Elementary Mathematics

Quadratic polynomials

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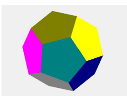

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1. The linear function

Earlier we have been treating at the linear function

$$f(x) = ax + b\ ,\ \ x \in R$$

Where a and b are arbitrary real numbers.

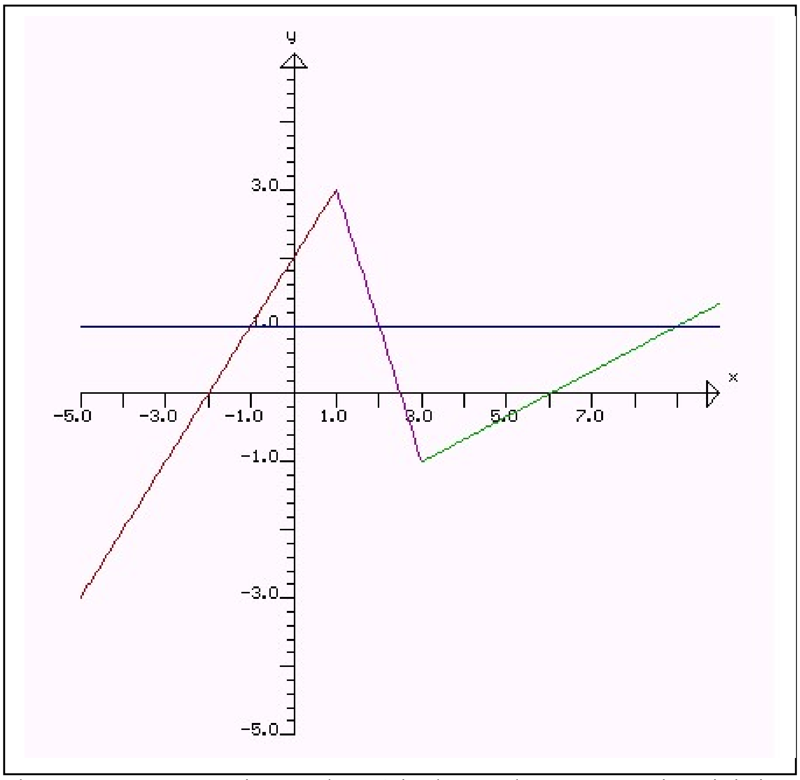
The constant a is called the slope of the linear function, and b is the intersection of the graph with the y-axis.
The function is increasing for $a > 0$, decreasing for $a < 0$, and f is constant equal to b for $a = 0$
.
The graph is a straight line, which intersects the y-axis in b, and for a $\neq 0$ intersects the x-axis in $-\frac{b}{a}$.

1.1 Sectional linear functions

If a function is composed of several linear functions it is called a sectional linear function.
The sections of linear functions do not necessarily need to be connected, in which case the function is said to be discontinuous in these points. A sectional linear function is often given by a "fork-rule"

Example

$$f(x) = \begin{cases} x+2 & for \quad x<1 \\ -2x+5 & for \quad 1 \leq x \leq 3 \\ \frac{1}{3}x-2 & for \quad x>3 \end{cases}$$



In we insert x = 1 in the two upper expressions and $x = 3$ in the two lower expressions it is immediately seen that the graph is connected.

.
Since $f(0) = 2$. the graph intersects the y-axis at $y = 2$ and that it intersects the x-axis in the points::

$$x + 2 = 0 \Leftrightarrow x = -2 \quad \text{and} \quad -2x + 5 = 0 \Leftrightarrow x = \tfrac{5}{2} \quad \text{and} \quad \tfrac{1}{3}x - 2 = 0 \Leftrightarrow x = 6$$

As an example, we shall investigate where the graph intersects the line $y = 1$.

$$f(x) = 1 \Leftrightarrow (x + 2 = 1 \wedge x < 1) \vee (-2x + 5 = 1 \wedge 1 \le x \le 3) \vee (\tfrac{1}{3}x - 2 = 1 \wedge x > 3) \Leftrightarrow x = -1 \vee x = 2 \vee x = 9$$

It is necessary to include the definition intervals of the linear function, when solving the equation, this we can see, if we try to solve the inequality: $f(x) < -2$.

$$f(x) < -2 \Leftrightarrow (x + 2 < -2 \wedge x < 1) \vee (-2x + 5 < -2 \wedge 1 \le x \le 3) \vee (\tfrac{1}{3}x - 2 < -2 \wedge x > 3)$$

$$\Leftrightarrow (x < -4 \wedge x < 1) \vee (x > \tfrac{7}{2} \wedge 1 \le x \le 3) \vee (x < 0 \wedge x > 3) \Leftrightarrow x < -4$$

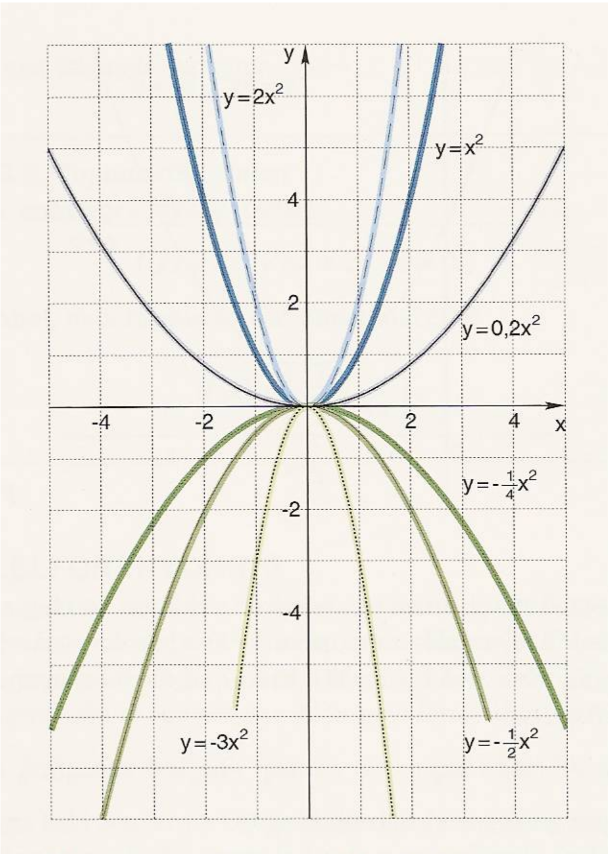
The solution to these equations can of course be located by looking at the graph.

2. The quadratic polynomial

A quadratic polynomial is a function which has the form:

$$f(x) = ax^2 + bx + c \ ,$$

where a , b and c are real numbers and $a \neq 0$. a is the coefficient to x^2 , b is the coefficient to x and c is the constant term. When the terms are arranged with respect to the decreasing power of x the polynomial is said to ordered. We shall first consider the case: $b = c = 0$.



$$f(x) = ax^2$$

Initially we shall look at $f(x) = x^2$. Two opposite numbers will have the same functional value.

x	$\pm\frac{1}{2}$	± 1	$\pm\frac{3}{2}$	± 2	± 3
y	$\frac{1}{4}$	1	$\frac{9}{4}$	4	9

The graphs, which is shown to the left are called *parabolas*.
If we look at $f(x) = \frac{1}{2}x^2$ and $f(x) = 2x^2$, then the graphs looks like $f(x) = x^2$, but where the function values are either halved or doubled.
All graphs $f(x) = ax^2$ passes through (0,0), which is the top point of the parabola.
If we compare $f(x) = x^2$ to $f(x) = -x^2$, then the sign changed for all functional values so $f(x) = -x^2$ is just a mirroring of $f(x) = x^2$ in the x-axis.
So we conclude that for $a > 0$:

$$f(x) = ax^2$$

is a parabola having top point in (0,0), with upwards branches, and for $a < 0$ a parabola having top point in (0,0), with downwards branches.

We shall then show that the graph for :

$$f(x) = ax^2 + bx + c$$

Is the same as the graph for

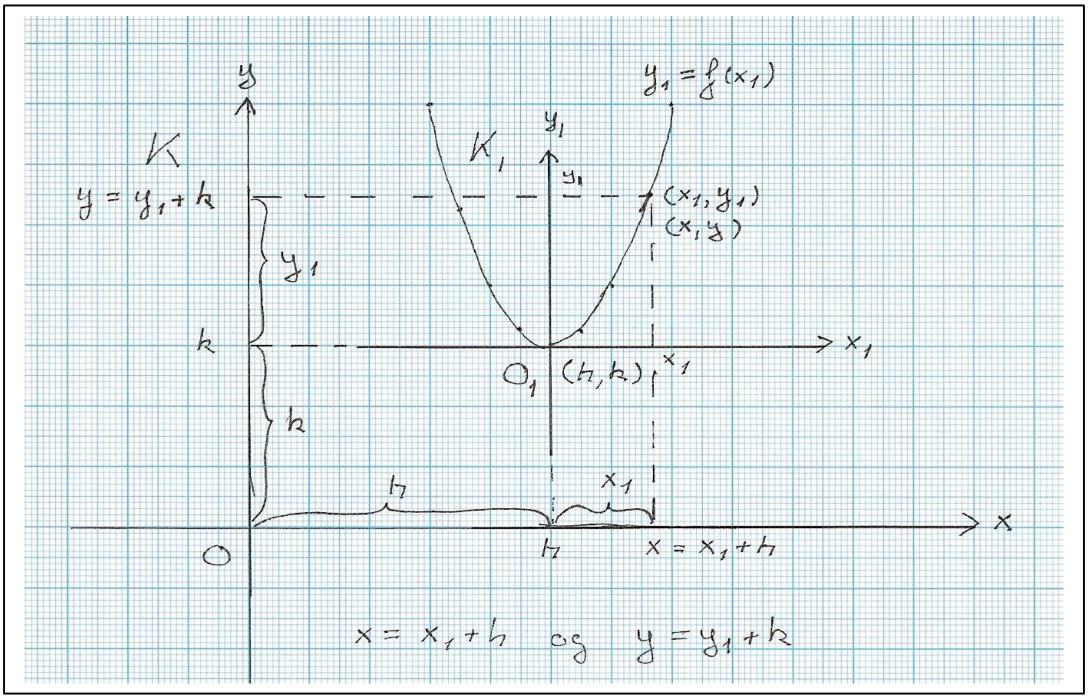
$$f(x) = ax^2 \ ,$$

but parallel displaced to $T = \left(-\frac{b}{2a}, -\frac{-d}{4a}\right)$ where $d = b^2 - 4ac$ is the discriminant for the quadratic equation:

$$ax^2 + bx + c = 0$$

To show this, we must first establish the relation between two coordinates system parallel displaced to each other.

2.1 Parallel displacement of the coordinate system



In the figure above, we have drawn two coordinate systems K and K_1. We wish to determine the connection between the coordinates to the *same* point having coordinates (x, y) in K and coordinates (x_1, y_1) in K_1.
The coordinates of the origin of K_1 is (h,k) in K. If you look at the figure, it is straightforward to see that with the position of K_1 as well as the point in the first quadrant that we have:

$$x = x_1 + h \quad \text{and} \quad y = y_1 + k \quad \Leftrightarrow \quad x_1 = x - h \quad \text{and} \quad y_1 = y - k$$

If K_1 or the point in question or both do not lie in the first quadrant it is perhaps less obvious, but it follows from the insertion rule for three points on an oriented line, a rule we have proven earlier.

If we now look at the equation $y_1 = f(x_1)$, which represents the graph for the function $y = f(x)$, drawn in K_1, and we want to determine the equation for the graph in K, then we only have to insert the two expressions for x_1 and y_1 given above. So we have:

$$y_1 = f(x_1) \quad \Leftrightarrow \quad y - k = f(x - h)$$

$y - k = f(x - h)$ is therefore the equation for $y_1 = f(x_1)$ in K.

The transformation may also be perceived in the manner of a parallel displacement the length h on the x-axis and k on the y-axis of the graph of a function. The graph will have the equation $y - k = f(x - h)$. In the following we shall stick to this interpretation without introducing the coordinate system K_1.

Examples:

1. Determine the equation of the graph $y = \sqrt{x}$, when it is parallel displaced to (-3,2). According to what we have found above the equation becomes.

$$y - 2 = \sqrt{x+3}\,.$$

2. The graph for 2. order polynomial $y = 2x^2$ is parallel displaced to (-1,2). The graph will now have the equation.

$$y - 2 = 2(x+1)^2 \Leftrightarrow y = 2x^2 + 4x + 4\,.$$

We notice that the moved graph is the equation of a general 2. degree polynomial.

2.2 Displacement of a parabola. Formula for the top point

We shall then show that the graph of a general 2-degree polynomial:

$$y = ax^2 + bx + c$$

Is a parallel shift of $y = ax^2$ to

$$T(-\frac{b}{2a}, -\frac{d}{4a})$$

where $d = b^2 - 4ac$ is the discriminant.

Since the parabola $y = ax^2$ has its top point in (0, 0), T must be the coordinates to the top point for the displaced parabola. To show this, we rewrite $y = ax^2 + bx + c$ in the following way:

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$$y = ax^2 + bx + c \quad \Leftrightarrow \quad y = a(x^2 + \frac{b}{a}x + \frac{c}{a})$$

We then rewrite the first two terms in the parenthesis to the square of a two term quantity.
To accommodate this x^2 must be the square of the first term and $\frac{b}{a}x$ must be the double product.
Further $\frac{b}{2a}$ must be the second term since: $2x\frac{b}{2a} = \frac{b}{a}x$.
We then add and subtract $(\frac{b}{2a})^2$ into the parenthesis, and convert it to a two term expression squared.

$$y = a((x + \frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a})$$

When putting the last two terms on a common fraction line:

$$y = a((x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a}) \quad \Leftrightarrow$$
$$y = a((x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}) \quad \Leftrightarrow$$
$$y = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}$$

$d = b^2 - 4ac$, is recognized as the discriminant. Moving the second term to the left side we have:

$$y + \frac{d}{4a} = a(x + \frac{b}{2a})^2 \quad \Leftrightarrow \quad y - (-\frac{d}{4a}) = a(x - (-\frac{b}{2a}))^2$$

If we hold this together with $y = ax^2$ and the formulas $y = f(x)$ and $y - k = f(x - h)$ for the parallel displacement of the graph of a function, the we can see that the graph of $y = ax^2 + bx + c$ is a parallel displacement of the graph of $y = ax^2$ to

$$T = (h,k) = (-\frac{b}{2a}, -\frac{d}{4a}) \ , \qquad where \ d = b^2 - 4ac$$

This last formula is usually denoted the *top point formula*.

Example
Determine the top point for the parabola $y = -2x^2 + 3x + 1$, and describe which parabola it is a parallel displacement for..

$$T(-\frac{b}{2a}, -\frac{d}{4a}) = (-\frac{3}{-4}, -\frac{17}{-8}) = (\frac{3}{4}, \frac{17}{8})$$

The parabola is a displacement of $y = -2x^2$ to the top point $(\frac{3}{4}, \frac{17}{8})$

If you should draw the parabola $y=-\frac{1}{2}x^2+2x-3$ by hand (although no one does it any longer), then it is most easily done by finding the top point, which in this case is (2,1), and then from that point draw the parabola $y=-\frac{1}{2}x^2$.

2.3 Factorizing a quadratic polynomial

A number r is said to be a *root* in a polynomial $p(x)$ if $p(r) = 0$.
Determining the roots of a polynomial is the same as finding the intersection of $p(x)$ with the x-axis.
We shall then show some theorems about the roots in a quadratic polynomial, which as we know, may have two roots if $d > 0$, only one root if $d = 0$ or no roots if $d < 0$.

First we look at the case where $d>0$, where the polynomial has two roots r_1 and r_2.

r_1 and r_2 are roots in: $p(x)=ax^2+bx+c \quad \Leftrightarrow \quad ax^2+bx+c=0 \quad \Leftrightarrow$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \quad \Leftrightarrow \quad x=r_1 \vee x=r_2 \quad \Leftrightarrow$$
$$x-r_1=0 \vee x-r_2=0 \quad \Leftrightarrow \quad (x-r_1)(x-r_2)=0$$

By multiplying the parentheses and collecting the terms, we have:

$$x^2-(r_1+r_2)x+r_1r_2=0$$

If we compare it to: $x^2+\frac{b}{a}x+\frac{c}{a}=0$, we see that we must have:

$$r_1+r_2=-\frac{b}{a} \quad og \quad r_1\cdot r_2=\frac{c}{a}$$

This can be formulated as:

In the ordered (by decreasing powers of x) and reduced (the coefficient to x^2 is 1) quadratic equation, the sum of the roots is equal to the coefficient to x with opposite sign, and the product of the roots is equal to the last term in the equation.

This theorem is often used to guess the roots in a quadratic equation

Example

1) Guess the roots in the quadratic equation: $x^2 + 2x - 15$. We should think of two numbers having the sum -2 and the product -15. The only possibility is 3 and -5 (Since the equation can have at most 2 roots)

2) If the roots are not integral numbers it is only a little more difficult.
   $x^2+\frac{3}{2}x-1=0$. We can see that: $-2+\frac{1}{2}=-\frac{3}{2} \quad and \quad -2\cdot\frac{1}{2}=-1$, so the roots are -2 and $\frac{1}{2}$.

We have established above that:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \quad (x - r_1)(x - r_2)$$

Multiplying the equation by a, we find:

$$ax^2 + bx + c = a(x - r_1)(x - r_2)$$

This is called the *factorization* of the quadratic polynomial, and it is a special case of a more comprehensive theorem about factorization of higher degree polynomials .

If $r_1 = r_2 = r$, which corresponds to the case $d = 0$, then the polynomial has only one root, and the factorization becomes

$$ax^2 + bx + c = a(x - r)(x - r) = a(x - r)^2$$

If the polynomial has no roots, then it cannot be factorized into first degree polynomials.

Example

The polynomial $f(x) = -3x^2 - 3x + 6$ has the roots 1 and -2, and it can therefore be factorized:

$$f(x) = -3x^2 - 3x + 6 = -3(x+2)(x-1)$$

2.4. The sign variation for the quadratic polynomial.

The task is to determine the variation of the sign of a polynomial, which amounts to the same as solving the 3 inequalities.

$$ax^2 + bx + c = 0 \quad \text{and} \quad ax^2 + bx + c > 0 \quad \text{and} \quad ax^2 + bx + c < 0$$

It is possible to solve these inequalities algebraic, (but it is actually much easier to it graphically), so we settle for just one example $a < 0$, $d > 0$ and the roots are r_1 and r_2, and $r_1 < r_2$.

$$\begin{aligned}
&ax^2 + bx + c > 0 \quad \Leftrightarrow \quad a(x - r_1)(x - r_2) > 0 \quad \Leftrightarrow \\
&(x - r_1) > 0 \ \wedge \ (x - r_2) < 0 \quad \vee \quad (x - r_1) < 0 \ \wedge \ (x - r_2) > 0 \quad \Leftrightarrow \\
&x > r_1 \ \wedge \ x < r_2 \quad \vee \quad x < r_1 \ \wedge \ x > r_2 \quad \Leftrightarrow \\
&r_1 < x < r_2
\end{aligned}$$

So the polynomial is positive between the roots and otherwise negative.

Below we have shown the 6 possibilities of location of the graphs for the quadratic polynomial, arranged together with the sign of a, and the cases $d > 0$, $d = 0$ and $d < 0$.
Once you have identified which kind of polynomial you are dealing with, and found the roots, you may directly identify the intervals, where the polynomial is positive or negative.

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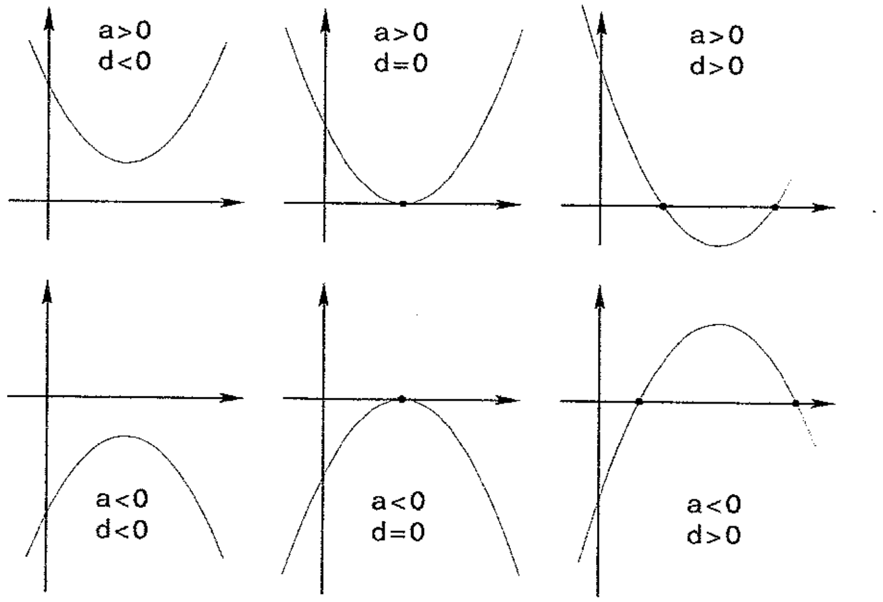
When $a > 0$ the branches go up, and when $a < 0$ the branches go down, and when $d > 0$ the polynomial intersects the x-axis in two points, $d = 0$ only in one point and when $d < 0$ there is no intersection with the x-axis



The method is best illustrated by doing some examples.

Example

1. Solve the inequality: $-3x^2 - 3x + 6 > 0$. The discriminant is: $d = 9+72=81 > 0$, so we have the case $a < 0$ and $d > 0$. From the figure above we see that the polynomial is positive between the roots, which are -2 and 1.

$$-3x^2 - 3x + 6 > 0 \quad \Leftrightarrow \quad -2 < x < 1$$

2. Solve the inequality: $2x^2 - 3x + 5 > 0$. $d = 9 - 40 = -31 < 0$. The case is then: $a > 0$ and $d < 0$, so the branches go up, and the parabola does not intersect the x-axis. So we therefore have.

$$2x^2 - 3x + 5 > 0 \quad \Leftrightarrow \quad x \in R$$

3. Solve the inequality: $4x^2 - 3x - 1 > 0$. $d = 9+16 = 25$ and the roots are 1 and $-\frac{1}{4}$. So we have the case: $a > 0$ and $d > 0$, The branches go up, and the parabola intersects the x-axis in two points.

$$4x^2 - 3x - 1 > 0 \;\Leftrightarrow\; x < -\frac{1}{4} \;\vee\; x > 1$$