Elementary Mathematics

Geometry with dividers and ruler

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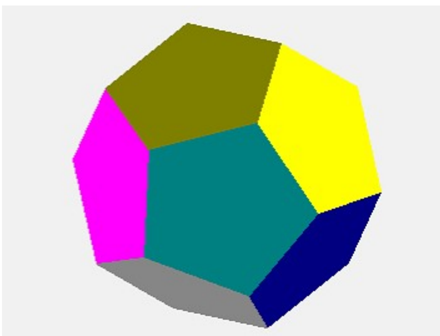

Ole Witt-Hansen 2006

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Chapter 1 The foundation of geometry

1. The axioms of geometry

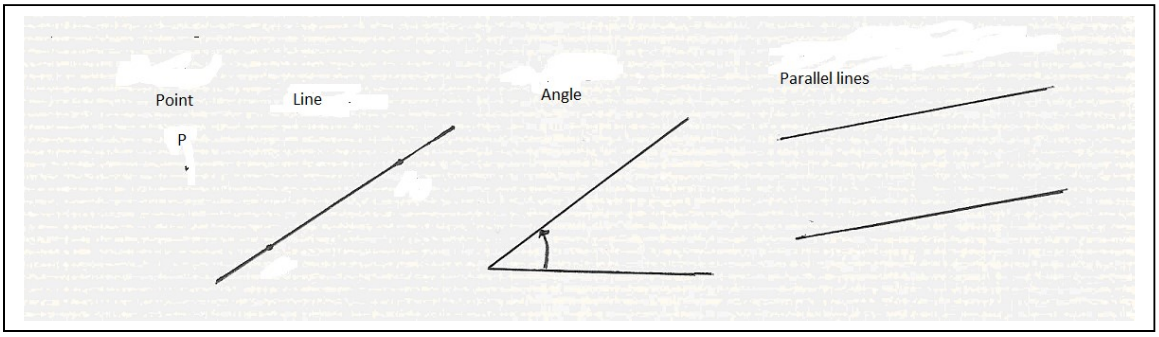
The classical geometry is concerned with properties of points, lines and geometrical figures in the plane. We shall not give an abstract rigorous definition of these geometrical objects, but settle for a more descriptive **definition**.

(1.1) A **point** has no extension, but only marks a position in the plane.

(1.2) A straight line segment is the shortest path between two points. If you extend the line beyond the two points you get a **straight line**. A line is only extended in one direction.

(1.3) Two semi-lines with a common origin are said to form an **angle**.

(1.4) Two lines are **parallel** if they do not intersect each other.



What applies to all mathematical endeavours, and is inherited from the axioms of Euclid, the geometry is built **axiomatic and deductively**. Axiomatic means that the theory is founded on some obvious assumptions called axioms. Axioms cannot be proven from more fundamental assumptions but one axiom can replace another, which makes the replaced axiom a theorem. **Deductively** means that the theory is built step by step from the axioms using logical inferences, called mathematical proofs. The inferences you proof are the called sentences or **theorems**.

The axioms may be chosen with certain arbitrariness. Theorems can be switched with axioms, which then become theorems. The axioms below are not the same as the ones proposed by Euclid but the content is the same.

(A1) Through two points can be drawn exactly one line.

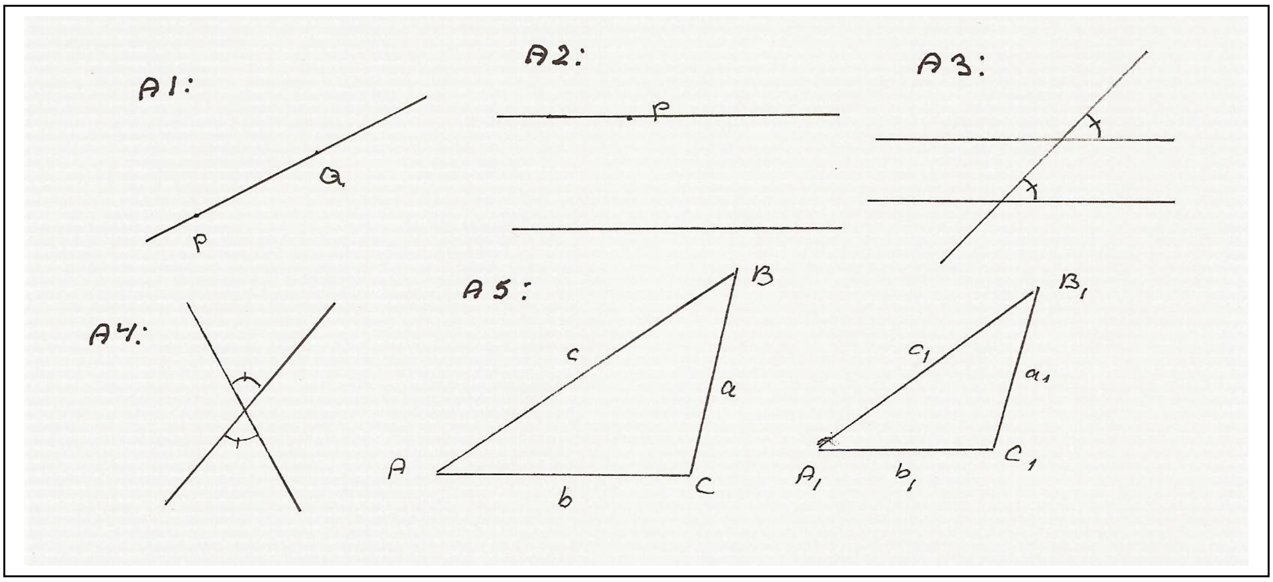
(A2) Through a point may be drawn exactly on line parallel to a given line.

(A3) When two parallel lines are intersected by a third line, then even lying angles are equal to each other.

(A4) When two lines intersect, then the opposite angles are the same.

(A5) The ratio between the even lying sides in two single-angled triangles is constant.

Below we have tried to illustrate the content of the five axioms. Notice especially A5.



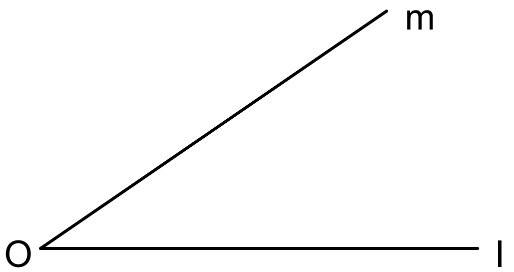
$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$$

Often the proportionality is written in another way:

$$\frac{a}{a_1} = \frac{b}{b_1} \Leftrightarrow \frac{a}{b} = \frac{a_1}{b_1}$$

Correspondingly for the two other sides.

1.1 Angles

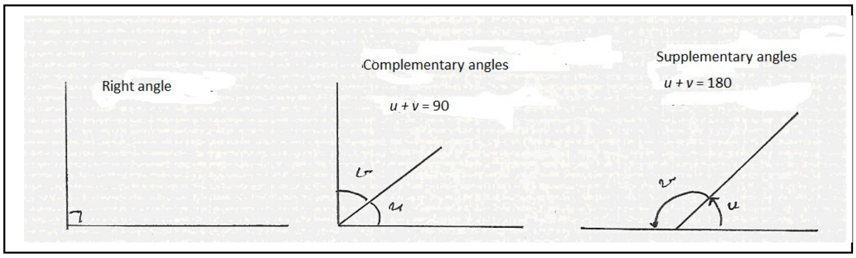


Two semi-lines having common origin are said to form an angle. The origin of the two semi-lines is said to be the vertex of the angle. Seen from the vertex, we speak of the left and right leg of the angle.
To define the number of degrees of an angle, we draw a circle with centre in the vertex *O* and arbitrary radius.
The circle arc is then divided in 360 equal pieces, where each one is one degree (1^0)
Using 1^0 as the unit, the arc *b* is then measured by the number of degrees spanned. The result is called the degrees of the angle. So angles are measured in degrees.

The number of degrees g^0 represents the same fraction of 360^0 that the arc *b* represents of the circumference of the circle. If *r* is the radius of the circle, this can be expressed as:

$$\frac{g^0}{360^0} = \frac{b}{2\pi r} \qquad \text{Or} \qquad g^o = \frac{b}{2\pi r} \cdot 360^o$$

The degrees of half the circumference is 180^o, corresponding to that the semi-lines *l* and *m* form a straight line. If the two semi-lines cover a quarter of the circle, then the degrees are 90^o. This is called a **right angle**. Two lines are then **perpendicular** to each other, and they are said to be **orthogonal**.



Two angles which together are 180^0 are called supplementary angles, and two angles which together are 90^0 are called complementary angles.

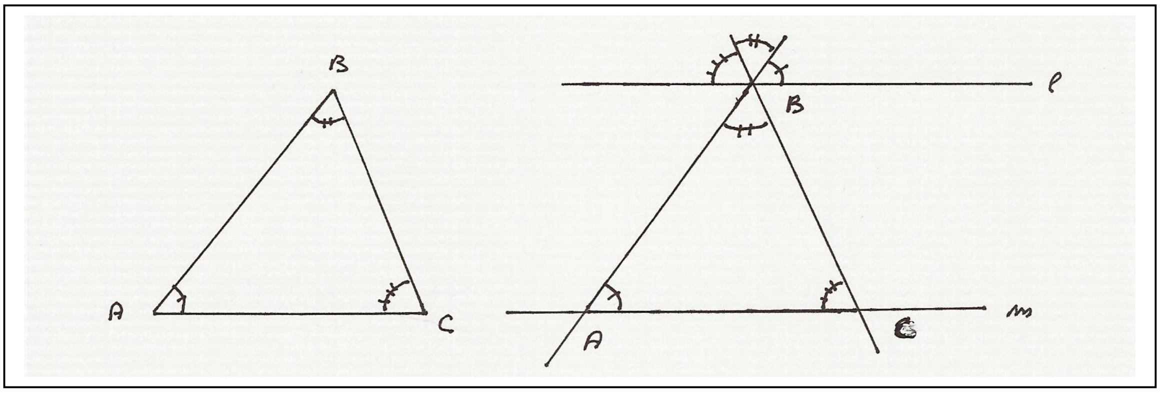
2. Some simple geometrical theorems

We shall first show a theorem, which appears obvious, which it also is, but we do it to demonstrate that it can be derived as a consequence of axiom *A2*. That theorem may in fact replace axiom *A2*, and this would make *A2* to a theorem. The reason for proving this theorem is also that it is frequently applied.

2.1 Theorem: If two lines both are parallel to a third line, then they are mutual parallel.

Proof: (indirectly).
If the two lines were not parallel, then they would intersect each other. Through the point of intersection, there are then *two* lines parallel with the third line; in contradiction to *A2.*Therefor the two lines must be parallel. The next theorem is well known.

2.2 Theorem: The sum of the three angles in a triangle is 180^o



The proof can be read from the figure above. There may be drawn only one line through *B* parallel to *AC*. (Axiom *A*2). Furthermore, the line segments *AC, AB* and *BC* are prolonged outside the

points *A, B* and *C*. The line *AB* intersects two parallel lines and therefore the even lying angles are the same. (Axiom *A*3).
The Line *BC* also intersects two parallel lines and we retrieve the angle *C* above *B*. Finally *AB* and *BC* intersect each other in *B*, and opposite angles are the same (axiom A4).
So we retrieve the angle *B* at the top of *B*. As it seen from the figure the sum of the three marked angles is 180^{o}, and therefore also $A + B + C = 180^{o}$.

Chapter 2 Geometrical constructions

1. The congruence sentences

Geometrical construction is traditionally conducted only with **dividers** and **rulers**. The ruler is used only for drawing straight lines. In principle the ruler is without printed measurement units. Measurement of a line segment is done with the dividers.
This has a more profound reason. In the trigonometry, we learn to calculate the unknown pieces in geometrical figures. The connection to the geometry is that, if you may calculate the unknown pieces in a geometrical figure by trigonometry, then it is in principle also possible to construct the geometrical figure with a dividers and a ruler.

Congruent means equal in a strict sense. If two figures are congruent it means that they may cover each other exactly, by a translation.
A translation can be a movement along a straight line, a rotation about a point, or a mirroring in a line. A combination of the three translations is also a translation.

2. The four sentences of congruence

K1: Two triangles are congruent if the have 3 sides in common.

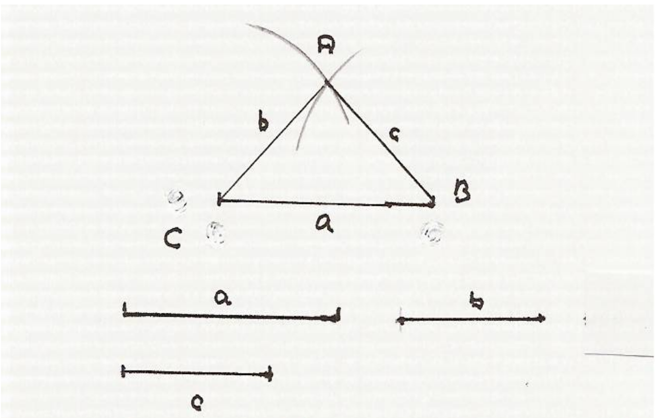
K2: Two triangles are congruent if the have 2 sides and the intermediate angle in common.

K3: Two triangles are congruent if they have 1 side and the two adjacent anglers in common.

K4: Two triangles are congruent if the have 1 side, an adjacent and the opposite angle in common.

The proof for the congruence sentences is conducted by constructing the triangle and verifying that there is one and only one solution.

Proof of K1: Below is drawn the three sides *a, b* and *c*.



Explanation: The side $a = |BC|$ is allocated. With the centre in *C* and radius *b*, is drawn a circle. With the centre in *B* and radius *c*, is drawn another circle. *A* is then determined as the intersection between the two circles. Since two circles can have at most two intersection points, the construction may have at most two solutions. From the construction it is evident that the two solutions are congruent by mirroring in the line through *BC*.

The triangle *ABC* is therefore uniquely determined by the three sides *a, b* and *c*, and two triangles having three sides in common are therefore congruent.

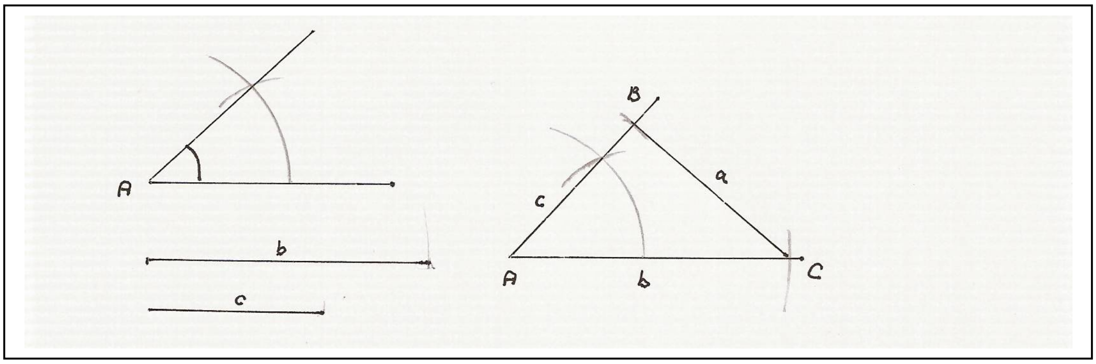
Discussion: From the construction it appears that a condition for solution is that $a < b + c$. A similar condition must apply for the two $c < a+b$ and $b < a + c$. If these inequalities are solved with respect to *c* one finds:

$$c < a + b \quad \wedge \quad c > a - b \quad \wedge \quad c > b - a$$

These 3 inequalities can be brought into one double inequality that in all cases is the condition for a solution.

$$|a - b| < c < a + b$$

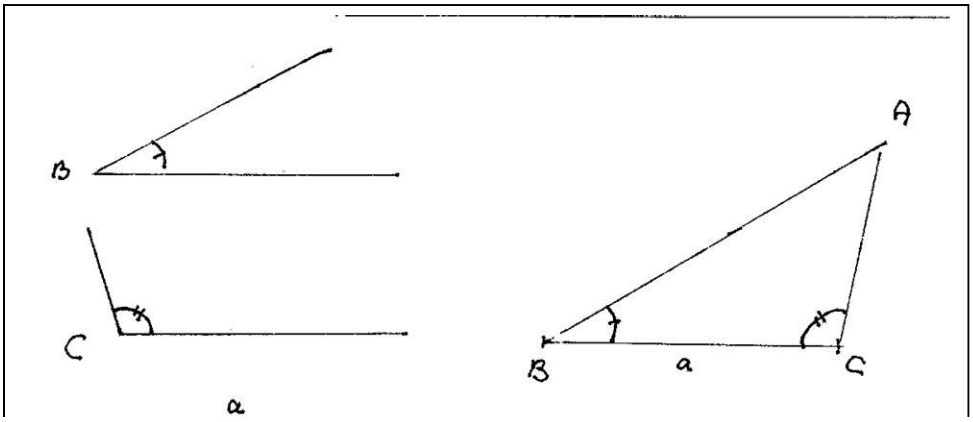
Proof of K2:



Explanation: The angle *A* and the two sides *b* and *c* are given. To construct the same angle with vertex in *A* is done the following. The side *b* with end points *A* and *C* is drawn
With the centre in the vertex of the given angle and in *A* are drawn circles with the same radius. The arc that the given angle spans on the circle is measured with the dividers, and is marked on the other circle. The side *c* is then drawn from *A* to the intersection point of the two circles. The end of *c* is then connected to *B* to finish the triangle. The construction has always one and only one solution.

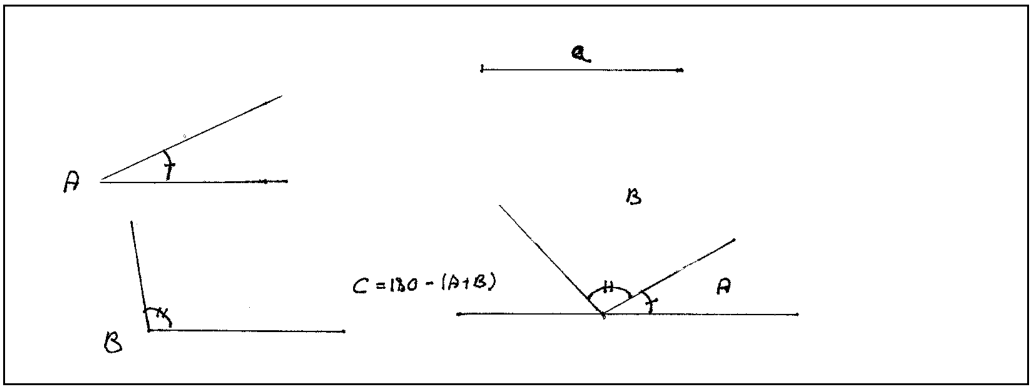
Remark: Above we have in detailed demonstrated how to construct an angle congruent with a given angle. In the following, we shall omit the details and just write e.g. the angle *A* is drawn.

Proof of K3: Below are the three given pieces. A side *a,* and the two adjacent angles. Also is shown the construction of the triangle.



Explanation: The side *a* is then drawn having the end points *B* and *C*. In *B* and *C* respectively are constructed the angle *B* and the angle *C*. *A* is the intersection point between the left leg of *B* and the right leg of *C*.
Discussion: The constructing has always only one solution if $B+C < 180^o$.

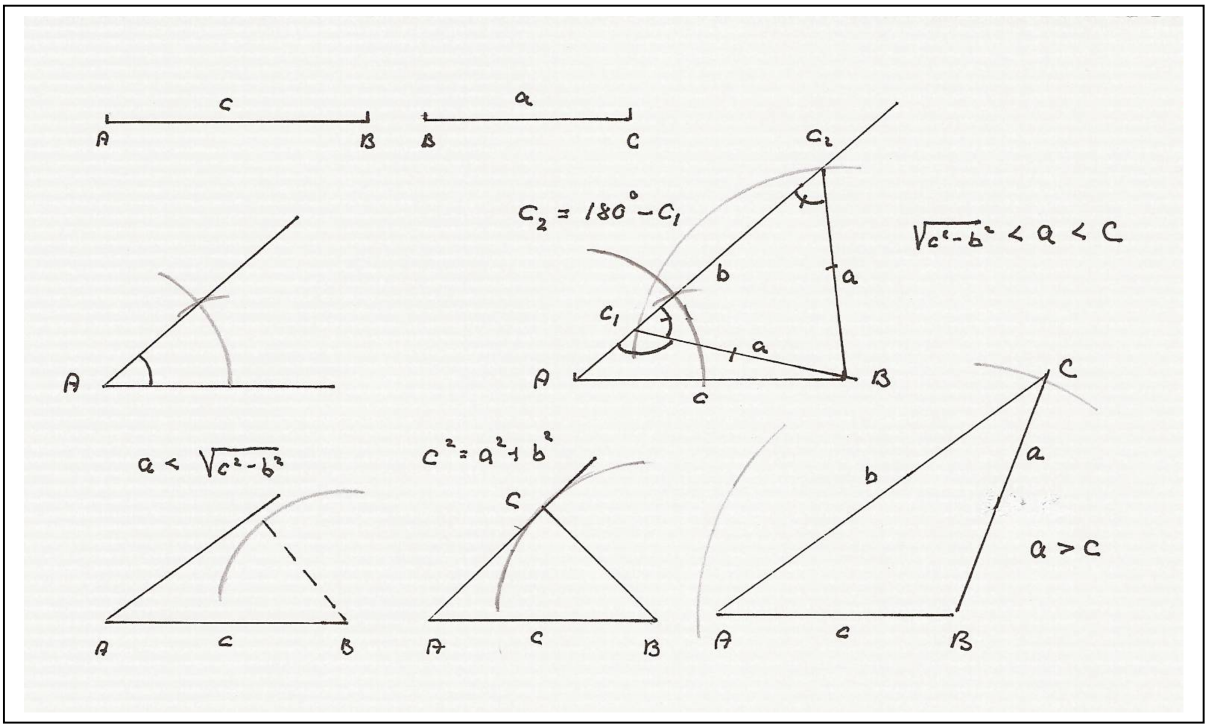
Proof of K4: If two triangles have two angles in common they have all three angles in common, so K4 can directly be related to K3, once you know how to construct an angle *w* being $180^o - (u+v)$, when *u* and *v* are given angles. How to do this is shown below.



Explanation: Let the side *a* together with the angles *A* and *B* be given. We want to construct the angle *C*.
The angle A is drawn and the right leg is prolonged to the left. The angle *B* is constructed from the left leg of *A*. The angle *C* is the found between the left leg of *B* and the prolongation of the right leg of *A*. Since now you know a side and two adjacent angles, *a, B* and *C*), the triangle may be constructed.

3. The five cases of constructing a triangle

In the geometry one speaks about the five cases of a triangle, and it refers to the five different ways to construct a triangle (with dividers and ruler), when 3 sides, two sides and an angle or one side and two angles are given. (Three angles is not an option).
The first four cases of a triangle are the four cases of congruence that we have treated above, where the construction has at most one solution, (apart from mirroring).
We shall then turn to the fifth case of a triangle, where an angle, an adjacent and an opposite side are given e.g. $\angle A$, c and a. Illustrated in the figure below.



Explanation: The angle A is drawn. From the right leg of A *is* drawn c, which also determines B. With the centre in B and radius a is drawn a circle. C lies where the circle intersects the left leg of A.

Discussion: From the construction appears that in the case chosen there are two solutions. The number of solutions depends however on the size of the pieces from which the triangle is constructed.

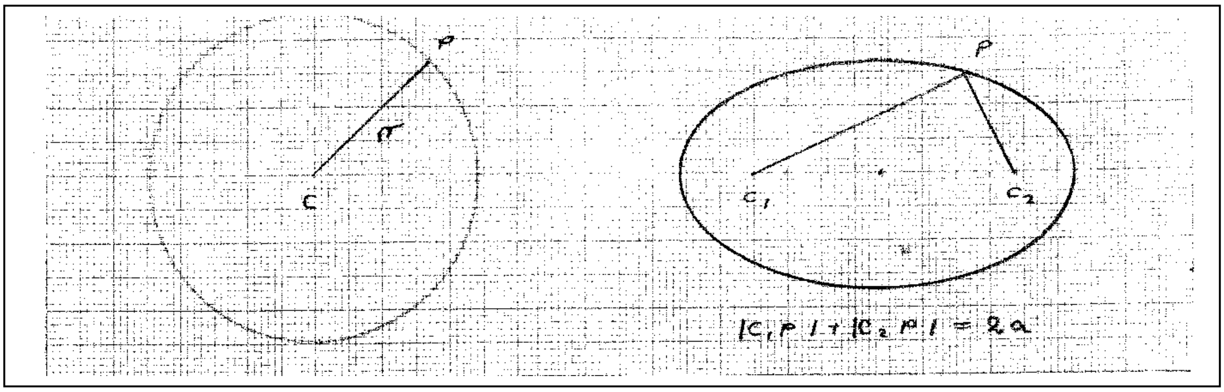
1. If the side a is too short, the circle then the circle will not intersect the left leg of $\angle A$ and the construction of the triangle has no solution. The limiting case is when the left leg of $\angle A$ is a tangent to the circle. In that case the triangle is right angled, and $a = \sqrt{c^2 - b^2}$, (*since* $c^2 = a^2 + b^2$).

2. If $a > c$ then the circle will only have one intersection with the left leg of $\angle A$.

If we summarize the results: If $a < \sqrt{c^2 - b^2}$ there are no solutions. If $a = \sqrt{c^2 - b^2}$, then there is only one solution. If : $\sqrt{c^2 - b^2} < a < c$, then there are two solutions, and if $a \geq c$ there is only one solution.

4. Geometrical locus

A geometrical locus is a somewhat antiquated designation for a set of points having a specific geometric property. We shall first define the concept of a circle and that of an ellipse.



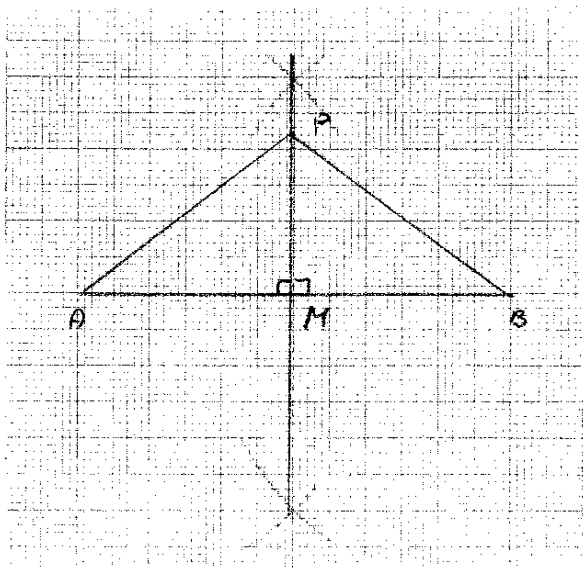
A **circle** is the geometrical locus for the points having the same distance from a given point *C*. The point is the **centre** of the circle and the distance is called the **radius**.

An **ellipse** is the geometrical locus for the points where the sum of the distances from two points is constant equal to $2a$. The two focus points are called the focus points for the ellipse and $2a$ is called the major axis of the ellipse.

A circle is, as you know, drawn using dividers, while a mathematical ellipse may be drawn by placing the end points of a cord with the length $2a$ at two points. With a pencil sliding in the cord, holding the cord tight, you let it pass through a circumference of 360^{o}. In this manner gardeners make elliptic flowerbeds.

4.1 Bisectors

A **bisector** is a line perpendicular to a line segment through its midpoint.
If the line segment is $|AB|$, then we shall show that every point on the bisector, has the same distance to A and B.



Proof: According to the definition $|AM| = |BM|$. P is an arbitrary point on the bisector.
The triangles ΔAPM and ΔBPM have the side $|PM|$ in common, furthermore $|AM| = |BM|$ and $\angle AMP = \angle BMP = 90^{o}$.
The two triangles therefore have two sides and the intermediate angle in common, and they are congruent, according to K2. Therefore also $|AP| = |BP|$, which completes the proof.

The bisector can therefore be described as the geometrical locus for the points which have the same distance from two given points.
This definition is used, when constructing the bisector on a given line segment.

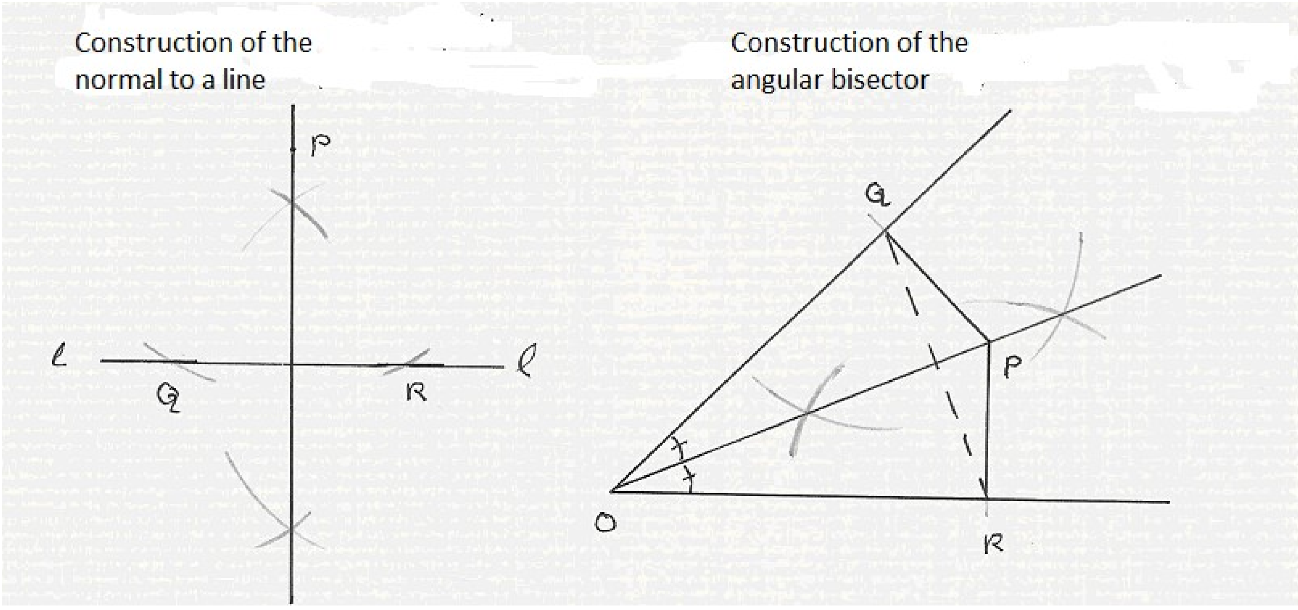
The construction of the bisector on a line segment AB is then performed as follows.
With the centre in A and B respectively and with radius $r > \frac{1}{2}|AB|$ are drawn two circles. The circles will intersect each other in two points which have the same distance to A and B, and they therefore lie on the bisector. If you draw the line through these points you have constructed the bisector on AB.

The same construction may be applied to establish the perpendicular in a point P. If the dividers are placed in the point P, and are used to mark two points A and B at the same distance on each side of the point P, then the bisector on AB will be the perpendicular to AB through P.

4.2 Angular bisector

By the angular bisector we understand a line which divides the angle in halves.
We shall then show that the **angular bisector** is the geometric locus for the points that have the same distance to the two semi-lines that form the angle.

Before we show this, we must, however, specify what we understand by the distance from a point to a line. A *normal* to a line is a (semi-) line perpendicular to it.
For the distance from a point to a line, we therefore understand the distance measured on a normal to the line, through the point.



In the figure to the left is shown how to construct the distance from a point *P* to a line *l*.
With centre *P*, is drawn a circle which intersects the line *l* in two points *Q* and *R*. These two points have the same distance from *P*, so *P* lies on the bisector for *QR*. If the bisector for *QR* is constructed, we have at the same time constructed a normal to *l* through *P*.

In the figure to the right, we assumed that we have the angular bisector for the angle *O*.
We shall start by proving that that any point on the angular bisector has the same distance to the two legs of the angle. In the figure we shall prove that $|PQ| = |PR|$.

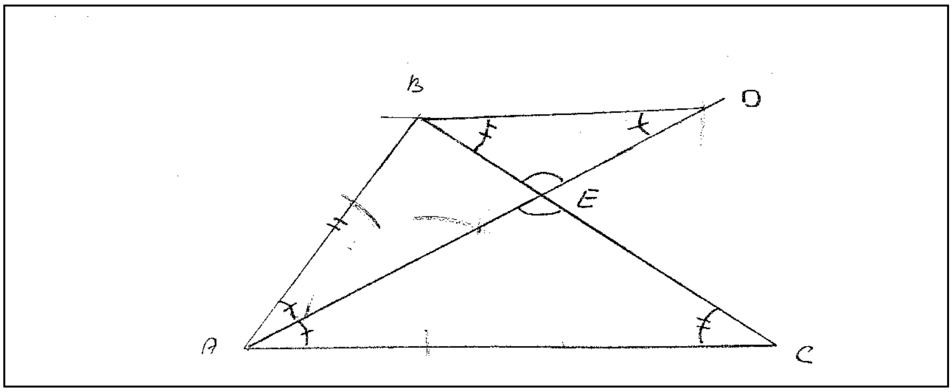
If we consider the triangles ΔOPQ and ΔOPR, then they have the side $|OP|$ in common, they are both right angled, and $\angle POQ = \angle POR = ½ \angle O$. Thus the triangles have a side $|OP|$, an adjacent and an opposite angle in common, so they are congruent: $|PQ| = |PR|$. Thus we have proved:

The angular bisector is the geometrical locus for the points, which have the same distance to two lines.

Furthermore we have $|OQ| = |OR|$ (because the triangles are congruent). The points *O* and *P* lie at the same distance from *Q* and *R*, and they therefore lie on the bisector for *QR*.
This can be applied to construct the angular bisector for a given angle as follows:

With the dividers and with centre at *O* is marked two points *Q* and *R* on the legs of the angle at the same distance from *O*. When constructing the bisector on *QR*, one has at the same time constructed the angular bisector for the angle *O*.
About the angular bisector for one of the angles in a triangle the following theorem applies.

The angular bisector divides the opposite side in the same ration as the two adjacent sides



Proof: In the triangle *ABC* we have drawn the angular bisector to the angle *A*, and it is prolonged out over the side *a*. Through *B* is drawn a line parallel with the side *b*. The intersection between that line and the angular bisector is *D*. We can now see that $\angle DAC = ½A = \angle ADB$.
Furthermore the angle $\angle CBD = \angle BCA$, so $\Delta AEC \sim \Delta DEB$. From this follows:

$$\frac{|AC|}{|DB|} = \frac{|EC|}{|EB|}$$

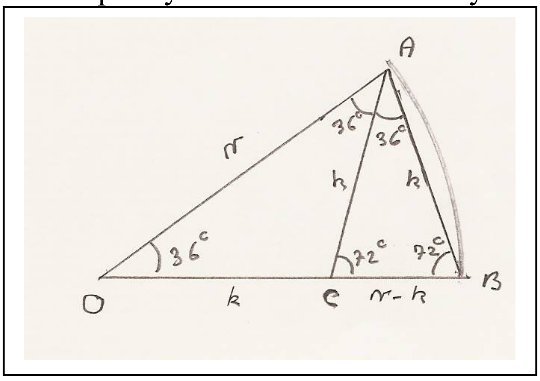
But $\angle DAB = \angle ADB$, so ΔABD is an isosceles triangle $|AB| = |DB|$. It then follows

$$\frac{|AC|}{|AB|} = \frac{|EC|}{|EB|},$$

This completes the proof.

5. Construction of the chords k_{10} and k_5.

The length of the chord in a regular decagon is designated k_{10}. It spans an arc $360^0/10 = 36^0$.
For simplicity we denote the chord by *k*.



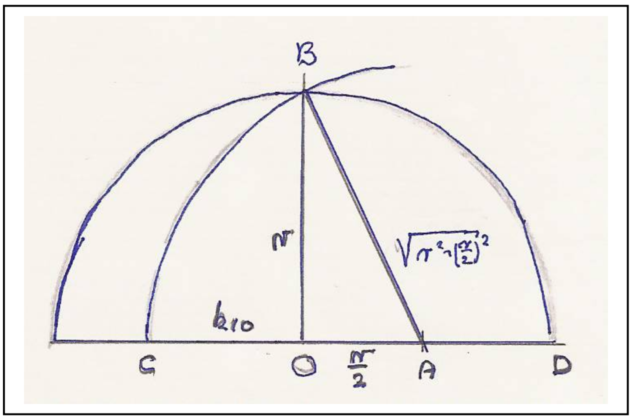
In the figure is drawn the chord as base in an isosceles triangle where the sides are the radii *r*, and the angle with vertex *O* is 36^0.
The angles at the base line are therefore 72^0. If one of these angles are halved by drawing the line *AC,* then *ΔABC* becomes isosceles, since $B = C = 72^0$.
So $|AC| = k$. *ΔACO* is also isosceles, since $O = \angle OAC = 36^0$, so that $|OC| = |AC| = k$.
We shall then apply the theorem about the angular bisector in the triangle *ΔOAB*.

$$\frac{|AO|}{|AC|} = \frac{|OC|}{|CB|} \quad \Leftrightarrow \quad \frac{r}{k} = \frac{k}{r-k} \quad \Leftrightarrow \quad k^2 + rk - r^2 = 0$$

Solving this quadratic equation discarding the negative solution:

$$k = \frac{-r + \sqrt{r^2 + 4r^2}}{2} = \frac{r}{2}(\sqrt{5} - 1)$$

This expression can then be used to construct k_{10} using dividers and ruler.



First is drawn a semi circle having radius r. At the centre O is created the normal, intersecting the circle in B. A is the midpoint of OD. Having centre in A is then drawn a circle through B, which intersects the diameter in the point C.

The radius in this circle is $|AB| = \sqrt{(\frac{r}{2})^2 + r^2} = \frac{r}{2}\sqrt{5}$.

Thus we find: $|OC| = \frac{r}{2}\sqrt{5} - r = \frac{r}{2}(\sqrt{5} - 1)$,

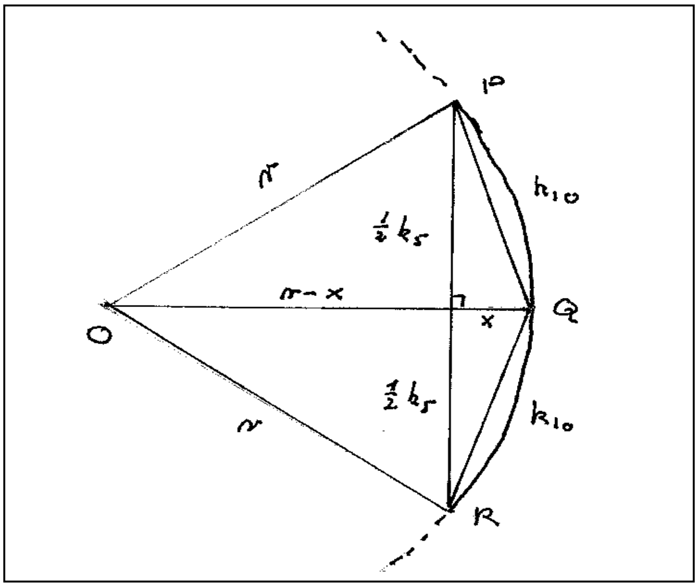
According to the result above this is the length of the chord k_{10}.

Having constructed k_{10}, it is easy to construct k_5, which is the side in the very familiar pentagon, since having constructed the decagon, you may just connect the line between two corners jumping one corner in the decagon. In the same manner we may construct k_{20} by halving k_{10} and then draw a line from the midpoint to the centre of the circle and then find the intersecting point of this line with the circle.

Then we have divided the angle 36^0 into two halves, giving the chord k_{20}.

It is only slightly more complicated to establish an expression for k_5 without using trigonometry.

We draw a circle with radius r and allocate two adjacent chords k_{10} .The line connecting the end points of the two chords is k_5.



We consider two right angled triangles. One triangle with the hypotenuse OP and another with the hypotenuse PQ. According to Pythagoras' theorem: (We have put r =1).

$$(1-x)^2 + (\tfrac{1}{2}k_5)^2 = 1 \;\Leftrightarrow\; x^2 - 2x + (\tfrac{1}{2}k_5)^2 = 0$$

and:

$$x^2 + (\tfrac{1}{2}k_5)^2 = k_{10}^2$$

By subtracting the upper equation from the lower equation we have:

$$2x = k_{10}^2 \;\Leftrightarrow\; x = \tfrac{1}{2}k_{10}^2$$

Inserting this in the lower equation, we get:

$(\tfrac{1}{2}k_5)^2 = k_{10}^2 - \tfrac{1}{4}k_{10}^4 = \tfrac{1}{4}k_{10}^2(4 - k_{10}^2)$. And then inserting $k_{10} = \tfrac{1}{2}(\sqrt{5} - 1)$ and reducing:

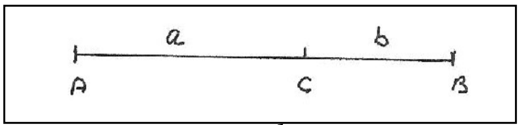
$$(\tfrac{1}{2}k_5)^2 = \tfrac{1}{4}k_{10}^2(4+k_{10}^2) = \tfrac{1}{4}(\tfrac{1}{2}(\sqrt{5}-1))^2(4-(\tfrac{1}{2}(\sqrt{5}-1))^2)$$
$$k_5{}^2 = \tfrac{1}{4}(5+1-2\sqrt{5})(4-\tfrac{1}{4}(5+1-2\sqrt{5})) = \tfrac{1}{2}(3-\sqrt{5})(4-\tfrac{1}{2}(3-\sqrt{5})) = \tfrac{1}{4}(3-\sqrt{5})(8-3+\sqrt{5})$$
$$k_5{}^2 = \tfrac{1}{4}(3-\sqrt{5})(5+\sqrt{5}) = \tfrac{1}{4}(15-5-2\sqrt{5}) = \tfrac{1}{4}(10-2\sqrt{5})$$

$k_5 = \tfrac{1}{2}\sqrt{10-2\sqrt{5}}$ And using the radius *r*. $k_5 = \dfrac{r}{2}\sqrt{10-2\sqrt{5}}$

Which is recognized as the same expression, as is found using trigonometry.

6. The golden cut

The golden cut is the designation for dividing a line segment into two pieces *a* and *b*, which fulfils the relation:



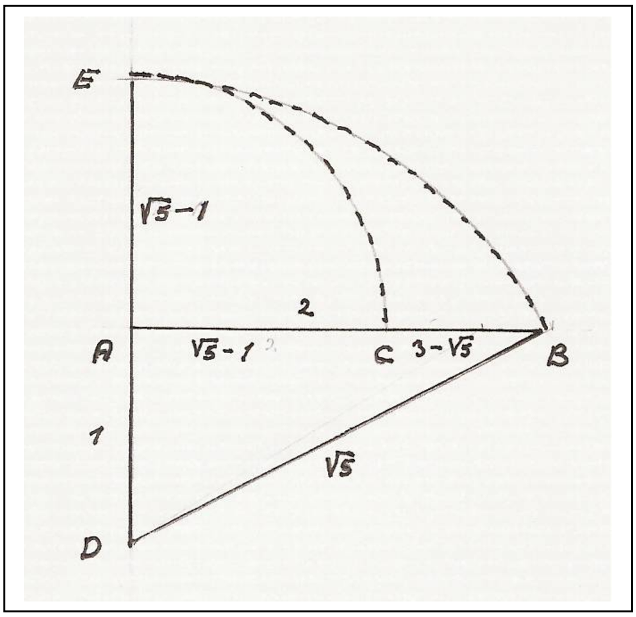
$$\frac{a}{b} = \frac{a+b}{a}$$

The equation $\dfrac{a}{b} = \dfrac{a+b}{a}$ can be rewritten as: $\dfrac{a}{b} = 1+\dfrac{b}{a}$. If we put: $\dfrac{a}{b} = x$ then the golden cut will fulfil the equation:

$$x = 1+\frac{1}{x} \Leftrightarrow x^2 - x - 1 = 0 \Leftrightarrow x = \frac{1\pm\sqrt{5}}{2}.$$

The ratio between the two pieces is therefore: $\dfrac{a}{b} = x = \dfrac{1+\sqrt{5}}{2} \approx 1{,}618$

The geometrical division of a line segment in the golden cut is not entirely obvious. The construction below can be traced back to Euclid.



The line segment is *AB*. Without restriction, (and because it facilitates the calculations), we put
$|AB| = 2$.
In the point *A* is raised a normal to the side *AB*, and we mark the distance1 downwards on the normal. The endpoint is marked *D*. *ΔABD* is right angled having the catheti 1 and 2.
Thus we have: $|BD| = \sqrt{1^2+2^2} = \sqrt{5}$.
With the centre in *D* and radius $|BD| = \sqrt{5}$ is then drawn a circle. The point where the circle intersects the normal in *A* is *E*. We see that $|AE| = \sqrt{5}-1$. With centre in *A* and radius $|AE|$ is drawn a circle. The intersection with *AB* is *C*.

The assertion is that *C* divides *AB* in the golden cut.

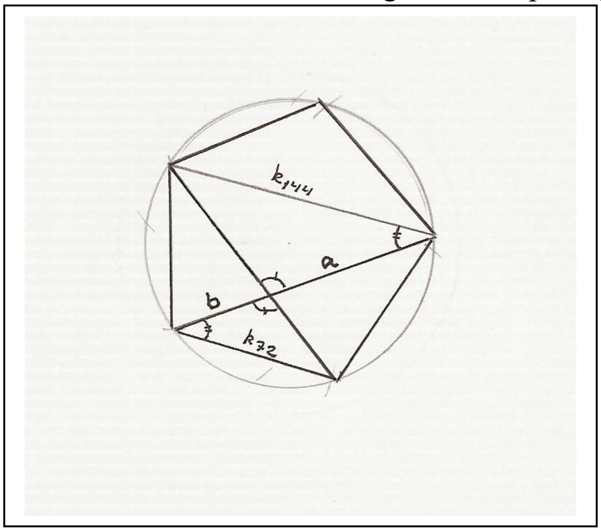
$$|AC| = |AE| = \sqrt{5}-1 \quad \text{and consequently} \quad |CB| = 2 - |AC| = 3-\sqrt{5}.$$

We then calculate the ratio:

$$\frac{|AC|}{|CB|} = \frac{\sqrt{5}-1}{3-\sqrt{5}} = \frac{(\sqrt{5}-1)(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{5-3+2\sqrt{5}}{3^2-5} = \frac{2+2\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2}$$

Recognized as the golden cut!

The golden cut comes up in numerous connections, especially in classical painting, in nature, but also in more imaginative and speculative areas. We shall not deal with the latter, but restrict ourselves to show that the diagonals in a pentagon are divided in a ratio that is the golden cut.



The side in a pentagon is a chord which spans a angle of 72^0. According to the chord formula:

$$k = 2R\sin\frac{v}{2}$$

The side in a pentagon is: $k_{72} = 2R\cdot\sin 36$.
The diagonal in a pentagon is a chord which spans an angle of 144^0, so the length of the diagonal is $k_{144} = 2R\cdot\sin 72$. A diagonal is parallel to the opposite side in the pentagon
The two triangles shown in the figure are therefore singled-angled triangles, so the ratio between k_{144} and k_{72} are the same as the ratio between the pieces a and b.

$$\frac{k_{144}}{k_{72}} = \frac{a}{b} \quad \Leftrightarrow \quad \frac{a}{b} = \frac{2R\sin 72}{2R\sin 36} = \frac{\cos(90-72)}{\sin 36} = \frac{\cos 18}{2\sin 18\cos 18} = \frac{1}{2\sin 18}$$

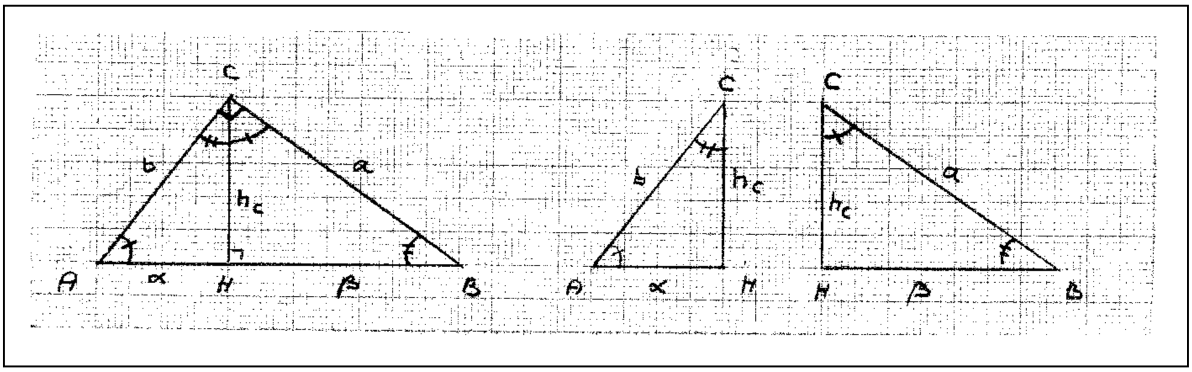
We have applied the formula: $\sin 2v = 2\cdot\sin v\cdot\cos v$. $\sin 18^0$ is, however, already known for the geometrical derivation of the chord k_{10} (the chord in a regular decagon). $\sin 18 = \frac{\sqrt{5}-1}{4}$.
When this is inserted we find:

$$\frac{a}{b} = \frac{k_{144}}{k_{72}} = \frac{1}{2\sin 18} = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2} \quad \text{(The golden cut)}$$

Chapter 3. Triangles and squares

1. The right angled triangle

A right angled triangle is a triangle, where one of the angles is 90^0. Traditionally the right angle is denoted *C*, so that the opposite side, the hypotenuse is *c*. The two other angles are *A* and *B*, and their opposite sides *a* and *b* are called catheti.



The height h_C is a normal from *C* on the side *c*. In the figure *H* is its foot point on *AB*.
Since $C = 90^o$ and $A + B + C = 180^o$ we have $A + B = 90^o$ or equivalently $B = 90^o - A$ or $A = 90^o - B$. In the figure both of the triangles ΔACH and ΔCBH are right angled, since $H = 90^o$ in both cases.
Furthermore $\angle ACH$ (the angle *C* in ΔACH) is equal to 90^o - *A* equal to *B*, so $\angle ACH = \angle B$.
In the same manner is seen that $\angle BCH = \angle A$ in ΔCBH. The two triangles *ACH* and *CBH* are redrawn to the right in the figure. What we see is that these two triangles are singled-angled, since they both have a right angle, an angle *A* and an angle *B*. Written symbolically:

(4.1) $$\Delta ABC \sim \Delta ACH \sim \Delta CBH$$

According to the axiom A5, the ratio between even lying sides is constant for these three triangles. Traditionally the pieces in which the height h_C divides the hypotenuse are called α and β.

To derive some sentences for the right angled triangle we shall establish the complete set of ratios, which are a consequence of the three even angled triangles. To write these ratios, we need only to look at (4.1) and not at the figure. We do not intend to use all the equations, but it is close to impossible to know which ones before the equations are established.

$$\Delta ABC \approx \Delta ACH \Rightarrow \frac{|AB|}{|AC|} = \frac{|AC|}{|AH|} = \frac{|BC|}{|CH|} \Leftrightarrow \frac{c}{b} = \frac{b}{\alpha} = \frac{a}{h_c}$$

$$\Delta ABC \approx \Delta CBH \Rightarrow \frac{|AB|}{|CB|} = \frac{|AC|}{|CH|} = \frac{|BC|}{|BH|} \Leftrightarrow \frac{c}{a} = \frac{b}{h_c} = \frac{a}{\beta}$$

$$\Delta ACH \approx \Delta CBH \Rightarrow \frac{|AC|}{|CB|} = \frac{|AH|}{|CH|} = \frac{|CH|}{|BH|} \Leftrightarrow \frac{b}{a} = \frac{\alpha}{h_c} = \frac{h_c}{\beta}$$

From the third line we find: $\dfrac{\alpha}{h_c} = \dfrac{h_c}{\beta}$, which gives:

$h_c^2 = \alpha\beta$ This equation expresses that:

In a right angled triangle, the height is the geometrical mean between the pieces in which it divides the hypotenuse.
(The geometrical mean between two numbers *a* and *b* is $\sqrt{ab}$. The arithmetic mean is $\frac{1}{2}(a+b)$)

The sentence can be applied if someone wants to construct the geometrical mean between two pieces. However it requires also the knowledge of the concept of the perspective arc over a line segment. We shall return to this issue later.
From the first line we have: $\dfrac{c}{b} = \dfrac{b}{\alpha} \Leftrightarrow b^2 = c\alpha$ and from the second line: $\dfrac{c}{a} = \dfrac{a}{\beta} \Leftrightarrow a^2 = c\beta$.
If these two lines are added:

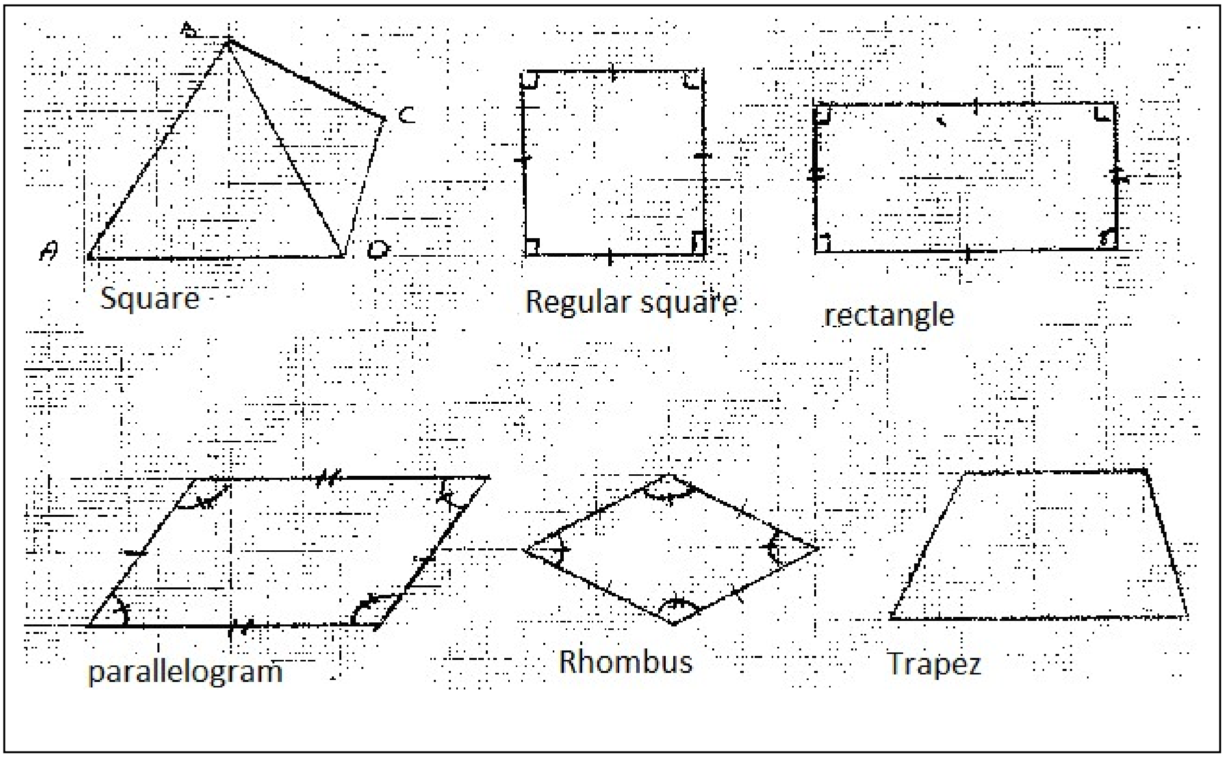
(4.2) $a^2 + b^2 = c\beta + c\alpha = c(\beta + \alpha) = c\,c = c^2 \quad \Rightarrow \quad a^2 + b^2 = c^2$

Pythagoras theorem is probably the most well known mathematical theorem of all.

In a right angled triangle the square of the hypotenuse is equal to the sum of squares of the catheti.

Notice that Pythagoras' theorem is founded only on the axiom A5, (and a little ingenuity).

2. Squares



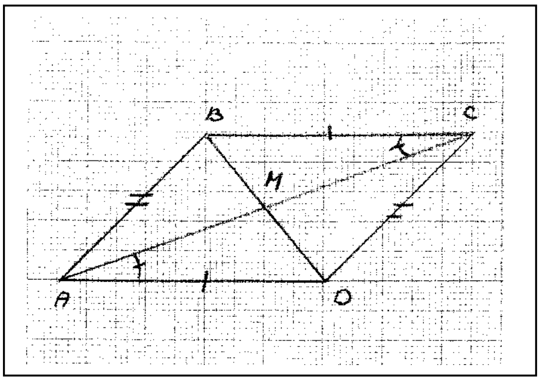
The figure above shows an arbitrary square together with 5 special squares having the well known names as shown in the figure. Their properties should be well known, as they appear from the figure. We shall then show some small sentences.

The sum of the angles in an arbitrary square is 360^{o}.

A line which connects two opposite angles in a square is called a **diagonal**. In the first figure is drawn a diagonal between *B* and *D*.
The diagonal divides the square in two triangles. The sum of the angles in the square is equal to the sum of the angles in the triangles *ABD* and *BDC*. The sum of the four angles is thus 360^{o}.

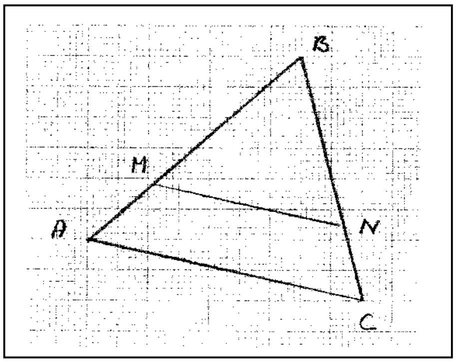
It is then easy to show that the sum of angles in an *n*-sided polygon is (n-2)*180^{o}. The *n*-side polygon can namely be separated into *n*-2 triangles, by drawing diagonals. The sum of angles in the *n*-sided polygon is thus the sum of the angles in the *n*-2 triangles.

In a parallelogram the diagonals halves each other.



In the figure is the intersection point of the diagonals is denoted *M*. $\angle CAD = \angle ACB$ caused by the axioms A3 and A4. Correspondingly $\angle DBC = \angle BDA$. Furthermore $|BC| = |AD|$ and $|AB| = |CD|$ (according to the definition of a parallelogram). Then it then follows that ΔAMD is congruent with ΔCMB, (according to K3).
Then we can conclude that $|AM| = |CM|$ and $|MD| = |MB|$. The diagonal AC halves the diagonal BD. In the same manner we may show that the diagonal *BD* halves the diagonal *AC*.
Thus we have shown that in a parallelogram the diagonals halve each other.

3. Transverses



In the triangle *ABC* is drawn a line segment *MN*, which is parallel to *AC*. Such a line is called a **transverse**.
Since $\Delta ABC \sim \Delta MBN$ we have:

$$\frac{MB}{AB} = \frac{BN}{BC} = \frac{MN}{AC}$$

Especially if $|MB| = ½|AB|$. It follows that $|BN| = ½\,|BC|$ and $|MN| = ½|AC|$. The transverse is then called a mid point transverse, since it connects the middle of the sides *AB* and *BC*. Notice that the length of the mid point transverse is $½|AC|$.

Conversely we have the **midpoint transverse theorem**:

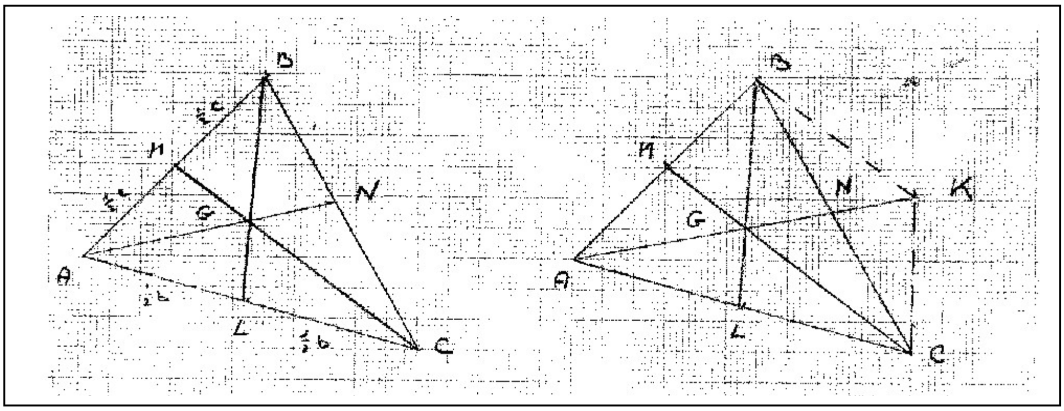
If a line connects the midpoints of the sides in a triangle it will be parallel with the third side and having half the length.

This can be derived from the fact, that there is only one line which connects the midpoints of the two sides in a triangle and that is the midpoint transverse.

4. Medians

A line which connects the vertex of an angle to the mid point of the opposite side is called a **median**. We shall then prove the theorem:

The three medians in an arbitrary triangle intersect in the same point, and the intersection point divides any of the medians in the ratio 2:1, measured from the angle.



Proof: Let *L* and *M* be the midpoints of the two sides *b* and *c*. We draw the medians *BL* and *CM*. Their point of intersection is *G*. We then draw the line from *A* through *G*. (But we do not know if it is a median…yet)). The line intersects *BC* in *N*. (We do not know if *N* is the midpoint of *BC* yet) We then prolong the line beyond the point *N* to the point *K*, such that $|AG| = |GK|$.

We then consider the triangle *ABK*. Here *GM* is a midpoint transverse, since *M* is the midpoint of *AB* and *G* is the midpoint of *AK* (according to the construction). It then follows:

$$|GM| = \tfrac{1}{2}|BK| \quad \text{and} \quad GM \neq BK \quad \Leftrightarrow \quad CG \neq BK.$$

In quite the same manner we consider the triangle ΔAKC:

$$|GL| = \tfrac{1}{2}|KC| \quad \text{and} \quad GL \neq KC \quad \Leftrightarrow \quad BG \neq KC.$$

So we have: $CG \neq BK$ and $BG \neq KC \Leftrightarrow$ *CGBK* is a parallelogram.

In a parallelogram the diagonals halves each other, so *N* must be the midpoint *BC*.
The line from *A* through *G* must therefore be the median from *A*.

The medians in a triangle intersect in the same point.

That was the first part of the theorem.

Since *GM* is the midpoint transverse in Δ*ABK,* then |*GM*| = ½|*BK*| = ½|*CG*|. (|*BK*| = |*CG*|, since in a parallelogram the opposite two sides are equal to each other).

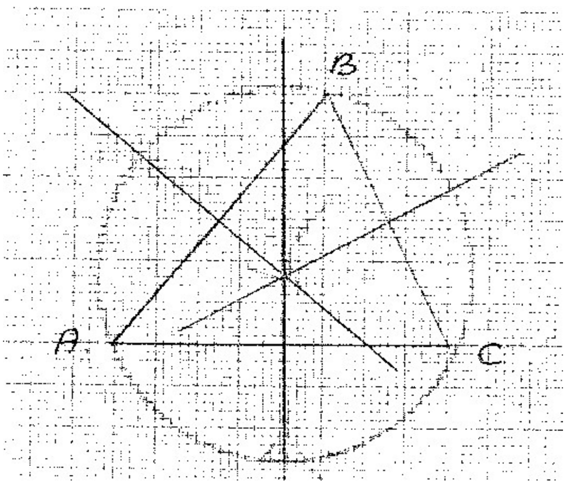
So *G* divides the midpoint median *CM* in the ratio 2:1. In the same manner we find that *G* divides the median *BL* in the ratio 2:1, and this holds true for the median *AN* according to the construction.

The intersection point of the three medians divides any of the medians in the ratio 2:1, measured from the angle.

And that was the second part of the theorem.

5. The circumscribed circle of the triangle

The bisectors in a triangle intersect in the same point, and this point is the centre of the circumscribed circle of the triangle.

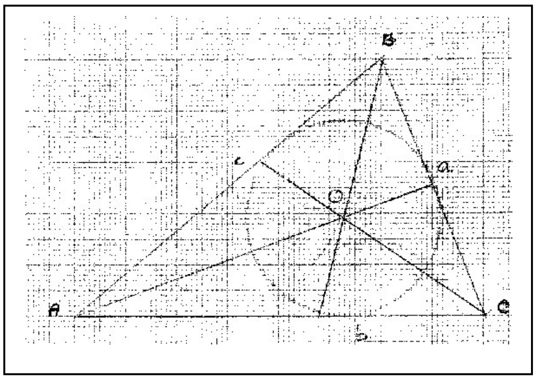


Proof: In the figure are constructed the bisectors on the sides *AB* and *BC*. Their intersection is *O*. We have then: |*OA*| = |*OB*| and |*OB*| = |*OC*|, from which follows that: |*OA*| = |*OC*|. Thus *O* lies on the bisector for *AC*, so the three bisectors in a triangle intersect in the same point.
Since *A, B* and *C* are situated in the same distance from *O* they lie on a circle with centre *O*, and this is called the circumscribed circle of the triangle.

5.1 The inscribed circle of a triangle

By the inscribed circle of a triangle, we understand a circle which tangents the three sides of the triangle. We shall prove:

The angular bisectors in a triangle intersect each other in the same point, and this point is the centre of the inscribed circle of the triangle.



Proof: In the figure there are constructed the angular bisectors from *A* and *C*. Their point of intersection is *O*. *O* has the same distance from the two sides *b* and *c*, since *O* lies on the angular bisector from *A*.
The point *O* has also the same distance from the two sides *a* and *b*, since *O* lies on the angular bisector from *C*. Dist(*O,b*) = Dist(*O,c*) and Dist(*O,a*) = Dist(*O,b*), so Dist(*O,c*) = Dist(*O,a*), and *O* must also lie on the angular bisector from *B*. The angular bisectors thus intersect in the same point.

If one draws a circle with centre in O, having radius $r = |aO| = |bO| = |cO|$, (where $|aO|$ is the distance from the side a to the point O etc.), then this circle will tangent the three sides in the triangle, and it is called the **inscribed circle of the triangle**.

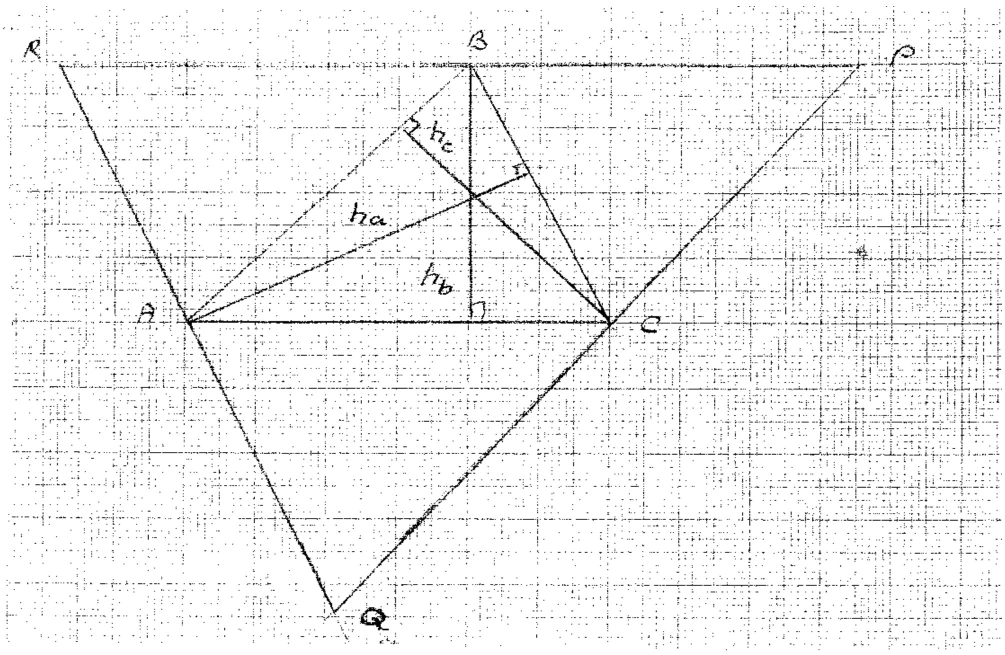
In connection with the inscribed circle, we shall prove a small theorem about the relation between the area of a triangle T, the radius r of the inscribed circle, and half the perimeter of the triangle $s = ½(a + b + c)$.

$$T = r \cdot s$$

If the triangle is divided into three triangles AOC, COB and AOB, then the area of each of these triangles can be calculated as (half the height times the baseline) $\frac{1}{2} rb$, $\frac{1}{2} ra$ and $\frac{1}{2} rc$.
So the area of the triangle is

$$T = \tfrac{1}{2} rb + \tfrac{1}{2} ra + \tfrac{1}{2} rc = \tfrac{1}{2} r(b + a + c) = \tfrac{1}{2} r2s = r \cdot s$$

6. Heights



We shall then show that:

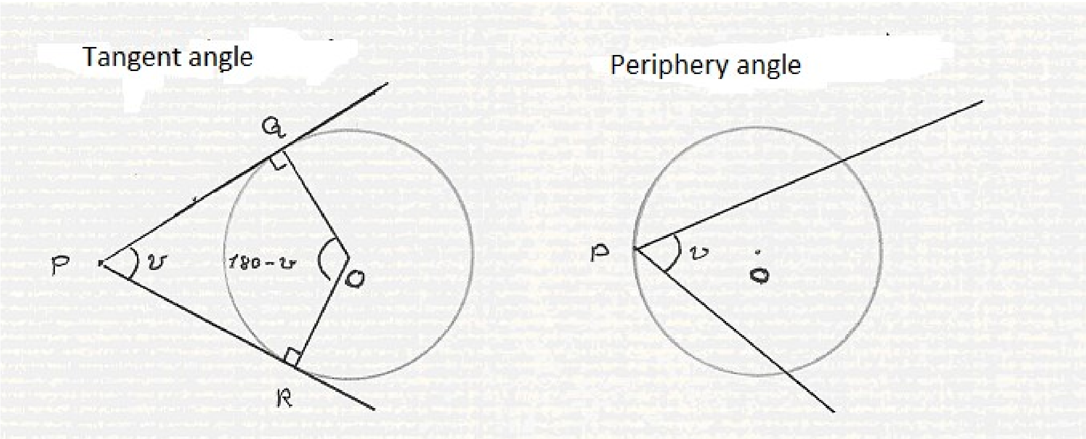
In a triangle the three heights intersect each other in the same point.

The proof is a bit more circumstantial than the proofs concerning bisectors and angular bisectors, and the proof requires a trick.
Through the three angular corners of the triangle are drawn lines parallel to the opposite sides. These lines form a new triangle PQR.
First we consider the polygon $ABPC$. In this polygon we have $BP \neq AC$ and $AB \neq PC$. ("$\neq$" means parallel) The polygon is a parallelogram so $|BP| = |AC|$ and $|PC| = |AB|$. Next we consider the polygon $ARBC$. In this polygon we have $AR \neq BC$ and $RB \neq AC$.
The polygon is a parallelogram such that $|RB| = |AC|$ and $|AR| = |BC|$.

Holding the two polygons together $|BP| = |AC|$ and $|RB| = |AC|$. Thus we find that $|BP| = |RB|$, such that B is the midpoint of PR. The height from b, that is, h_b is the bisector on the side PR in the triangle PQR, and correspondingly for the two other heights. Since the three heights are h_a, h_b and h_c are also bisectors in the triangle PQR, they intersect each other in the same point, which completes the proof.

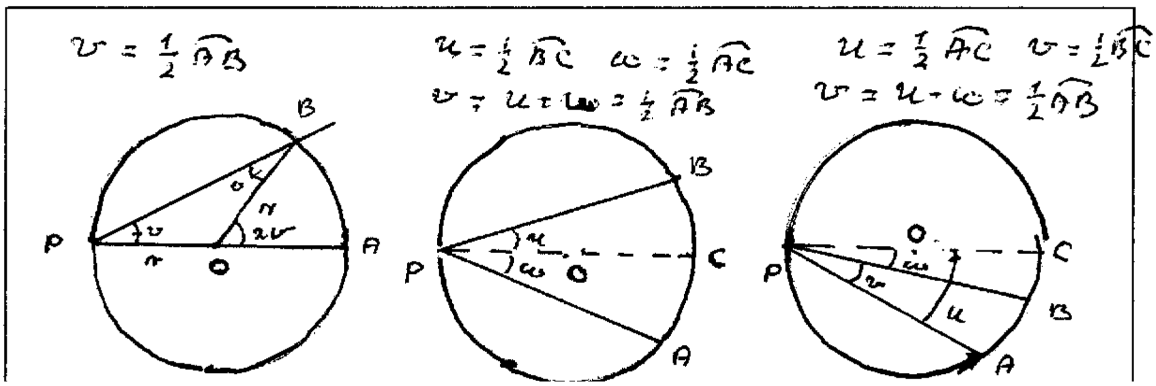
Chapter 4 Circles

1. Tangent angles and periphery angles



We begin with some definitions:

A **tangent** to a circle is a line which intersects the circle in only one point.
A **secant** is a line which intersects the circle in two points.
A **chord** is a line segment which connects two points on the periphery of a circle.
If the chord passes through the centre of the circle, it is called a **diameter**.
A **centre angle** is an angle which has its vertex in the centre of the circle. It is measured by the angle it spans on the circle. (Definition of the degree measure)
A **tangent angle** is an angle where its legs are tangents to the circle.
A **periphery angle** is an angle which has its vertex on the periphery.



First we shall prove that a **tangent angle is measured by 180^o – (the arc it spans).**

Proof: In the figure: $\angle OQP = \angle ORP = 90^o$, so we have $O + P = 180^o$, which should be proved.

We shall then prove:

A **periphery angle is measured by half the angle it spans.**

The proof comes in three parts: First we look at the case, where the one leg of the periphery angle passes through the centre *O* of the circle. (The figure to the left)
The angle *v* has vertex at *P*. We have drawn a helpline *BC*. The angles with vertices in *P* and *B* are both equal to *v*, since the triangle *OPB* is isosceles, and $\angle AOB$ is the supplement angle to $\angle POB = 180^{o} - 2v$, so that $\angle AOB = 2v$. So the periphery angle *v* having vertex in *P* spans the arc *AB*, which is 2*v*, so in this case we may conclude that a periphery angle is measured by half the arc that it spans.

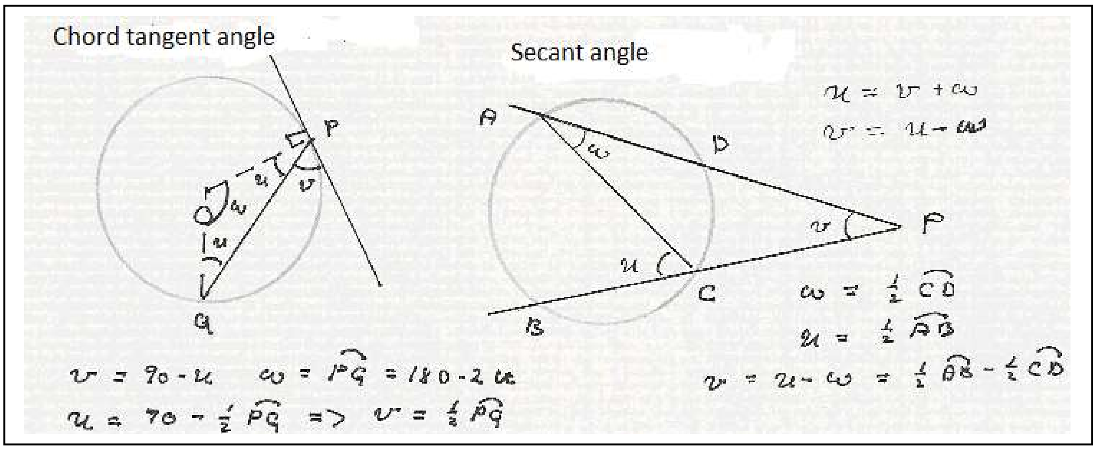
The periphery angle in the figure in the middle has its two legs at each side of the line that connects *P* with the centre *O* of the circle. If this line is prolonged, we have divided the periphery angle into two angles each having a leg passing through the centre of the circle. Since we have already shown that such periphery angles are measured by half the arc that they span, we have also in this case, since $v = u + w = ½AB + ½CB = ½AB$.

Finally we look at the figure to the right. Here both legs are on the same side of *OP*. Again we prolong the line *PO* so that it intersects the circle in *C*. We then have: $v = u - w = ½CB - ½ CA = ½AB$, which completes the proof.

We need, however, to treat two more kinds of angles in connection with a circle.

A **chord tangent angle** is an angle, where one leg is a tangent and the other leg is a chord.

A chord tangent angle is measured by half the arc it spans



This follows from the statement about the periphery angles, since the chord tangent angle is a limiting case for a periphery angle, where the one chord degenerates into a tangent.
But it ca also be shown directly as illustrated in the figure. If the two radii to *P* and *Q* are drawn, and the angles at the baseline are *u*, then $u = 90 - v$. $w = PQ = 180 - 2u$ => $u = 90 - ½PQ$.
So $v = ½PQ$, which is half the arc it spans.

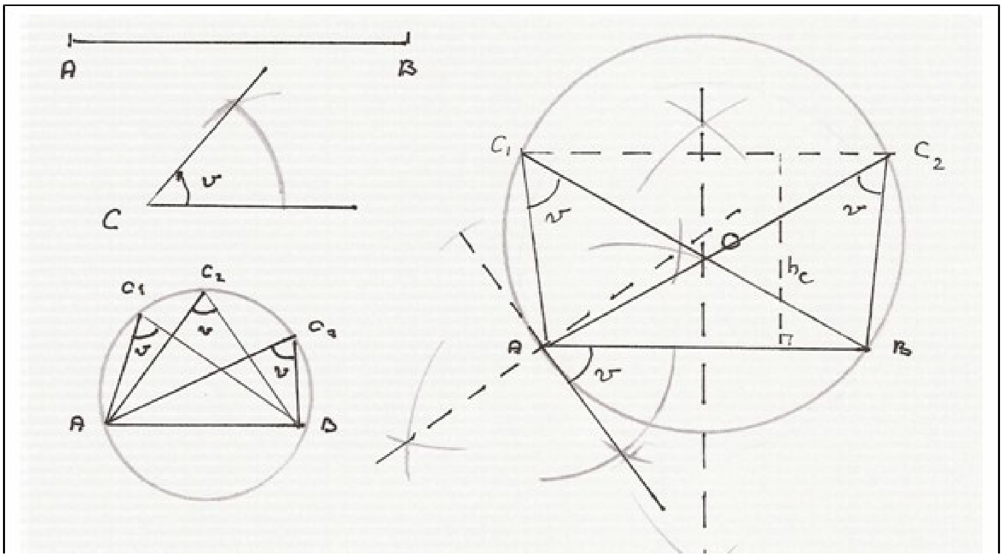
A **secant angle** is an angle, where the two legs are secants in a circle: (Figure to the right).
We shall then prove that:

A secant angle is measured by half of the difference between the two arcs the angle spans on the circle.

Proof: (The figure to the right) The legs of the secant angle intersect the circle in the points *A, B, C* and *D*. We the draw a helpline *AC*, which defines the two angles *u* and *w*. *u* is the supplement angle to ∠*PCA*, so *u* = *v*+*w*. From this follows: *v* = *u*-*w* = ½*AB* - ½*CD*, completing the proof. (We have used that *u* and *w* are periphery angles).

2. Perspective angle arc

In the figure below is drawn a circle with a chord *AB*. *AB* cuts off the arcs *AB* (and 360^{o} - *AB*) in the circle. All the periphery angles C_1, C_2 and C_3 cuts off the same arc *AB* and therefore they are equal. The arc *AB* (and 360^{o} - *AB*) are called the perspective angle arcs for the chord *AB*, since *AB* is seen under the same angle for every point, which lie on the arc *AB*.



2.1 Construction of the perspective arc

It is not entirely uncomplicated to construct the perspective arc under which a given line segment is seen under a given angle. The construction is shown in the figure to the right.
The line segment is *AB* and the angle is *v* = *C*.
The angle *v* is set in *A*, such that the chord *AB* is the left leg of *C*. A circle which passes through *A* and *B*, and which has the right leg of the angle *C* as a tangent in *A*. will have the angle *v* as a chord tangent angle, and the arc *AB* (below) will therefore be the double of *v*.

The circle mentioned will therefore be the perspective arc, we are looking for.

The circle can then be constructed, since the centre must lie on the bisector to *AB* as well as on a normal to the right leg of the angle in *A*, so by constructing the bisector on *AB* and the normal in *A*, the centre of the circle lies on the intersection of the two lines.

If *c* = |*AB*| and ∠*C* are known in a triangle, then it requires another information to locate *C* and thereby to construct the triangle Δ*ABC*. It might be h_c (the height on the side *c*) or the median m_c on the side *c*.

Exercises:

a) Construct ΔABC, when $\angle C = 45^{\circ}$, $c = 6$ and $m_c = 5$. b) , and $\angle C = 45^{\circ}$, $c = 6$ and $h_c = 4$

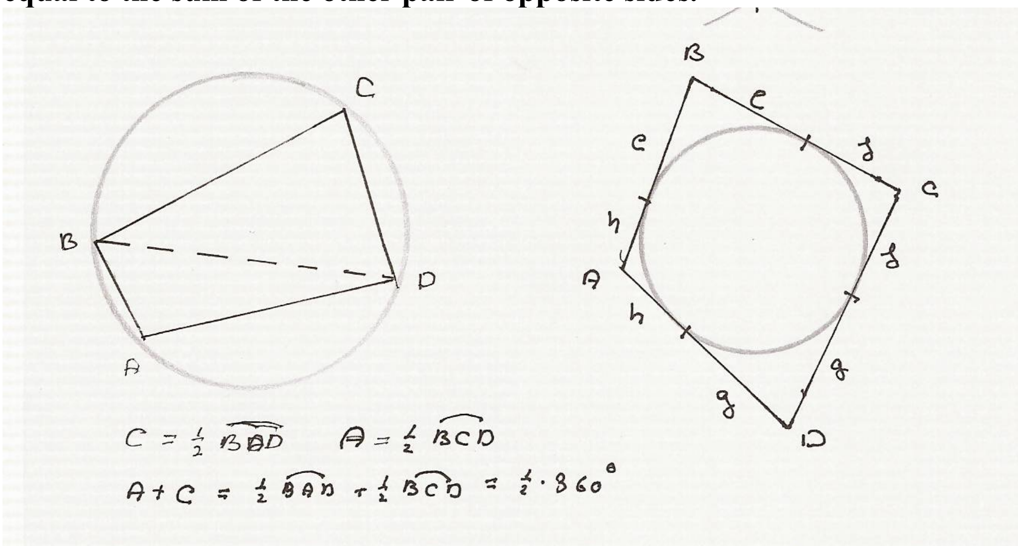
4. Inscribed and circumscribed squares

We have already shown that any triangle has an inscribed and a circumscribed circle, but it is easy to realize that this does not necessarily apply for an arbitrary square. The square can namely be divided into two triangles, and these triangles must have the same circumscribed circle to ensure that the square has a circumscribed circle. And this is naturally not always the case.

A square can be **inscribed** if it has a **circumscribed circle**, that is, a circle that passes through the four corners of the square.
A square can be **circumscribed**, if it has an **inscribed circle**, that is, a circle which has the four sides of the square as tangents. We shall then prove the following:

A square has a circumscribed circle if and only if the sum of opposite angles are 180^0.

A square has an inscribed circle if and only if the sum of the one pair of opposite sides is equal to the sum of the other pair of opposite sides.



Proof:
In the figure to the left is shown a square with a circumscribed circle. The angles $\angle A$ and $\angle C$ are both periphery angles, measured by half the arc that they span. Together they span the whole circle 360^0 so they must together be 180^0. So $A+C = 180^0$, and likewise $B + D = 180^0$.

In the figure to the right is shown a square with an inscribed circle, where *A, B, C* and *D* are four tangent angles. (For a tangent angle the distance to the two intersection points with the circle are the same). The distances on the tangents from *A, B, C* and *D* to the intersections with the circle are denoted *e, f, g* and *h*. Then it is seen:

$$|AB| + |CD| = h+e+g+f, \quad \text{and} \quad |BC| + |AD| = e+f+g+h,$$

It then follows:

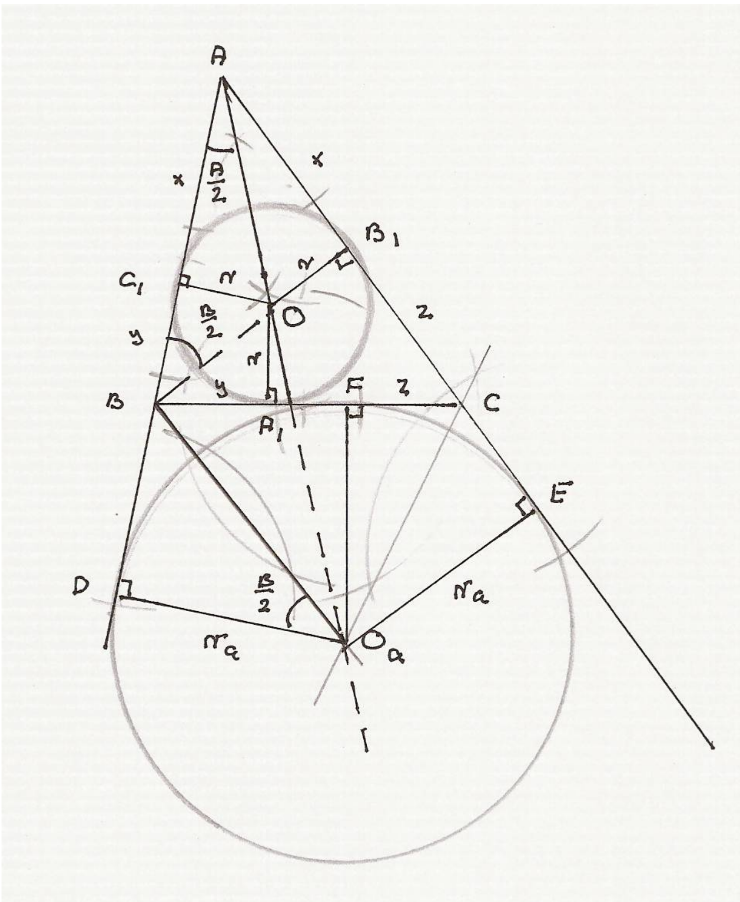
$$|AB| + |CD| = |BC| + |AD|$$ which completes the proof.

5. Herons formula

There are numerous theorems in the classical geometry, of which we have only mention the most notorious.
In a textbook of geometry for the 9th grade in 1961 (performed with dividers and ruler only!) the book was terminated by a frightening theorem called *Herons formula*, which expresses the area T of a triangle exclusively by the sides a, b and c in a triangle.
The half perimeter of a triangle is denoted s, so that $2s = a + b + c$. Herons formula states:

$$T = \sqrt{s(s-a)(s-b)(s-c)}$$

In the figure below is drawn a triangle, its inscribed circle together with the outer touching circle on the side a. The outer touching circle has the prolongation of the two sides b and c as tangents.



The centre O_A of the outer touching circle lies on the angular bisector for $\angle A$, together with the angular bisectors for $\angle B$ and $\angle C$. Radius in the outer touching circle is r_a.
The touching points of the inscribed circle with the sides a, b, c are denoted A_1, B_1 and C_1.
The distance from the points A, B, C to the touching point with the inscribed circle are denoted x, y og z. We have: $y+z = a$, $x+z = b$ and $x+y = c$. From which follows

$$2x+2y+2z = a+b+c = 2s \quad \Rightarrow \quad x + y + z = s$$

When $y+z = a$, is subtracted from this equation we find: $x = s-a$ and two analogous equations:

$x = s\text{-}a$, $y = s\text{-}b$ and $z = s - c$. Furthermore we have, since $AD = AE$:

$$AD = AB + BD = AB + BF \quad \text{and} \quad AE = AC + CE = AC + CF.$$

By addition of the two equations:

$$AD + AE = 2AD = 2AE = AB + AC + BF + CF = c + b + a = 2s \Rightarrow$$

$$AD = AE = s$$

The distance from a angular vertex to the touching point for the outer toucing circle on the opposite side is equal half the perimeter for the triangle. So

$$BD = AD - AB = s - c. \quad CE = AE - AC = s - b$$

We are now ready to establish some ratios between sides in two singled-angled-triangles.

Since the angular bisectors for two supplement angles are orthogonal (together are the two half angles for 180^0 equal to 90^0)

$$\angle OBC_1 = 90^o - \angle O_aBD \Rightarrow \angle BO_aD = \angle OBC_1 = ½B.$$

We then have: $\Delta BOC_1 \sim \Delta O_aBD$, which gives:

$$\frac{|OC_1|}{|BD|} = \frac{|BC_1|}{|O_aD|} \Rightarrow \frac{r}{s-c} = \frac{s-b}{r_a}$$

In the same manner: $\Delta AOC_1 \sim \Delta AO_aD$, resulting in

$$\frac{|OC_1|}{|O_aD|} = \frac{|AC_1|}{|AD|} \Rightarrow \frac{r}{r_a} = \frac{s-a}{s}$$

When the two equations: $\frac{r}{s-c} = \frac{s-b}{r_a}$ and $\frac{r}{r_a} = \frac{s-a}{s}$ are multiplied with each other and r_a is cancelled we find an expression for the radius r in the inscribed circle:

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}$$

Finally applying $T = r \cdot s$, we arrive at Herons formula

$$T = \sqrt{s(s-a)(s-b)(s-c)}$$

6. About proving the inverse of a theorem

Although it may seem obvious, then sometimes you may feel inclined to prove the inverse of a theorem. Here we shall illustrate the method by proving the inverse theorem to Pythagoras' theorem.
If we for a triangle with the sides *a, b, c* have: $c^2 = a^2 + b^2$, then $\angle C = 90^o$.

Proof: We form a right angled triangle with the catheti *a* and *b*. For this triangle applies $c^2 = a^2 + b^2$. But since this triangle and the first triangle have 3 sides in common they are congruent, and therefore the first triangle is right angled.