$, $i(x) = 4\cos(\frac{1}{3}x + 2)$.