Eigenvalue problems
In
Linear algebra

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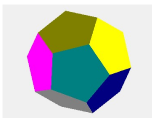

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1. Linear algebra with matrices

A vector $\vec{a}$ is a quantity having *n* coordinates (for geometrical vectors usually two or three): $\vec{a} = (a_1, a_2, a_3, ..., a_n)$. In the plane: $\vec{a} = (a_1, a_2)$, and in 3-dim space $\vec{a} = (a_1, a_2, a_3)$.
For a quantity to be a vector it is also required to have certain transformation properties for example by rotations.
The scalar product of two vectors $\vec{a} = (a_1, a_2, a_3, ..., a_n)$ and $\vec{b} = (b_1, b_2, b_3, ..., b_n)$ is defined as:

(1.1) $$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

If $(\vec{e}_1, \vec{e}_2, \vec{e}_3, ..., \vec{e}_n)$ is a base of mutually orthogonal unit vectors, spanning the vector space in question, then the meaning of the coordinates are: $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3, ..., a_n\vec{e}_n$.
Unit vectors means that the lengths of the base vectors are 1. $|\vec{e}_k| = 1$, and mutually orthogonal means $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and 0 otherwise.

An n x n matrix A_{ij} $(i, j = 1..n)$ has two indices and n^2 elements. It is often written in "matrix form" with rows and columns. Below is illustrated a 3 x 3 matrix. A matrix is (in these notes) denoted with a double underscore.

(1.2) $$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The transposed matrix is marked with a hyphen. The transposed matrix is the matrix where the row index and the column index are switched: $A'_{ij} = A_{ji}$.

(1.3) $$\underline{\underline{A}}' = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Each of the rows and each of the columns, can be considered as a vector, and is also in this context marked with an double underscore: $\underline{\underline{x}}$ denotes a column vector and $\underline{\underline{x}}$' then denotes a row vector having the same coordinates.
The scalar product between two vectors $\underline{\underline{a}}$ and $\underline{\underline{b}}$ in matrix form is then written: $\vec{a} \cdot \vec{b} = \underline{\underline{a}}'\underline{\underline{b}}$

Multiplication of a matrix with a column vector results in a column vector:

(1.4) $$\underline{\underline{y}} = \underline{\underline{A}}\underline{\underline{x}} \quad \Leftrightarrow \quad y_i = \sum_{j=1}^{n} a_{ij} x_j$$

That is, the *i*'th component is the scalar product of the *i*'th row of the matrix with the vector $\underline{\underline{x}}$.

Multiplication of a matrix with a row vector results in a row vector:

(1.5) $$\underline{\underline{y}}' = \underline{\underline{x}}'\underline{\underline{A}} \quad \Leftrightarrow \quad y_j = \sum_{j=1}^{n} a_{ij} x_i$$

That is, the j'th component of $\underline{\underline{y}}'$ is the scalar product of the j'th column of the matrix with the vector $\underline{\underline{x}}$'.

Multiplication of two matrices:

(1.6) $$\underline{\underline{C}} = \underline{\underline{A}}\underline{\underline{B}} \quad \Leftrightarrow \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

That is, the i, j'th element of the product matrix is the scalar product of the i'th row of $\underline{\underline{A}}$ and the j'th column of $\underline{\underline{B}}$.
The transposed of a product of matrix is the transposed of the two matrices in reverse order.

(1.7) $$(\underline{\underline{A}}\underline{\underline{B}})' = \underline{\underline{B}}'\underline{\underline{A}}'$$

The unit matrix $\underline{\underline{E}}$ is a matrix with 1 in the diagonal, and otherwise 0. If e_{ij} is the element in $\underline{\underline{E}}$, then $e_{ij} = \delta_{ij}$.
The product of any matrix with the unit matrix is the matrix itself:

(1.8) $$\underline{\underline{A}}\,\underline{\underline{E}} = \underline{\underline{A}} \text{ and } \underline{\underline{E}}\,\underline{\underline{A}} = \underline{\underline{A}}.$$

1.1 Determinants

The concept of the determinant of matrix is well known from linear algebra, and shall not be treated here in detail.
(An introduction to determinants is found in: www.olewitthansen.dk "Vectors in space").
The determinant is formally written:

(1.9) $$\det(\underline{\underline{A}}) \quad \text{or} \quad |\underline{\underline{A}}|$$

The determinant of the unit matrix is 1: $\det(\underline{\underline{E}}) = 1$

The determinant is nonzero, if and only if the rows and columns in the matrix are linear independent. The matrix is then called *regular*.
For a 3x3 matrix the evaluation of a determinant goes as follows.

(1.10) $$\vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1$$

Each term is produced by multiplying a factor from each of the three columns and from each of the three rows. And the sign of the term is determined depending of whether the permutation (1,2,3) is even or odd. (A permutation (*i,j,k*) is even if it requires a even number of permutation of neighbour elements to restore (1,2,3), otherwise odd.
This procedure for evaluating determinants may easily be generalized to an *n x n* determinant.

For two matrices: $\underline{\underline{A}}$ and $\underline{\underline{B}}$, one may show that the determinant of the product matrix $\underline{\underline{AB}}$ is the products of the determinants $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

(1.11) $$\det(\underline{\underline{AB}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$$

The general proof, however, requires an association to a volume measure concept, spanned by the vectors in the matrices, which is beyond the scope of this article.

However it may be shown directly using 2 x 2 matrices:

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

The product matrix is:

(1.12) $$\underline{\underline{AB}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

The determinants are:

$$\det\underline{\underline{A}} = a_{11}a_{22} - a_{21}a_{12} \qquad \text{and} \qquad \det\underline{\underline{B}} = b_{11}b_{22} - b_{21}b_{12}$$

and therefore

$$\det(\underline{\underline{A}})\det(\underline{\underline{B}}) = (a_{11}a_{22} - a_{21}a_{12})(b_{11}b_{22} - b_{21}b_{12}) = a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{21}b_{12} - a_{21}a_{12}b_{11}b_{22} + a_{21}a_{12}b_{21}b_{12}$$

Whereas:

$$\det(\underline{\underline{AB}}) = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22})$$
$$= a_{11}b_{11}a_{21}b_{12} + (a_{11}b_{11}a_{22}b_{22}) + (a_{12}b_{21}a_{21}b_{12}) + a_{12}b_{21}a_{22}b_{22} - a_{21}b_{11}a_{11}b_{12} - (a_{21}b_{11}a_{12}b_{22}) - (a_{22}b_{21}a_{11}b_{12}) - a_{22}b_{21}a_{12}b_{22}$$
$$= a_{11}b_{11}a_{22}b_{22} - a_{22}b_{21}a_{11}b_{12} - a_{21}b_{11}a_{12}b_{22} + a_{21}b_{11}a_{12}b_{22} = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$$

It is then seen that four of the terms cancel e.g. $a_{11}b_{11}a_{21}b_{12} - a_{21}b_{11}a_{11}b_{12}$, and we are left with the former result.

The determinant of a matrix and the determinant of its transposed matrix are the same. They will namely have the same factors in each term just appearing in the reversed order.

If the determinant of a matrix $\underline{\underline{A}}$ is nonzero, $\underline{\underline{A}}$ has an inverse matrix $\underline{\underline{A}}^{-1}$, defined by:

(1.13) $$\underline{\underline{A}}\,\underline{\underline{A}}^{-1}=\underline{\underline{A}}^{-1}\,\underline{\underline{A}}=\underline{\underline{E}}$$

The algebraic determination of the inverse matrix is beyond the scope of this article.

It then follows from the rule of the determinant of a product, that:

(1.14) $$\det(\underline{\underline{A}}^{-1})\det(\underline{\underline{A}})=1$$

2. Linear functions and Eigenvalues.

A *linear* function $y = f(x)$ in an *n*-dimensional space, may be written as a matrix equation:

(2.1) $$\underline{y}=\underline{\underline{A}}\underline{x} \quad \Leftrightarrow \quad y_i=\sum_{j=1}^{n} a_{ij}x_j$$

An *eigenvector u* to a linear function is a vector which is mapped into a constant times itself.

(2.2) $\underline{\underline{A}}\underline{u}=\lambda\underline{u}$ or if we transpose this equation: $\underline{u}'\underline{\underline{A}}'=\lambda\underline{u}'$

The constant λ is called an eigenvalue of the linear function (the matrix equation), and we can see from (2.2) that a matrix and its transposed matrix have the same eigenvalues.

The equation: $\underline{\underline{A}}\underline{u}=\lambda\underline{u}$ can be written as: $(\underline{\underline{A}}-\lambda\underline{\underline{E}})\underline{u}=\underline{0}$, but this requires that $\det(\underline{\underline{A}}-\lambda\underline{\underline{E}})=0$. Written out, (and we settle for a 3 x 3 matrix).

(2.3) $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}=0$$

This 3 (*n*) degree polynomial will according to the fundamental theorem of algebra have 3 (*n*) (complex) roots. So that any (*n x n*) matrix has *n* (in general complex) eigenvalues.

For two real eigenvalues $\lambda_i \neq \lambda_j$ the corresponding eigenvectors u_i *and* u_j are orthogonal:

(2.4) $$\underline{u}'_i\,\underline{\underline{A}}\,\underline{u}_j=\lambda_j\underline{u}'_i\cdot\underline{u}_j \quad \text{and} \quad \underline{u}'_i\,\underline{\underline{A}}\,\underline{u}_j=\lambda_i\underline{u}'_i\cdot\underline{u}_j$$

(2.5) $$\lambda_j\underline{u}'_i\cdot\underline{u}_j=\lambda_i\underline{u}'_i\cdot\underline{u}_j \quad\Rightarrow\quad (\lambda_j-\lambda_i)\underline{u}'_i\cdot\underline{u}_j=0 \quad\Rightarrow\quad \underline{u}'_i\cdot\underline{u}_j=0$$

A matrix and its transposed matrix have the same eigenvalues. This follows from the fact that the determinant of a matrix and its transposed matrix are the same.

Complex conjugation is denoted with an asterisk *. $(a + ib)^* = (a - ib)$, where *i* here denotes the complex unit $i^2 = -1$.

However, working with complex numbers the scalar product of two vectors $u_1 \cdot u_2$ must be replaced by $u_1^* \cdot u_2$ (since otherwise the length of a vector would not be a real positive number).

Also working with complex numbers, the transpose of a matrix, must be replaced by its Hermitian conjugate $\underline{\underline{A}}^{\#}$, which is the transposed complex conjugated matrix.
Below is a summery of most of the common operations with a matrix $\underline{\underline{A}}$.

Transposed: $\underline{\underline{A}}'$: $a'_{ij} = a_{ji}$
Complex conjugate: $\underline{\underline{A}}^*$ (complex conjugation of all elements).
If $\underline{\underline{A}}^* = \underline{\underline{A}}$ then $\underline{\underline{A}}$ is real.
Symmetric if: $\underline{\underline{A}}' = \underline{\underline{A}}$
Hermitian if : $\underline{\underline{A}}^{\#} = \underline{\underline{A}}$
Orthogonal if: $\underline{\underline{A}}^{-1} = \underline{\underline{A}}'$. i.e. $\underline{\underline{A}}'\underline{\underline{A}} = \underline{\underline{E}}$
Unitary if: $\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{\#}$
Diagonal if $a_{ij} = 0 \; for \; i \neq j$
Trace of a matrix: $Tr\underline{\underline{A}} = \sum_i a_{ii}$
Calculation rules that apply for matrices:

$$(\underline{\underline{A}}\underline{\underline{B}})^{-1} = (\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}) \;,\quad (\underline{\underline{A}}\underline{\underline{B}})' = (\underline{\underline{B}}'\underline{\underline{A}}') \;,\; \det(\underline{\underline{A}}\underline{\underline{B}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$$

3. Coordinate transformations

We have seen that a linear function can be represented by a matrix equation:

(3.1) $$\underline{y} = \underline{\underline{A}}\underline{x} \quad \Leftrightarrow \quad y_i = \sum_{j=1}^{n} a_{ij} x_j$$

Now we shall look into how the vectors and matrices change under a coordinate transformation.
Assume that the base vectors are: $(\vec{e}_1, \vec{e}_2, \vec{e}_3, .., \vec{e}_n)$, and that the new base is: $(\vec{e}_1', \vec{e}_2', \vec{e}_3', .., \vec{e}_n')$.
Notice:
<u>Vectors in the new base are in the following marked with a hyphen It does not mean transposed</u>.

Since any vector in the vector space may be expanded on the base, then the relation between the two bases is:
(3.2) $$\vec{e}_j' = \sum_i t_{ij} \vec{e}_i \,,$$

where $\underline{\underline{T}} = (t_{ij})$ is the regular transformation matrix, and:

(3.3) $$\underline{e}' = \underline{e}\,\underline{\underline{T}} \quad \Leftrightarrow \quad \underline{e} = \underline{e}'\underline{\underline{T}}^{-1}$$

The same vector expanded on the two bases results in:

(3.4) $$\vec{x} = \sum_i x_i \vec{e}_i = \sum_i x_j ' \vec{e}_j ' = \sum_i x_j ' \sum_i t_{ij} \vec{e}_i \quad \Rightarrow \quad x_i = \sum_j t_{ij} x_j '$$

Or in matrix form:

(3.5) $$\underline{x} = \underline{\underline{T}}\, \underline{x}' \quad \Leftrightarrow \quad \underline{x}' = \underline{\underline{T}}^{-1} \underline{x}$$

In this context, vectors which appear to the left of a matrix are always row vectors, whereas vectors to the right are always column vectors:

(3.6) $$\underline{y} = \underline{\underline{A}}\, \underline{x} \qquad \text{and} \qquad \underline{y}' = \underline{\underline{A}}' \underline{x}'$$

If we in the first equation express $\underline{x} = \underline{\underline{T}}\, \underline{x}'$ and $\underline{y} = \underline{\underline{T}}\, \underline{y}'$ by the transformed coordinates, we find:

(3.7) $$\underline{\underline{T}}\, \underline{y}' = \underline{\underline{A}}\, \underline{\underline{T}}\, \underline{x}' \quad \Leftrightarrow \quad \underline{y}' = \underline{\underline{T}}^{-1} \underline{\underline{A}}\, \underline{\underline{T}}\, \underline{x}'$$

And the matrix representation for the linear function $\underline{y} = \underline{\underline{A}}\, \underline{x}$ in the marked coordinates is thus:

(3.8) $$\underline{y}' = \underline{\underline{A}}' \underline{x}' \quad \textit{where} \quad \underline{\underline{A}}' = \underline{\underline{T}}^{-1} \underline{\underline{A}}\, \underline{\underline{T}}$$

If $(\vec{e}_1, \vec{e}_2, \vec{e}_3, .., \vec{e}_n)$ is an orthonormal base, and we require that the new coordinate system: $(\vec{e}_1', \vec{e}_2', \vec{e}_3', .., \vec{e}_n')$ also form an orthonormal base, then the following must hold:

$$\delta_{ij} = \vec{e}_i' \cdot \vec{e}_j' = \left(\sum_k t_{ki} e_k \right) \left(\sum_m t_{mj} e_m \right) = \sum_k (t_{ki})^* t_{kj} = (\underline{\underline{T}}^* \underline{\underline{T}})_{ij}$$

Thus $\underline{\underline{T}}^* \underline{\underline{T}} = \underline{\underline{E}}$ and $\underline{\underline{T}}$ must be a orthogonal (unitary if complex) matrix.

4. Hermitian matrices

A Hermitian matrix H is a matrix, which is equal to its transposed, and complex conjugate matrix $H^{\#} = (H')^*$. If h are the elements of H, then $h_{ij} = (h_{ji})^*$.
For a real matrix, Hermitian is the same as symmetric.

The scalar product of two Hermitian vectors is (as mentioned above) defined as:

(4.1) $$u^{\#} \cdot v = \sum_i u_i^{\#} v_i$$

We shall then prove some theorems about eigenvalues and eigenfunctions of an Hermitian matrix. Let H be a Hermitian matrix, and let u_1 and u_2 be two eigenvectors belonging to the eigenvalues λ_1 and λ_2.

(4.2) $$Hu_1 = \lambda_1 u_1 \qquad \text{and} \qquad Hu_2 = \lambda_1 u_2$$

If we take the scalar product with u_2 in the first equation and the scalar product with u_1 in the second equation, we have:

(4.3) $$u_2^{\#} H u_1 = \lambda_1 u_2^{\#} u_1 \qquad u_1^{\#} H u_2 = \lambda_2 u_1^{\#} u_2$$

The left sides of the equations are complex conjugates, since:

(4.4) $$\begin{aligned}(u_2^{\#} H u_1)^* &= \sum_{ij} (u_{2i}^{\ *} H_{ij} u_{1j})^* \\ &= \sum_{ij} (u_{2i} (H_{ij})^* u_{1j}^{\ *}) = \sum_{ij} (u_{1j}^{\ *} H_{ji} u_{2i})\end{aligned}$$

That is:

(4.5) $$(u_2^{\#} H u_1)^* = (u_1^{\#} H u_2)$$

Therefore from: $u_2^{\#} H u_1 = \lambda_1 u_2^{\#} u_1$ and $u_1^{\#} H u_2 = \lambda_2 u_1^{\#} u_2$:

$$(\lambda_1 u_2^{\#} u_1)^* = (\lambda_2 u_1^{\#} u_2) \iff (\lambda_1^{\ *} u_2 u_1^{\#}) = (\lambda_2 u_1^{\#} u_2)$$

which gives:

(4.6) $$(\lambda_1^{\ *} - \lambda_2) u_1^{\#} u_2 = 0$$

Let us first assume that $\lambda_1 = \lambda_2$ and $u_1 = u_2 \neq 0$ then: $u_1^{\#} u_2 = u_1^{\#} u_1 = |u|^2 > 0$, therefore: $(\lambda_1^{\ *} = \lambda_1)$.

The eigenvalues of a Hermitian matrix are real.

Next suppose that $\lambda_1 \neq \lambda_2$. It then follows from (4,6) that $u_1^{\#} u_2 = 0$. As it is the case for real matrices:

(4.7) The eigenvectors belonging to different eigenvalues are orthogonal.

As a consequence the *n* different eigenvectors from a Hermitian matrix (when normalized) can be used as a base, which result in a diagonal matrix.

If several linear independent eigenvectors share the same eigenvalue, the eigenvalue is said to be degenerate.
We have proved above that the eigenvectors belonging to different eigenvalues are orthogonal, but what about linear independently eigenvectors to a degenerate eigenvalue?

It is however well known from linear algebra that you may form a orthonormal base in a *n*-dimensional vector space from *n* linear independent vectors. We shall settle for showing the procedure for three vectors. The method is known as the *Gram-Schmidt orthogonalization method*.

Suppose that we have three linear independent vectors $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ sharing the same (degenerate) eigenvalue λ. We will then proceed to form an orthogonal set. $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

First we notice, that any linear combination of $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ will also be an eigenvector with eigenvalue λ.

So we put: $\vec{v}_1 = \vec{u}_1$ and $\vec{v}_2 = \vec{u}_2 + \alpha\vec{v}_1$, and then determine α such that $\vec{v}_2 \cdot \vec{v}_1 = 0$

$$\vec{v}_2 \cdot \vec{v}_1 = 0 \quad \Leftrightarrow \quad (\vec{u}_2 + \alpha\vec{v}_1) \cdot \vec{v}_1 = 0 \quad \Leftrightarrow \quad \alpha = -\frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}$$

We continue to put: $\vec{v}_3 = \vec{u}_3 + \beta\vec{v}_1 + \gamma\vec{v}_2$ and determine β and γ so $\vec{v}_3 \cdot \vec{v}_1 = 0$ *and* $\vec{v}_3 \cdot \vec{v}_2 = 0$

$$\vec{v}_3 \cdot \vec{v}_1 = 0 \quad \Leftrightarrow \quad (\vec{u}_3 + \beta\vec{v}_1 + \gamma\vec{v}_2) \cdot \vec{v}_1 = 0 \quad \beta = -\frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}$$

$$\vec{v}_3 \cdot \vec{v}_2 = 0 \quad \Leftrightarrow \quad (\vec{u}_3 + \beta\vec{v}_1 + \gamma\vec{v}_2) \cdot \vec{v}_2 = 0 \quad \gamma = -\frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}$$

Obviously this procedure can be accomplished for any number of linear independent vectors.

The eigenvalues of a matrix are independent of the coordinate system.

Suppose that we make a coordinate transformation to the (hyphen) system :

(4.8) $$\underline{\underline{u}} = \underline{\underline{T}}\,\underline{\underline{u}}' \quad \Leftrightarrow \quad u' = \underline{\underline{T}}^{-1}\underline{\underline{u}}\,.$$

And suppose that λ is an eigenvalue of the matrix $\underline{\underline{A}}$ with the eigenvector $\underline{\underline{u}}$: $\underline{\underline{A}}\,\underline{\underline{u}} = \lambda\underline{\underline{u}}$, which implies:

(4.9) $$\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}}\,\underline{\underline{T}}^{-1}\underline{\underline{u}} = \lambda\underline{\underline{T}}^{-1}\underline{\underline{u}} \quad \Leftrightarrow \quad \underline{\underline{A}}'\underline{\underline{u}}' = \lambda\underline{\underline{u}}'\,.$$

Thus, if $\underline{\underline{u}}$ is an eigenvector of the matrix $\underline{\underline{A}}$, then its transformed eigenvector $\underline{\underline{u}}' = \underline{\underline{T}}^{-1}\underline{\underline{u}}$ is an eigenvector to the matrix $\underline{\underline{A}}'$ in the transformed base and having the same eigenvalue.
If we make a transformation to a base, where the rows in the matrix $\underline{\underline{A}}$ are the are the normalized eigenvectors.: $(\vec{e}_1', \vec{e}_2', \vec{e}_3', .., \vec{e}_n')$, then we have:

(4.10) $$\underline{\underline{e}}_i\underline{\underline{A}}\,\underline{\underline{e}}_j = \lambda_j\underline{\underline{e}}_i \cdot \underline{\underline{e}}_j = \lambda_j\delta_{ij}$$

If $\underline{\underline{e}}_i$ is to be considered as one row of a transformation matrix: $\underline{\underline{T}} = (\underline{\underline{e}}_j)$, then when we transform $\underline{\underline{A}}$, we find: $\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}} = \lambda_j\delta_{ij}$ a diagonal matrix, having the eigenvalues in the diagonal.

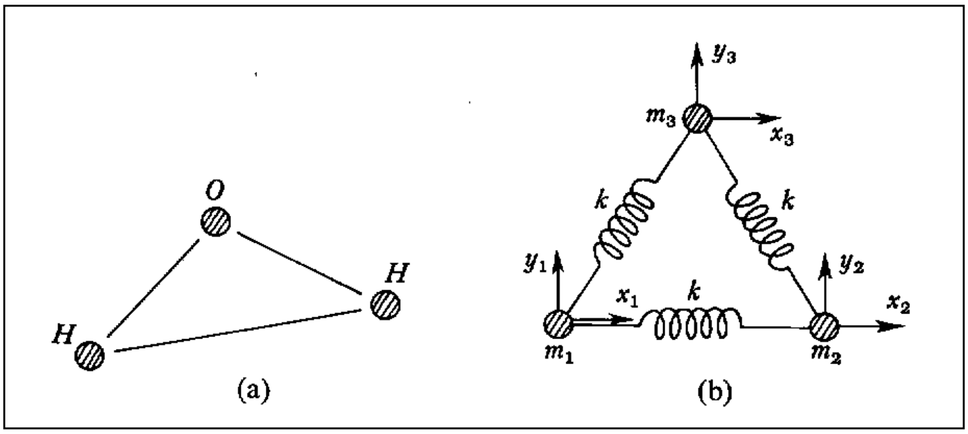
For a diagonal matrix $\underline{\underline{A}}$, having the eigenvalues in the diagonal, we have (trivially):

(4.11) $$Tr(\underline{\underline{A}}) = \sum_{i=1}^{n}\lambda_i \quad \text{and} \quad det(\underline{\underline{A}}) = \prod_{i=1}^{n}\lambda_i$$

In the physics of harmonic oscillations of a composite system one often encounters the problem of determining the modes or eigenvibrations of the system. This is done by finding the eigenvalues of the left side of the equation of motion, and determining the corresponding eigenvectors.

5. Example: Eigenvalues of a composite vibrating system.

We shall look at a vibrating system that could be a water molecule or three masses connected by springs, as shown in the figure below.



The system to the right consists of three masses m_1, m_2, m_3, connected in a triangle with springs having spring constants k_1, k_2, k_3. The three masses have the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3). When the system is at rest the distances between the masses are denoted:

$$l_1 = l_{12},\ \ l_2 = l_{23},\ \ l_3 = l_{31}\,.$$

The kinetic and potential energy are then given by:

(5.1) $$T = \sum_{i=1}^{3} \tfrac{1}{2} m_i (\dot{x}_i^{\,2} + \dot{y}_i^{\,2})$$

(5.2) $$U = \sum_{i=1}^{3} \tfrac{1}{2} k_i (l_i - \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}\ \ )^2 \qquad \text{where } (x_4,y_4) = (x_1,y_1)$$

To try to analyze this system with this choice of coordinates will lead to six nonlinear coupled differential equations, so we start by inventing new coordinates, which makes the potential energy a diagonal matrix.
The new coordinates are the displacement vectors between the three masses.
If $\vec{x}_1 = (x_1, y_1)$, $\vec{x}_2 = (x_2, y_2)$, $\vec{x}_3 = (x_3, y_3)$, and we take the centre of mass as the origin of the coordinate system, we must have:

(5.3) $$m_1\vec{x}_1 + m_2\vec{x}_2 + m_3\vec{x}_3 = (m_1 + m_2 + m_3)\vec{X}_{CM} = \vec{0}\,.$$

We then define three new coordinates:

$$\vec{q}_1 = \vec{x}_2 - \vec{x}_1\,,\quad \vec{q}_2 = \vec{x}_3 - \vec{x}_2\ \ ,\vec{q}_3 = \vec{x}_3 - \vec{x}_1\,,$$

And we assign the coordinates to the q vectors: $\vec{q}_1 = (q_1, q_2)$, $\vec{q}_3 = (q_3, q_4)$, $\vec{q}_3 = (q_5, q_6)$.
To get further we must express the x-vectors by the q-vectors. To do so we have the equations:

$$\vec{q}_1 = \vec{x}_2 - \vec{x}_1 , \quad \vec{q}_2 = \vec{x}_3 - \vec{x}_2 , \quad \vec{q}_3 = \vec{x}_3 - \vec{x}_1$$

together with the constraint:

$$m_1\vec{x}_1 + m_2\vec{x}_2 + m_3\vec{x}_3 = \vec{0}$$

Using the last equation to find: $m_1\vec{x}_1 = -m_2\vec{x}_2 - m_3\vec{x}_3$ and multiplying $\vec{q}_1 = \vec{x}_2 - \vec{x}_1$ by m_1 we find:

$m_1\vec{q}_1 = m_1\vec{x}_2 - m_1\vec{x}_1 \quad \Rightarrow \quad m_1\vec{q}_1 = m_1\vec{x}_2 + m_2\vec{x}_2 + m_3\vec{x}_3$, and two similar equations:
So we have the three set of equations:

$$(5.4) \qquad \begin{aligned} m_1\vec{q}_1 &= (m_1 + m_2)\vec{x}_2 + m_3\vec{x}_3 \\ m_2\vec{q}_2 &= m_1\vec{x}_1 + (m_2 + m_3)\vec{x}_3 \\ m_3\vec{q}_3 &= (m_1 + m_3)\vec{x}_1 + m_2\vec{x}_2 \end{aligned}$$

These three linear equations may be solved in a normal manner, but the solutions expressed by m_1, m_2, m_3 are not so interesting (but it can be found), so we just write:

$$\vec{x}_1 = b_1\vec{q}_1 + b_2\vec{q}_2 + b_3\vec{q}_3$$

$$\vec{x}_2 = c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3$$

$$\vec{x}_3 = d_1\vec{q}_1 + d_2\vec{q}_2 + d_3\vec{q}_3$$

The choice of these coordinates has the advantage that the potential energy is simply written as:

$$U = \tfrac{1}{2}k_1(l_1 - q_1)^2 + \tfrac{1}{2}k_2(l_2 - q_2)^2 + \tfrac{1}{2}k_3(l_3 - q_3)^2$$

For simplicity we shall replace l - q by q, since it makes no difference in the expression for the kinetic energy as (l - q)' = q'. So doing that, the potential energy becomes

$$(5.5) \qquad U = \tfrac{1}{2}k_1 q_1{}^2 + \tfrac{1}{2}k_2 q_2{}^2 + \tfrac{1}{2}k_3 q_3{}^2$$

Written with the q coordinates, the kinetic energy becomes.
(A period above a variable, as usual means differentiation with respect to time).

$$(5.6) \qquad \begin{aligned} T &= \sum_{i=1}^{3} \tfrac{1}{2} m_i \dot{\vec{x}}_i{}^2 = \tfrac{1}{2}m_1(b_1\dot{\vec{q}}_1 + b_2\dot{\vec{q}}_2 + b_3\dot{\vec{q}}_3)^2 + \tfrac{1}{2}m_2(b_1\dot{\vec{q}}_1 + b_2\dot{\vec{q}}_2 + b_3\dot{\vec{q}}_3)^2 + \tfrac{1}{2}m_3(b_1\dot{\vec{q}}_1 + b_2\dot{\vec{q}}_2 + b_3\dot{\vec{q}}_3)^2 \\ T &= \tfrac{1}{2}m_1(b_1{}^2\dot{q}_1{}^2 + b_2{}^2\dot{q}_2{}^2 + b_3{}^2\dot{q}_3{}^2 + 2b_1b_2\dot{\vec{q}}_1 \cdot \dot{\vec{q}}_2 + 2b_1b_3\dot{\vec{q}}_1 \cdot \dot{\vec{q}}_3 + 2b_2b_3\dot{\vec{q}}_2 \cdot \dot{\vec{q}}_3) + \tfrac{1}{2}m_2(\ldots) + \tfrac{1}{2}m_3(\ldots) \end{aligned}$$

$\vec{q}_1 = (q_1, q_2)$, $\vec{q}_3 = (q_3, q_4)$, $\vec{q}_3 = (q_5, q_6)$, so: $\dot{\vec{q}}_1^{\,2} = \dot{q}_1^{\,2} + \dot{q}_2^{\,2}$, $\dot{\vec{q}}_2^{\,2} = \dot{q}_3^{\,2} + \dot{q}_4^{\,2}$ and $\dot{\vec{q}}_3^{\,2} = \dot{q}_5^{\,2} + \dot{q}_6^{\,2}$, and the scalar products: $\dot{\vec{q}}_1 \cdot \dot{\vec{q}}_2 = q_1 q_3 + q_2 q_4$ $\dot{\vec{q}}_1 \cdot \dot{\vec{q}}_3 = q_1 q_5 + q_2 q_6$ $\dot{\vec{q}}_2 \cdot \dot{\vec{q}}_3 = q_3 q_5 + q_4 q_6$

Thus the kinetic and potential energies are bilinear forms in the coordinates $\dot{q}_i$ and q_i , i =1..6. We then rewrite the kinetic and potential energy:

(5.7) $$T = \sum_{i,j=1}^{6} t_{ij} \dot{q}_i \dot{q}_j \qquad \text{and} \qquad U = \sum_{i=1}^{6} \tfrac{1}{2} k_i q_i^2$$

When written out in matrix form:

$$T = (\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3 \quad \dot{q}_4 \quad \dot{q}_5 \quad \dot{q}_6) \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \\ t_{61} & t_{62} & t_{63} & t_{64} & t_{65} & t_{66} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{pmatrix} \qquad T = \underline{\underline{\dot{q}}}' \underline{\underline{T}} \underline{\underline{\dot{q}}}$$

$$U = (q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6) \begin{pmatrix} \frac{1}{2}k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}k_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}k_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} \qquad U = \underline{\underline{q}}' \underline{\underline{U}} \underline{\underline{q}}$$

Once we have found the eigenvalues λ_i of $\underline{\underline{T}}$ and the corresponding eigenvectors $\underline{\underline{p}}_i$, we may use the eigenvectors in a coordinate transformation to diagonalize $\underline{\underline{T}}$.
We have already seen that the matrix $\underline{\underline{S}}$, where the rows of $\underline{\underline{S}}$ are the normalized eigenvector, will diagonalize $\underline{\underline{T}}$, so that the diagonal elements are the eigenvalues: $\underline{\underline{S}}^{-1} \underline{\underline{T}} \underline{\underline{S}} = \lambda_i (\delta_{ij})$
Calling the transformed coordinates $\underline{\underline{\overline{q}}}$, we then have:

(5.8) $$\underline{\underline{q}} = \underline{\underline{S}} \underline{\underline{\overline{q}}} \quad \Leftrightarrow \quad \underline{\underline{\overline{q}}} = \underline{\underline{S}}^{-1} \underline{\underline{q}}$$

For a unitary matrix $\underline{\underline{S}}$, we have $\underline{\underline{S}}\underline{\underline{S}}' = \underline{\underline{E}}$ so that $\underline{\underline{S}}' = \underline{\underline{S}}^{-1}$.
This comes about because $\underline{\underline{S}}$ is symmetric and the rows in $\underline{\underline{S}}$ are the orthogonal eigenvectors, so that: $\underline{\underline{p}}_i' \cdot \underline{\underline{p}}_j = \delta_{ij}$.
In the transformed system we have:

(5.8) $$T = \underline{\underline{\dot{q}T\dot{q}}} = \underline{\underline{\dot{\overline{q}}'S}}^{-1}\underline{\underline{TS\dot{\overline{q}}}} = \underline{\underline{\dot{\overline{q}}'\overline{T}}}\,\dot{\overline{q}}$$

Where $\underline{\underline{\overline{T}}}$ is now a diagonal matrix, with the diagonal elements equal to the eigenvalues.
We have denoted the eigenvalues: $(\frac{1}{2}\mu_1, \frac{1}{2}\mu_2, \frac{1}{2}\mu_3, \frac{1}{2}\mu_4, \frac{1}{2}\mu_5, \frac{1}{2}\mu_6)$.

The potential energy, will transform in the same manner: $\underline{\underline{qS}}^{-1}\underline{\underline{USq}} = \underline{\underline{\overline{q}'\overline{U}\,\overline{q}}}$

To comply with the Lagrange formalism and for conceptual reasons, we shall keep the letter q for the transformed coordinates, so from now on: $\underline{\underline{q}}$ is the same as $\underline{\underline{\overline{q}}}$, whereas the previous $\underline{\underline{q}} = \underline{\underline{S\overline{q}}}$.
With these formal changes we obtain the following expressions for the potential and kinetic energies.

(5.9) $$U = \sum_{i,j=1}^{6} u_{ij} q_i q_j \quad \text{and} \quad T = \sum_{i=1}^{6} \tfrac{1}{2}\mu_i \dot{q}_i^{\,2}$$

The Lagrange equation: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} - \dfrac{\partial L}{\partial q_i} = 0$ then becomes:

(5.10) $$\mu_i \ddot{q}_i = \sum_{j=1}^{6} u_{ij} q_j$$

We have thus reduced the 6 nonlinear differential equations to 6 linear 2. order differential equations with constants coefficients. Such a system, may be transformed into a system of linear equations by setting:

(5.11) $$q_i = a_i e^{i\omega_i t}.$$

We then find the equations:

$$-\mu_i \omega_i^{\,2} a_i e^{i\omega_i t} = \sum_{j=1}^{6} u_{ij} a_j e^{i\omega_j t}$$

or written in matrix form

$$\begin{pmatrix} -\mu_1\omega_1^{\,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_2\omega_2^{\,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_3\omega_3^{\,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_4\omega_4^{\,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_5\omega_5^{\,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_6\omega_6^{\,2} \end{pmatrix} \begin{pmatrix} a_1 e^{i\omega_1 t} \\ a_2 e^{i\omega_2 t} \\ a_3 e^{i\omega_3 t} \\ a_4 e^{i\omega_4 t} \\ a_5 e^{i\omega_5 t} \\ a_6 e^{i\omega_6 t} \end{pmatrix} =$$

Eigenvalues in linear algebra

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\ u_{21} & u_{22} & u_{23} & u_{24} & u_{25} & u_{26} \\ u_{31} & u_{32} & u_{33} & u_{34} & u_{35} & u_{36} \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} & u_{46} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} & u_{56} \\ u_{61} & u_{62} & u_{63} & u_{64} & u_{65} & u_{66} \end{pmatrix} \begin{pmatrix} a_1 e^{i\omega_1 t} \\ a_2 e^{i\omega_2 t} \\ a_3 e^{i\omega_3 t} \\ a_4 e^{i\omega_4 t} \\ a_5 e^{i\omega_5 t} \\ a_6 e^{i\omega_6 t} \end{pmatrix}$$

This can be reduced into one matrix equation:

$$\begin{pmatrix} u_{11}-\mu_1\omega_1^2 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\ u_{21} & u_{22}-\mu_2\omega_2^2 & u_{23} & u_{24} & u_{25} & u_{26} \\ u_{31} & u_{32} & u_{33}-\mu_3\omega_3^2 & u_{34} & u_{35} & u_{36} \\ u_{41} & u_{42} & u_{43} & u_{44}-\mu_4\omega_4^2 & u_{45} & u_{46} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55}-\mu_5\omega_5^2 & u_{56} \\ u_{61} & u_{62} & u_{63} & u_{64} & u_{65} & u_{66}-\mu_6\omega_6^2 \end{pmatrix} \begin{pmatrix} a_1 e^{i\omega_1 t} \\ a_2 e^{i\omega_2 t} \\ a_3 e^{i\omega_3 t} \\ a_4 e^{i\omega_4 t} \\ a_5 e^{i\omega_5 t} \\ a_6 e^{i\omega_6 t} \end{pmatrix} = 0$$

But this linear system has only a zero solution if the determinant of the matrix is zero.

$$\begin{vmatrix} u_{11}-\mu_1\omega_1^2 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\ u_{21} & u_{22}-\mu_2\omega_2^2 & u_{23} & u_{24} & u_{25} & u_{26} \\ u_{31} & u_{32} & u_{33}-\mu_3\omega_3^2 & u_{34} & u_{35} & u_{36} \\ u_{41} & u_{42} & u_{43} & u_{44}-\mu_4\omega_4^2 & u_{45} & u_{46} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55}-\mu_5\omega_5^2 & u_{56} \\ u_{61} & u_{62} & u_{63} & u_{64} & u_{65} & u_{66}-\mu_6\omega_6^2 \end{vmatrix} = 0$$

From this we may (in principle) determine the 6 eigenfrequencies $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6)$ and (in principle) transforming the solution back, arriving at the solution to the original problem, finding:

$$\vec{x}_1 = (x_1(t), y_1(t))\ ,\ \vec{x}_2 = (x_2(t), y_2(t))\ ,\ \vec{x}_3 = (x_3(t), y_3(t))$$

expressed by linear combinations of: $a_1 e^{i\omega_1 t}, \ldots, a_6 e^{i\omega_6 t}$.

Although this delivers a theoretical solution, it is not realistic for a practical specific solution. Even if the solution was found it would be far from transparent, and you would have to resort to a graphical representation. But then (in the age of mathematical computers), you might as well solve the basic coupled differential equation numerically from the start.

6. Graphical solution to the vibrating triangle

The kinetic and potential energy are then given by:

$$T = \sum_{i=1}^{3} \tfrac{1}{2} m_i (\dot{x}_i^{\,2} + \dot{y}_i^{\,2})$$

$$U = \sum_{i=1}^{3} \tfrac{1}{2} k_i \left(l_i - \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}\ \right)^2 \qquad \text{where } (x_4, y_4) = (x_1, y_1)$$

The Lagrange equations are, where $L = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} - \frac{\partial U}{\partial x_i} = 0 \quad \Leftrightarrow$$

$$m_i \ddot{x}_i + k_i \frac{(l_i - \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}\ )(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} - k_{i-1} \frac{(l_{i-1} - \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}\ )(x_i - x_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} - \frac{\partial U}{\partial x_i} = 0 \quad \Leftrightarrow$$

$$m_i \ddot{x}_i + k_i \left(\frac{l_i}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} - 1\right)(x_{i+1} - x_i) - k_{i-1}\left(\frac{l_{i-1}}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} - 1\right)(x_i - x_{i-1}) = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_i} - \frac{\partial U}{\partial y_i} = 0 \quad \Leftrightarrow$$

$$m_i \ddot{y}_i + k_i \left(\frac{l_i}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} - 1\right)(y_{i+1} - y_i) - k_{i-1}\left(\frac{l_{i-1}}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} - 1\right)(y_i - y_{i-1}) = 0$$

Writing out the 6 differential equations:

$$m_1 \ddot{x}_1 + k_1 \left(\frac{l_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} - 1\right)(x_2 - x_1) - k_3\left(\frac{l_3}{\sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}} - 1\right)(x_1 - x_3) = 0$$

$$m_1 \ddot{y}_1 + k_1 \left(\frac{l_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} - 1\right)(y_2 - y_1) - k_3\left(\frac{l_3}{\sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}} - 1\right)(y_1 - y_3) = 0$$

$$m_2 \ddot{x}_2 + k_2 \left(\frac{l_2}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}} - 1\right)(x_3 - x_2) - k_1\left(\frac{l_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} - 1\right)(x_2 - x_1) = 0$$

$$m_2 \ddot{y}_2 + k_2 \left(\frac{l_2}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}} - 1\right)(y_3 - y_2) - k_1\left(\frac{l_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} - 1\right)(y_2 - y_1) = 0$$

$$m_3\ddot{x}_3 + k_3\left(\frac{l_3}{\sqrt{(x_1-x_3)^2+(y_1-y_3)^2}}-1\right)(x_1-x_3) - k_2\left(\frac{l_2}{\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}}-1\right)(x_3-x_2) = 0$$

$$m_3\ddot{y}_3 + k_3\left(\frac{l_3}{\sqrt{(x_1-x_3)^2+(y_1-y_3)^2}}-1\right)(y_1-y_3) - k_2\left(\frac{l_2}{\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}}-1\right)(y_3-y_2) = 0$$

There are six coupled second order differential equations. These equations can not be solved analytically, but a graphical computer solution, can of course be found.
The solution is done with a DOS Turbo Pascal program from 1995 (with a home made "windows" interface). The program was developed with Win95, where it was possible to make a screen dump. This was no longer possible with WinXP, and after WinXP, the program cannot run at all.
To my knowledge there exists no mathematical computer program, having the same capabilities. (Although they probably exist).
For these reasons the graphs show, are taken with a cell-phone.
The three graphs are various stages of the oscillations of the three masses. The masses chosen are 100 g, and 200 g, and the spring constants are 100 N/m and 200 N/m. The three masses are initially situated at 10 cm from each other.
There motions are clearly chaotic, as it is mostly the case for a three body problems.