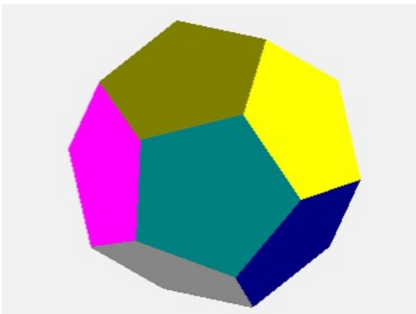
# Elementary Mathematics

# Differential Equations



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# Chapter 1. Differential equations

## 1. Differential equations of first order

A differential equation of first order is an equation, where the equation besides a function $y = f(x)$, then it also contains the differential quotient of this function $y' = f'(x)$.
If also the second derivative $y'' = f''(x)$ is part of the equation, then it is a second order differential equation. We recall the other way of writing the differential quotient:

(1.1) $$f'(x) = \frac{dy}{dx}$$

As well as the definition of the differential $dy$.

(1.2) $$dy = f'(x)dx\,.$$

Formally we may say that we get the differential $dy$ by multiplying the equation (1.1) by $dx$.

A first order differential equation may formally be written as:

(1.3) $$\frac{dy}{dx} = H(y,x)\,,$$

where $H$ is an arbitrary continuous function of $y$ and $x$.

To solve the differential equation means to determine any function $y = f(x)$, which satisfy (1.3).

$$f'(x) = H(f(x),x).$$

First we shall illustrate this by an example:

**1.3 Example.** We look at the differential equation:

$$\frac{dy}{dx} = \frac{y^2 + y}{x} + x$$

This differential equation cannot be solved by any standard method, but the aim is to demonstrate how you investigate whether a function is a solution to the equation or not. So we make a "guess" at a solution: $y = x\cdot\tan(x)$.
If we differentiate this function:

$$y' = x\cdot(1 + \tan^2 x) + \tan x = x\tan^2 x + \tan x + x = \frac{y^2}{x} + \frac{y}{x} + x = \frac{y^2 + y}{x} + x$$

So $x\cdot\tan(x)$ is indeed a solution to the equation. It is then easy to see, that all functions $y = x\cdot\tan(x+c)$ will be a solution as well.
It is characteristic that a solution to a first order differential equation is determined except by a constant.
On the other hand if you have a solution containing an arbitrary constant, then you have the complete solution.

## 1.1 Solution of first order differential equations

We have already in the integral calculus met differential equations where $y$ does not appear on the right side.

(1.4) $$\frac{dy}{dx} = f(x)$$

As you know, it has the solution.

(1.4) $$y = \int f(x)dx$$

Or, when writing out the meaning of the integral symbol

$$y = F(x) + c; \quad c \in R, \quad \text{where} \quad F'(x) = f(x)$$

A first order linear differential equation is a linear expression, where $y'$ and $y$ only appear in the first power. A general expression may be written as:

$$g_1(x)\frac{dy}{dx} + g_2(x)y = h_1(x)$$

If we confine ourselves to intervals, where $g_1(x) \neq 0$, we can divide by $g_1(x)$ and we get:

(1.5) $$\frac{dy}{dx} + g(x)y = h(x)$$

If $h(x) = 0$ then the equation is said to be homogenous.

$$\frac{dy}{dx} + g(x)y = 0$$

It is possible to solve any first order equation, but before we present the formula, we shall look at differential equations that can be separated in their dependence on $x$ and $y$.
The general form of such an equation can be written.

(1.6) $$g(y)\frac{dy}{dx} = h(x),$$

where $g(y)$ and $h(x)$ are now just assumed to be continuous.

Let $G(y)$ be an integral to $g(y)$ and let $H(x)$ be an integral to $h(x)$, such that:

$$G'(y) = g(y) \quad \text{and} \quad H'(x) = h(x)$$

We shall now show the following:

1) If $y$ is determined by the equation $G(y)=H(x) + c$ , then $y$ is a solution to the differential equation:

2) If $y = f(x)$ is a solution to the differential equation then it satisfies the equation: $G(f(x))=H(x) + c$.

When we have verified this, then we have proven that all solutions to the differential equation are given by:

(1.7) $$G(y) = H(x) + c.$$

We start by showing 1), as we differentiate $G(y) = H(x) + c$, according to the rules for differentiating a composite function:

$$G'(y)y' = H'(x) \quad \Leftrightarrow \quad g(y)y' = h(x) \quad \Leftrightarrow \quad g(y)\frac{dy}{dx} = h(x)$$

Which shows that $y$ given by $G(y) = H(x) + c$ is a solution to the differential equation.

Next we show 2), as we assume that $y = f(x)$, is a solution. To do this, we shall show that $G(f(x)) = H(x) + c$, where $c$ is a constant. Moving $H(x)$ to the other side: $G(f(x)) - H(x) = c$.
We differentiate then the left side of the equation:

$$(G(f(x)) - H(x))' = G'(f(x))f'(x) - H'(x) = g(f(x))f'(x) - h(x) = g(y)y' - h(x) = 0$$

In the last expression, we have applied that $y = f(x)$ is a solution, such that $g(y)y' = h(x)$.
The differential quotient $(G(f(x))-H(x))'$ is identically zero, which implies that:

$$G(f(x)) - H(x) = c \quad \Leftrightarrow \quad G(f(x)) = H(x) + c$$

which completes the proof.

In practice (1.6) is solved by separation of the variables followed by integration in a series of steps:

$$g(y)\frac{dy}{dx} = h(x) \quad \Leftrightarrow \quad g(y)dy = h(x)dx \quad \Leftrightarrow \quad \int g(y)dy = \int h(x)dx \quad \Leftrightarrow$$

(1.7) $$G(y) = H(x) + c$$

Where $G'(x) = g(x)$ and $H'(x) = h(x)$.

As we can see, we reach the complete solution by separation and integration.

Notice the important presence of the constant $c$, which can be determined by the claim that the solution $y = f(x)$ should pass through the point $(x_0, y_0)$, such that $y_0 = f(x_0)$.

**Explanatory note**:
When we did not use differentials from the start, it is because the nature of differentials as infinite small quantities, however, not zero does not have the mathematical rigor, which is claimed in modern mathematics, although in other sciences especially physics, one has carefree derived formulas using differentials for more than 200 years.
But this is a textbook on mathematics and not of mathematical applications in physics, and therefore we prefer to keep the mathematical rigor.

It is possible to show the following theorem, but we omit the proof, since it is rather technical and abstract

(1.8) The existence and uniqueness theorem for differential equations of first order.

*If $h(x)$ is defined and continuous in an interval I and $g(y)$ is defined and continuous in an interval J, then there exists exactly one solution $y = f(x)$, which goes through the point $(x_0, y_0)$, where* $x_0 \in I \wedge y_0 \in J$ .

## 1.1 Examples of first order differential equations

**1.8 Example.**
We shall determine the solutions to the differential equation

(1.9) $$\frac{dy}{dx} = ky ,$$

where $k$ is a non zero real constant.
From the differential calculus we know that exponential functions have the property that the differential quotient is proportional to the function itself.
We therefore "guess" the solution $y = c \cdot e^{kx}$. Differentiating we get:

$$y' = ce^{kx} \cdot k = k \cdot ce^{kx} = ky$$

Which shows that $y = c \cdot e^{kx}$ is a solution to the equation.
We shall then solve the equation, using the methods developed above, and establish the complete solution actually is $y = c \cdot e^{kx}$ , where $c \in R$ -

We separate the equation as we did in (1.7), and we consider two cases:
1) We can immediately see that $y = 0$ is a solution to the equation. This is called the zero solution.
2) $y \neq 0$. Then we separate in the dependence in $x$ and $y$, and integrate on both sides of the equation:

$$\frac{dy}{dx} = ky \quad \Leftrightarrow \quad \frac{1}{y}dy = kdx \quad \Leftrightarrow \quad \int \frac{1}{y}dy = \int kdx \quad \Leftrightarrow$$

$$\ln|y| = kx + c_1 \quad \Leftrightarrow \quad |y| = e^{kx+c_1} \quad \Leftrightarrow \quad y = e^{c_1}e^{kx} \quad \vee \quad y = -e^{c_1}e^{kx} \quad \Leftrightarrow$$
$$y = ce^{kx} \quad , \quad c \in R$$

The last rewriting, because $e^{c_1} \in R_+$ and $c = 0$ is the zero solution.
This way of solving differential equations is characteristic, and in the future we shall omit the details.
It is, however, important to remember the zero solution.

**1.10 Example**
The differential equation above is one of the most frequent in many sciences like physics, economy, chemistry and biology.
Often the variable $x$ represents the time, and therefore it is replaced with $t$.
Differentiating once with respect to time means to find the velocity, and differentiating twice with respect to time means to find the accelerations. These concepts are therefore often used for the first and second derivative.

The difference quotients $v = \frac{\Delta s}{\Delta t}$ *and* $a = \frac{\Delta v}{\Delta t}$ , becomes in the limit $\Delta t \to 0$ the differential quotients.

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \quad and \quad a = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

So when *y*' means the velocity, then the differential equation and its solution is written as:

(1.9) $$\frac{dy}{dt} = ky \Leftrightarrow y = ce^{kt} \ , \ c \in R$$

**1.11 Example:** Exponential growth.

For a population (bacteria or any other spices), having unlimited access to food, one may as a first approximation assume that in limited periods of time $\Delta t$, the increment in the population $\Delta y$ is proportional to the size of the population *y* and the period $\Delta t$. Mathematically this is expressed as:

$$\Delta y = ky\Delta t \Leftrightarrow \frac{\Delta y}{\Delta t} = ky$$

By letting $\Delta t$ go to zero, we regain the differential equation (1.9)

$$\frac{dy}{dt} = ky \Leftrightarrow y = ce^{kt} \ , \ c \in R$$

If a population has a growth which leads to the differential equation (1.9), we speak about *exponential growth*.

It appears that numerous biological organisms, including human populations in a limited period of time may be described by an exponential growth.
Likewise it is obvious that the exponential growth will break down, since all exponential functions goes (fast) to infinity from a certain stage.
In the section on exponential functions we saw, that one may calculate the doubling constant as: $T_2 = \frac{\ln 2}{k}$

**1.12 Example**
We shall then look at two types of differential equation, which are very similar to (1.9) and is solved in the same manner.

(1.13) $$\frac{dy}{dx} = ay + b$$

We introduce the auxiliary variable $z = y + \frac{b}{a} \Leftrightarrow az = ay + b$ and we have : $\frac{dz}{dx} = \frac{dy}{dx}$ and thus:

$$\frac{dy}{dx} = ay + b \Leftrightarrow \frac{dz}{dx} = az \Leftrightarrow z = ce^{ax} \Leftrightarrow y = ce^{ax} - \tfrac{b}{a}$$

We can see that the solution corresponds to an exponential growth, just displaced along the y-axis.

**1.13 Example.**
A lake which has a volume *V*, is added *u* litres of polluted water a day. The fraction of pollution is denoted *q*. From the lake is released the same amount of water *u* litres a day. We shall establish a differential equation for the fraction *y* of pollution of the lake, as a function of time.
The velocity *dy/dt*, with which the lake is polluted has two contributions. The inlet of polluted water, which is $q\frac{u}{V}$, and the outlet which is $y\frac{u}{V}$. The differential equation hereafter becomes:

$$\frac{dy}{dt} = q\frac{u}{V} - y\frac{u}{V} \Leftrightarrow \frac{dy}{dt} = \frac{u}{V}(q - y)$$

We can see that this differential equation is of the type (1.13), why we may immediately write the solution:

$$y = ce^{-\frac{u}{V}t} + q$$

Assuming that the lake is clean at $t = 0$, we find that $0 = c + q \;\Leftrightarrow\; c = -q$, where after we can write the solution.

$$y = -qe^{-\frac{u}{V}t} + q \quad \Leftrightarrow y = q(1 - e^{-\frac{u}{V}t})$$

Since $e^{-\frac{u}{V}t} \to 0 \;\; for \;\; t \to \infty$, the lake will end up with the same degree of pollution as the waste water which runs into the lake.
We shall finally determine the degree of pollution , when the inlet of water corresponds to half of the lakes volume.

This means that $ut = ½V$, so we find $y = q(1 - e^{-\frac{1}{2}}) = 0.393q$ corresponding to 39,3%

**1.14 Example.** Logistic growth.
The supposition of unlimited access to food is only realistic in shorter periods. We get a more realistic model if we assume that the rate of growth of the population is proportional to the size of the population (as is the case of exponential growth), but also proportional to an upper limit *M* for the population. Whereas this model reflects actual conditions, can only be decided by experience of course. Since we explicitly discuss growth, we shall use *t* (time) as the independent variable. The differential equation then get the appearance.

(1.15) $$\frac{dy}{dt} = ay(M - y)$$

Where $M - y$ represents the distance to the upper limit for the population.

The equation is then solved by separation, and we notice that $y = 0$ and $y = M$, are trivial solutions.

$$\int \frac{dy}{y(M - y)} = \int a dt$$

The first integral is not so trivial to evaluate, but it is done with a technique, which is called expansion on partial fractions. So we shall try to write the integrand as two fractions each having one of the factors in the denominator of the integrand..

$$\frac{1}{y(M-y)} = \frac{p}{y} + \frac{q}{M-y} = \frac{p(M-y)+qy}{y(M-y)} = \frac{pM+(q-p)y}{y(M-y)}$$

If this is a identity valid for all $y$, it then follows that $q = p$ and $pM = 1$, so $p = q = \frac{1}{M}$ . If we multiply by M we get the differential equation:

$$\int (\frac{1}{y} + \frac{1}{M - y}) dy = aM \int dt \quad \Leftrightarrow \quad \ln y - \ln(M - y) = aMt + k$$

Here we have assumed that $y > 0$ and $y < M$, which were the suppositions for the model.
By reduction and using the rules for the logarithmic function, we find:

$$\ln y - \ln(M-y) = aMt + k \quad \Leftrightarrow \quad \ln\frac{y}{M-y} = aMt + k \quad \Leftrightarrow \quad \frac{y}{M-y} = e^{aMt+k}$$

If we solve this equation with respect to *y* and put $e^k = c$, where *c* is a positive constant, we find the solution:

(1.15) $$y = cM\frac{e^{aMt}}{1+ce^{aMt}} \quad \Leftrightarrow \quad y = cM\frac{1}{e^{-aMt}+c}$$

Often you see that *c* is replaced by 1/c, where after the solution takes the simple form.

(1.15) $$y = \frac{M}{1+ce^{-aMt}}$$

The constants *a* and *M* are fixed by the model, whereas *c* is fixed by the size of the population at a given time.

**1.15 Example**

In a lake the concentration of phosphorus is a function of time. $y = f(t)$. In a model for the concentration of phosphorus in the lake it is assumed that a constant amount of phosphorus is delivered to the lake a day, while the same amount that is let out is proportional to the concentration of phosphorus in the lake.

We may the establish a differential equation which expresses that the relative concentration of phosphorus in the period of time $\Delta t$ is a constant times $\Delta t$ minus a constant times the phosphorus concentration *y* times $\Delta t$.

$$\frac{\Delta y}{y} = a\Delta t - by\Delta t \quad \Leftrightarrow \quad \frac{\Delta y}{\Delta t} = y(a-by) \quad \Leftrightarrow \quad \frac{\Delta y}{\Delta t} = by(\frac{a}{b} - y)$$

Letting $\Delta t$ go to zero, we get a differential equation which we recognize as the logistic equation.

$$\frac{dy}{dt} = by(\frac{a}{b} - y)$$

For the lake in question, we have: $b = 0.00001$ and $a/b = 200$. Furthermore *f* satisfies the equation $f(475) = 107$. We wish to determine the function *f*, and calculate $f(1000)$, together with the rate of growth for the phosphorus concentration at $t = 1000$. By inserting $f(475) = 107$ in the solution formula for the logistic equation we have:

$$107 = \frac{200}{1+ce^{-0{,}002\cdot 475}} \quad \Leftrightarrow \quad 107ce^{-0{,}95} = 200 - 107 \quad \Leftrightarrow \quad c = \frac{93}{107e^{-0{,}95}} = 2{,}20$$

and the solution becomes:

$$y = \frac{200}{1+2.20e^{-0.002t}} \qquad \text{with} \qquad f(1000) = \frac{200}{1+2.20e^{-2}} = 154$$

The rate of growth *dy/dt*, is determined by direct insertion in the differential equation.

$$\frac{dy}{dt} = 0.00001 \cdot y(200-y) = 0.00001 \cdot 154(200-154) = 0.0708$$

## 1.3 The complete solution to the first order linear differential equation

We want to solve the equation:

(1.16) $$\frac{dy}{dx} + g(x)y = h(x)$$

We assume that $g(x)$ and $h(x)$ are continuous functions, so they can be integrated.
If $G(x)$ is an integral to $g(x)$, then $G'(x) = g(x)$, and multiplying the equation with $e^{G(x)}$, we get:

$$e^{G(x)}\frac{dy}{dx} + e^{G(x)}g(x)y = e^{G(x)}h(x) \quad \Leftrightarrow \quad \frac{d}{dx}(e^{G(x)}y) = e^{G(x)}h(x)$$

Where the rewriting follows from: $\frac{d}{dx}(e^{G(x)}y) = e^{G(x)}\frac{dy}{dx} + e^{G(x)}G'(x)y = e^{G(x)}\frac{dy}{dx} + e^{G(x)}g(x)y$

The equation:

$$\frac{d}{dx}(e^{G(x)}y) = e^{G(x)}h(x)$$

can immediately be integrated to:

(1.17) $$e^{G(x)}y = \int e^{G(x)}h(x)dx \quad \Leftrightarrow \quad y = e^{-G(x)}\int e^{G(x)}h(x)dx$$

The equation (1.17) is the complete solution, but as you should remember, the integral always demands an arbitrary constant. (A constant of integration).

**1.18 Example.**

Determine the solution to the differential equation : $\frac{dy}{dx} + \frac{y}{x} = x^2$, where $x > 0$, which goes through (2,-3).

We find directly: $G(x) = \int \frac{1}{x}dx = \ln x \quad \Rightarrow \quad e^{G(x)} = e^{\ln x} = x \quad \Rightarrow \quad e^{-G(x)} = e^{-\ln x} = x$

When we insert in the solution formula, we find:

$$y = \frac{1}{x}\int x \cdot x^2 dx = \frac{1}{x}(\frac{1}{4}x^4 + c) \Leftrightarrow y = \frac{1}{4}x^3 + \frac{c}{x}$$

Solution through (2,-3) is found by insertion of $(x,y) = (2,-3)$ in the solution: $-3 = 2 + \frac{c}{2} \Leftrightarrow c = -10$

$$y = \frac{1}{4}x^3 - \frac{10}{x}$$

## 2. Second order differential equations

A Second order differential equation, is an equation which involves the second derivative of the function. In general this is a huge area in mathematics and especially physics, but since this is a text book of "elementary mathematics", we shall only be concerned with second order differential equations having the form shown below:

(2.1) $$\frac{d^2y}{dx^2} = \pm k^2 y \quad \text{where} \quad k \neq 0$$

The case where $k = 0$ is immediately seen to give the solution $y = c_1x + c_2$

Second order differential equations cannot be solved (except for linear differential equation having constant coefficients), but the task is the same, to investigate whether they have solutions, and subsequently to find all solutions.

First we shall show, (but it follows from the linearity of the equation):

*If $f_1$ and $f_2$ are solutions to the* differential equation (2.1)*, so is any linear combination of $f_1$ and $f_2$.*

(2.2) $$f(x) = c_1 f_1(x) + c_2 f_2(x)$$

First we write the differential equation on the form.

(2.3) $$\frac{d^2y}{dx^2} = my \quad \text{where} \quad m = \pm k^2$$

Then we differentiate (2.2) twice and get:

$$f''(x) = c_1 f_1''(x) + c_2 f_2''(x)$$

Since both $f_1$ *og* $f_2$ are solutions to (2.3), we have:

$$f_1''(x) = m f_1(x) \quad \text{and} \quad f_2''(x) = m f_2(x),$$

And therefore:

$$f''(x) = c_1 f_1''(x) + c_2 f_2''(x) = c_1\, m f_1(x) + c_2\, m f_2(x) = m(c_1 f_1(x) + c_2 f_2(x)) = m f(x)$$

Which shows that $f(x)$ is a solution to the differential equation (2.3).

## 2.1 The Wronski-determinant

In the following, we shall make use of the determinant-solution formula for two linear equations having two unknowns. The system of equations:

(2.4) $$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \text{has the determinant} \quad D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

If $D \neq 0$, then the system of equations (2,4) has exactly one solution given by:

(2.5) $$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

For the determination of the solutions to the differential equation:

$$\frac{d^2y}{dx^2} = my,$$

We introduce the so called *Wronski*-determinant of two differentiable functions.

(2.6) $$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

**2.7 Eksempel**
The Wronski-determinant may be evaluated for any two functions. E.g. $f(x) = x^2$ and $g(x) = \sin x$.

$$W(f,g) = \begin{vmatrix} x^2 & \sin x \\ 2x & \cos x \end{vmatrix} = x^2 \cos x - 2x \sin x$$

Generally the Wronski-determinant is a function of $x$, but we shall show a somewhat surprising theorem:

(2.8) *If $f_1$ and $f_2$ are solution to the differential equation* (2.3)*,*
*then the Wronski-determinant is a constant.*

$$W(f_1, f_2) = c \quad \Leftrightarrow \quad W'(f_1, f_2) = 0$$

Differentiating $W(f_1, f_2) = f_1 f_2' - f_1' f_2$ using the product rule for differentiating of two factors:

$$W'(f_1, f_2) = (f_1f_2' - f_1'f_2)' = f_1'f_2' + f_1f_2'' - f''_1 f_2 - f_1'f_2' = f_1f_2'' - f''_1 f_2$$

Using that $f_1$ and $f_2$ are solutions to the differential equation (2.3), such that:

$$f_1'' = m f_1 \quad \text{and} \quad f_2'' = m f_2$$

We find:

$$W'(f_1, f_2) = f_1f_2'' - f''_1 f_2 = f_1mf_2 - mf_1f_2 = m(f_1f_2 - f_1f_2) = 0$$

So the Wronski-determinant is constant using two arbitrary solutions to (2.3).

We are now ready to prove the main theorem, about the solutions to the differential equation (2.3).

*If $f_1$ and $f_2$ are solutions to the differential equation*

$$\frac{d^2y}{dx^2} = my$$

*Where we have $W(f_1, f_2) \neq 0$ (a non zero constant)), then all the solutions $y = f(x)$ can be written as a linear combination of the two solutions.*

$$f(x) = c_1 f_1(x) + c_2 f_2(x)$$

We have already shown that $f(x) = c_1 f_1(x) + c_2 f_2(x)$ is a solution if $f_1(x)$ and $f_2(x)$ are solutions, so we only need to show that all solutions may be written in this form.

If we momentarily release the claim that $c_1$ and $c_2$ constants, then it will always be possible to determine two functions $c_1 = c_1(x)$ and $c_2 = c_2(x)$, which satisfy the system of equations:

$$\begin{aligned} c_1 f_1(x) + c_2 f_2(x) &= f(x) \\ c_1 f_1'(x) + c_2 f_2'(x) &= f'(x) \end{aligned}$$

The determinant for this system of equations is:

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix},$$

which we have presupposed to be non zero.

The solutions $c_1 = c_1(x)$ and $c_2 = c_2(x)$ to the system of equations, can according to the solution formula for two equations with two unknowns be written as:

$$c_1 = \frac{\begin{vmatrix} f & f_2 \\ f' & f_2' \end{vmatrix}}{W(f_1, f_2)} = \frac{W(f, f_2)}{W(f_1, f_2)} \quad \wedge \quad c_2 = \frac{\begin{vmatrix} f_1 & f \\ f_1' & f' \end{vmatrix}}{W(f_1, f_2)} = \frac{W(f_1, f)}{W(f_1, f_2)}$$

Since $f_1$ , $f_2$ and $f$ all are supposed to be solutions, then all the involved determinants are constants, and consequently $c_1$ and $c_2$ are constants, which completes the proof.

We shall then solve the differential equation:

$$\frac{d^2y}{dx^2} = my \qquad \text{(for } m > 0 \text{ and for } m < 0\text{).}$$

**2. 8 Example.**

$m>0$: We set $m = k^2$ . then the differential equation reads:

$$\frac{d^2y}{dx^2} = k^2 y$$

Functions where the second derivative is proportional to the function itself are either exponential functions or sine-cosine functions, but when the constant is positive it is exponential functions.

By direct insertion it is verified that: $f_1(x) = e^{kx}$ and $f_2(x) = e^{-kx}$ are solutions, since

$$(e^{-kx})' = -ke^{-kx} \quad \text{and} \quad (e^{-kx})'' = (-ke^{-kx})' = k^2 e^{-kx}.$$

We then calculate the Wronski-determinant:

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{kx} & e^{-kx} \\ ke^{kx} & -ke^{-kx} \end{vmatrix} = -k - k = -2k \neq 0$$

Since the Wronski-determinant is non zero, we may write the complete solution:

(2.8) $$f(x) = c_1 e^{kx} + c_2 e^{-kx}$$

**2. 9 Example.**
$m < 0$: We put $m = -k^2$, and then the differential equation reads:

$$\frac{d^2y}{dx^2} = -k^2 y$$

Functions where the second derivative is proportional to the function itself are either exponential functions or sine-cosine functions, but when the constant is negative, then it is sine or cosine functions.

By inspection we notice that $f_1(x) = \cos kx$ and $f_2(x) = \sin kx$ are solutions, but we settle for showing one of them.

$$(\cos kx)' = -k \sin kx \quad \text{and} \quad (\cos kx)'' = (-k \sin kx)' = -k^2 \cos kx \quad .$$

Wronski-determinant for the two solutions.

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos kx & \sin kx \\ -k \sin kx & k \cos kx \end{vmatrix} = k\cos^2 kx + k\sin^2 kx = k(\cos^2 kx + \sin^2 kx) = k \neq 0$$

Since the Wronski-determinant is non zero, then we may write the complete solution:

(2.8) $$f(x) = c_1 \cos kx + c_2 \sin kx$$

## 2.2 Uniqueness of solutions

By a *line-element* , we understand a point $(x_0, y_0)$ that a solution passes through, together with a differential quotient (the slope of the tangent in that point).
A line-element can for example be written as $(x_0, y_0; \alpha)$. If a solution satisfies the line-element then we have: $f(x_0) = y_0$ and $f'(x_0) = \alpha$. We shall then show the following:

*To each line element* $(x_0, y_0; \alpha)$ *there is exactly one solution to the differential equation:* $\frac{d^2y}{dx^2} = my$ *, which goes through that line-element.*

We show that there is one and only one solution $y = f(x)$, satisfying: $f(x_0) = y_0$ and $f'(x_0) = \alpha$.

All solutions to the differential equation may be written:

$$f(x) = c_1 f_1(x) + c_2 f_2(x)$$

where $f_1(x)$ and $f_2(x)$ are two solutions, where the Wronski-determinant is non zero.
The conditions for a solution is: $f(x_0) = y_0$ and $f'(x_0) = \alpha$. This can be written in two equations:

$$c_1 f_1(x_0) + c_2 f_2(x_0) = y_0$$
$$c_1 f_1'(x_0) + c_2 f_2'(x_0) = \alpha$$

This may be considered as two linear equations having the unknowns $c_1$ and $c_2$.
The determinant of the equation system is however:

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix},$$

Which is exactly the Wronski-determinant for the two solutions, (which we haven proven is non zero).
Since the determinant for the equations is non zero, then the equations has always one and only one solution.

**2.10 Eksempel**

Determine for the differential equation: $y'' = \frac{1}{4}y$ the solution which goes through $A(0,6)$ and in the point $A$ has a tangent with slope equal to 1.
The equation has the form $y'' = k^2 y$ with $k = \frac{1}{2}$, so we may write the complete solution immediately:

$$f(x) = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} \quad \text{and} \quad f'(x) = \tfrac{1}{2}c_1 e^{\frac{1}{2}x} - \tfrac{1}{2}c_2 e^{-\frac{1}{2}x}$$

The conditions for a solution.

$$f(0) = 6 \quad \text{and} \quad f'(0) = 1 \quad \Rightarrow \quad \tfrac{1}{2}c_1 e^0 - \tfrac{1}{2}c_2 e^0 = 1$$
$$c_1 + c_2 = 6 \quad \wedge \quad \tfrac{1}{2}c_1 - \tfrac{1}{2}c_2 = 1 \quad \Leftrightarrow \quad c_1 = 4 \quad \wedge \quad c_2 = 2$$

The solution then becomes:

$$f(x) = 4e^{\frac{1}{2}x} + 2e^{-\frac{1}{2}x}$$

**2.11 Eksempel**

Determine for the differential equation: $y'' = -9y$ the solution where the graph passes through $P(\frac{\pi}{9}, 2\sqrt{3})$ and in the point $P$ has a tangent with slope equal to -6.
The equation has the form $y'' = -k^2 y$ with $k = \frac{1}{2}$, so we may write the complete solution immediately:

$$f(x) = c_1 \cos 3x + c_2 \sin 3x \quad \text{med} \quad f(x) = -3c_1 \sin 3x + 3c_2 \cos 3x$$

The solution conditions are:

$$f(\frac{\pi}{9}) = 2\sqrt{3} \quad and \quad f'(\frac{\pi}{9}) = -6 \quad \Leftrightarrow$$

$$c_1 \cos 3\frac{\pi}{9} + c_2 \sin 3\frac{\pi}{9} = 2\sqrt{3} \ \wedge \ -3c_1 \sin 3\frac{\pi}{9} + 3c_2 \cos 3\frac{\pi}{9} = -6 \ \Leftrightarrow$$

$$c_1 \cos\frac{\pi}{3} + c_2 \sin\frac{\pi}{3} = 2\sqrt{3} \;\wedge\; -3c_1 \sin\frac{\pi}{3} + 3c_2 \cos\frac{\pi}{3} = -6 \quad \Leftrightarrow$$

$$c_1 \frac{1}{2} + c_2 \frac{\sqrt{3}}{2} = 2\sqrt{3} \;\wedge\; -c_1 \frac{\sqrt{3}}{2} + c_2 \frac{1}{2} = -2 \quad \Leftrightarrow$$

$$c_1 + c_2\sqrt{3} = 4\sqrt{3} \;\wedge\; -c_1\sqrt{3} + c_2 = -4 \quad \Leftrightarrow$$

The equations are most easily solved by multiplying with $-\sqrt{3}$ adding the two equations gives:

$$4c_1 = 8\sqrt{3} \Rightarrow c_1 = 2\sqrt{3} \;\wedge\; c_2 = 2$$

The solution is hereafter:

$$f(x) = 2\sqrt{3}\cos 3x + 2\sin 3x$$

We have shown that the differential equation:

$$\frac{d^2y}{dx^2} = -k^2 y$$

has the complete solution:

$$f(x) = c_1 \cos kx + c_2 \sin kx$$

We wish to determine the value set for this function.
This my be obtained, by determining two numbers $A$ and $\varphi$, such that $c_1$ and $c_2$ may be written on the form:

$$c_1 = A\cos\varphi \quad \text{and} \quad c_2 = A\sin\varphi$$

From the two equations we get by division: $\tan\varphi = \dfrac{A\sin\varphi}{A\cos\varphi} = \dfrac{c_2}{c_1}$

Taking the square of the two equations and adding them: $A^2(\cos^2\varphi + \sin^2\varphi) = {c_1}^2 + {c_2}^2$,
So we have:

$$A = \sqrt{{c_1}^2 + {c_2}^2}\,.$$

Inserting the obtained expressions for $c_1$ and $c_2$ in the solution, we have:

$$f(x) = A(\cos\varphi\cos kx + \sin\varphi\sin kx) = A\cos(kx - \varphi)$$

The last rewriting is due to the first addition formula for cosine: $\cos(x-y) = \cos x\cos y + \sin x\sin y$.

From the last expression for the solution, it is obvious that the value set for the solution is $[-A, A]$, since the value set for cosine is [-1, 1].

In physics the solution is often written: $f(x) == A\cos(kx+\varphi_0)$ . Here $A$ is called the *amplitude* (in a harmonic oscillation), $\varphi = kx + \varphi_0$ is called the phase and $\varphi_0$ is called the initial phase.