Differential Geometry
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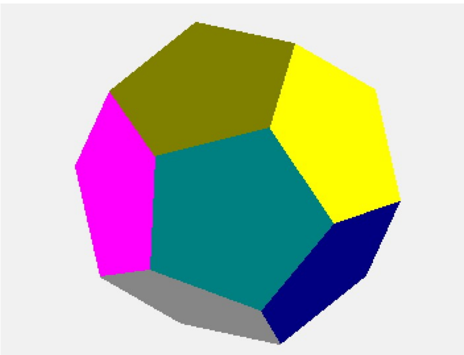

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Introduction

When I was an undergraduate student in mathematics and physics during the period 1964-1967 I had courses in mathematics for three years, which at that time - I believe were rather different from what is happening today, concerning the level of rigor and abstraction.
There were 4 subjects which especially fascinated me. One was vector analysis, which we used heavily in physics courses, the other three were the theory of analytic (complex) function, calculus of variations, and differential geometry.

The latter, I consider as most exacting. I did not follow lectures in general relativity during my graduate study, but 30 years later, when I was compelled to teach cosmology in high school I took up general relativity, and realized, that without a rather profound knowledge of differential geometry and tensor analysis, one was literally lost.
So I took the subject up again, and these notes represent some of my achievements.
I still consider differential geometry, as one of the most abstracts and challenging areas of mainstream mathematics.

Differential geometry is fascinating, because it delivers some deep theoretical results, applicable to an arbitrary (differentiable) surface, first developed and conceived by Gauss and Riemann.

When we analyze a plane curve, we usually do it, by analyzing its properties in a Euclidian plane, using an ordinary (Cartesian) coordinate system, where we describe the curve with two coordinates. Similarly, when we analyze the properties of a surface, we describe it in a coordinate system in a three dimensional space.
The obvious question is, whether it is possible to describe curves in a plane or surfaces in three dimensional space without resorting to measurement from without the geometrical object, but only by making measurements on the curve or in the surface itself - called the intrinsic geometry of the object,
To investigate this, is the objective of differential geometry, developed first by Gauss and later Riemann extended the results of Gauss to higher dimensions.

One may give a simple example, that illustrates, how you may decide, whether a surface is a Euclidian plane, or a curved surface, by measuring the ratio of the perimeter and the diameter of a circle, and compare it to pi. If it is a curved surface the ratio will always be less than pi.

This implies of course, that you have a way of measuring distances in the surface.

One of the main results of differential geometry is to find the shortest path between two points on the surface, which is called a geodesic.
That this can be obtained and expressed only by intrinsic geometry is closely connected to the concept of covariant differentiation. That the Gaussian curvature is an intrinsic geometric entity is the contents of Gauss' famous "teorema egregium".

Although – to my knowledge – differential geometry, has had only little practical application in the common geometrical world, is has since the appearance of Einstein's General theory of relativity, had a decisive influence on the development of that theory.
This is bound to the fact, that we have no higher dimensional space, where we can describe our three dimensional world or the four dimensional space time.

We have to resort to measurements in space-time itself to determine the geometrical structure of the universe.

There is another point of differential geometry. Apart from curves in the plane or in space, it is very difficult to give practical examples. The (non linear) differential equations in differential geometry are usually not to be solved analytically. Even to prove, that the shortest path on a sphere is a great circle, offers a challenge.
But the theoretical results however have had a vital importance in development of the general theory of relativity and cosmology.
We start out with the least complicated part of differential geometry, where the two concepts curvature and torsion is introduced.
Then we consider the geometry of two dimensional surfaces in ordinary Euclidian space. Here covariant derivative and geodesics are introduced. This part is however considerably more complex, than section one.

1. Parameter curves

First we shall introduce general properties of parameter curves in ordinary three-dimensional space, introducing the very important concept of curvature.
A curve in space is defined by a parametric representation of the coordinates. If P_t is a point at the curve, $P_t = P(t) = (x(t), y(t), z(t))$, where t is the parameter, belonging to some open interval on the real axis.
Introducing the three mutually orthogonal base vectors $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$, the position vector of the point P, may be written:

$$\overrightarrow{OP} = x_1(t)\,\vec{e}_1 + x_2(t)\,\vec{e}_2 + x_3(t)\,\vec{e}_3$$

There is an infinity of possible choices of parameter, since any increasing function of t may qualify. Specifically the length of the curve s, from one starting point, may be chosen as a parameter, s is called a natural parameter. It has great theoretical importance, but is usually useless for practical purposes since:

$$ds^2 = dx^2 + dy^2 + dz^2 \quad or \quad s = \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

Where a bullet over a letter, as usual means differentiation with respect to t (time in analytic mechanic).

To achieve the tangent vector to the curve, we form as usual the expression:

$$\vec{P}'(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{P(t)P(t+\Delta t)}}{\Delta t}$$

If the limit exists. If it does not, but the limit exist for $\Delta t \to 0+ \quad and \quad \Delta t \to 0-$ the curve has two semi tangents in that point, and the curve is said to have a tip at that point.
The tangent vector $\vec{v}(t) = \vec{P}'(t)$ is normally called the velocity.

Since $\vec{v}(t) = (\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t))$, the distance element is $ds = |\vec{v}|\, dt$.
As mentioned above, s (the length of the curve) can be taken as a parameter, named the natural parameter.

With this shift in parameter, we write the tangent vector as $\vec{v}(s)$. Obviously it is a unit vector, since

$$|\vec{v}(s)| = \frac{ds}{ds} = 1 \quad or \quad \frac{dP(t(s))}{dt} = \dot{P}(t(s))\frac{dt}{ds} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

1.1 Plane curves. Curvature

A curve in a two-dimensional Euclidian plane is assumed to have a natural parametric representation $\varphi(s)$, with a tangent vector $\vec{v}(s) = (v_1(s), v_2(s))$. Since $\vec{v}(s)$ is a unit vector, it may be written

$$\vec{v}(s) = (\cos(\varphi(s) + \varphi_0), \sin(\varphi(s) + \varphi_0))$$

Then we have $\varphi(s) = \cos^{-1}(v_1(s)) + const \quad or \quad \varphi(s) = \sin^{-1}(v_1(s)) + const$, which shows that $\varphi(s)$ is differentiable if $\vec{v}(s)$ is. $\varphi(s)$ marks the turn of the tangent vector as a function of the length of the curve. We may now define de *curvature* κ (kappa) as:

$$\kappa(s) = \frac{d\varphi(s)}{ds} \tag{1.1}$$

κ is the ration of tangent angular turn to the arc length. The normal vector to the curve is

$$\hat{v}(s) = (-\sin(\varphi(s) + \varphi_0), \cos(\varphi(s) + \varphi_0))$$

It then follows

$$\boxed{(1.2) \qquad \frac{d\vec{v}}{ds} = \hat{v}\frac{d\phi}{ds} = \kappa\hat{v} \qquad \text{and} \qquad \frac{d\hat{v}}{ds} = -\vec{v}\frac{d\phi}{ds} = -\kappa\vec{v}}$$

These relations are known as the Frenet formulas for plane curves.

The Frenet formulas look very simple, so we shall now proceed to find an expression for the curvature in the case, where the curve is given by an arbitrary parametric form.

$$\vec{x}(t) = (x_1(t), x_2(t))$$

From the Frenet formulas we have:

$$\frac{d\vec{v}_1}{ds} = \kappa\vec{v}_2 \qquad \Rightarrow \qquad \vec{v}_2 \cdot \frac{d\vec{v}_1}{ds} = \kappa$$

Since $\vec{v}_2$ is a unit vector

In the following, we denote the unit tangent vector $\vec{v}_1$ instead of $\vec{v}$ and the unit normal vector $\vec{v}_2$ instead of $\hat{v}$. It then follows:

$$\vec{v}_1(t) = \frac{(\dot{x}_1(t), \dot{x}_1(t))}{\sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2}} \qquad \text{and} \qquad \vec{v}_2(t) = \frac{(-\dot{x}_2(t), \dot{x}_1(t))}{\sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2}}$$

For convenience we drop the explicit dependence of *t*.

It reduces the volume of the calculations somewhat to notice:

$$\dot{s} = \frac{ds}{dt} = \sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2}$$

$$\frac{d\vec{v}_1}{ds} = \frac{d\vec{v}_1}{dt}\frac{dt}{ds} = \frac{1}{\dot{s}}\frac{(\ddot{x}_1 \cdot \dot{s} - \dot{x}_1 \cdot \ddot{s}\,,\, \ddot{x}_2 \cdot \dot{s} - \dot{x}_2 \cdot \ddot{s})}{\dot{s}^2}$$

Thus

$$\kappa = \vec{v}_2 \cdot \frac{d\vec{v}_1}{ds} = \frac{1}{\dot{s}^4}(-\dot{x}_2\ddot{x}_1 \cdot \dot{s} - \dot{x}_2\dot{x}_1 \cdot \ddot{s} + \dot{x}_1\ddot{x}_2 \cdot \dot{s} - \dot{x}_1\dot{x}_2 \cdot \ddot{s})$$

$$\kappa = \frac{1}{\dot{s}^4}(-\dot{x}_2\ddot{x}_1 \cdot \dot{s} + \dot{x}_2\dot{x}_1 \cdot \ddot{s} + \dot{x}_1\ddot{x}_2 \cdot \dot{s} - \dot{x}_1\dot{x}_2 \cdot \ddot{s})$$

(1.3) $$\kappa = \frac{\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1}{\dot{s}^3} \quad \Rightarrow \quad \kappa = \frac{\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1}{(\sqrt{\dot{x}_1{}^2 + \dot{x}_2{}^2})^3}$$

1.2 Examples

As an illustration of the theory, we shall find the curvature for three well known curves. The circle, the ellipse, and Archimedes spiral.

The circle

The circle has a parametric representation:

$$\varphi(t) = (r\cos\omega t, r\sin\omega t)$$

$\dot{x}_1 = -r\omega\sin\omega t \qquad \dot{x}_2 = r\omega\cos\omega t \qquad \ddot{x}_1 = -r\omega^2\cos\omega t \qquad \ddot{x}_2 = -r\omega^2\sin\omega t \qquad \dot{s} = r\omega$

$$\kappa = \frac{\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1}{(\sqrt{\dot{x}_1{}^2 + \dot{x}_2{}^2})^3} = \frac{-r\omega\sin\omega t \cdot r\omega^2\sin\omega t + r\omega\cos\omega t \cdot r\omega^2\cos\omega t}{(r\omega)^3} = \frac{1}{r}$$

Which indeed was to be expected, and for this reason, $\dfrac{1}{\kappa}$ is often called the curvature radius.

The ellipse

The ellipse forms a similar example however the curvature depends on position.

A parametric representation of the ellipse is:

$$\varphi(t) = (a\cos\omega t, b\sin\omega t)$$

Where *a* and *b* are the two semi axis. The calculations are quite similar to that of the circle.

$$\dot{x}_1 = -a\sin t \qquad \dot{x}_2 = b\cos t \qquad \ddot{x}_1 = -a\cos t \qquad \ddot{x}_2 = -b\sin t$$

$$\kappa = \frac{\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1}{(\sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2})^3} = \frac{ab\sin^2 t + ab\cos^2 t}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} == \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}$$

If $a = b = r$, it leads to $\kappa = 1/r$, as it should.

Archimedes spiral

Archimedes spiral in its simplest form has a parametric representation:

$$\varphi(t) = (t\cos t, t\sin t)$$

$$\dot{x}_1 = \cos t - t\sin t \qquad \ddot{x}_1 = -\sin t - \sin t - t\cos t \ = -2\sin t - t\cos t$$
$$\dot{x}_2 = \sin t + t\cos t \qquad \ddot{x}_2 = \cos t + \cos t - t\sin t \ = 2\cos t - t\sin t$$

$$\dot{x}_1(t)^2 + \dot{x}_2(t)^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 = 1 + t^2$$

$$\dot{x}_1\ddot{x}_2 = (\cos t - t\sin t)(2\cos t - t\sin t) = 2\cos^2 t + t^2\sin^2 t$$

$$\dot{x}_2\ddot{x}_1 = (\sin t - t\cos t)(-2\sin t - tcoost) = -2\sin^2 t - t^2\cos^2 t$$

$$\kappa = \frac{\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1}{(\sqrt{\dot{x}_1^{\,2} + \dot{x}_2^{\,2}})^3} = \frac{2+t^2}{(1+t^2)^{\frac{3}{2}}} \to \frac{1}{t} \text{ for large t,}$$

since t is the radius in the expanding circle.

2. Parameter curves in 3 dimensional space

From curves in a plane, we now turn to parameter curves in space. Besides curvature, we shall introduce the concept of *torsion*. If P_t is a point on the curve, the curve has a parametric representation:

$$\overrightarrow{OP_t} = \varphi(t) = (x_1(t), x_2(t), x_3(t))$$

We shall often omit the explicit dependence of t, when it is understood.
We shall also use the notation of differentials, (which usually requires, that the functions involved are two times differentiable). The infinitesimal distance element is given by:

$$ds^2 = dx_1^2 + dx_2^{\,2} + dx_3^{\,2}.$$

It then follows:

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{d\varphi}{dt}\right)^2 = \dot{x}_1^{\,2} + \dot{x}_2^{\,2} + \dot{x}_3^{\,2}$$

$\dfrac{d\varphi}{dt}$ is the tangent vector along the curve.
Introducing s as a natural parameter, the unit tangent vector becomes.

$$\vec{v}_1 = \frac{d\varphi}{ds} = \frac{d\varphi}{dt}\frac{dt}{ds} = \frac{1}{\dot{s}}\frac{d\varphi}{dt}$$

It is a unit vector since

$$d\varphi = d\vec{s} = (dx_1, dx_2, dx_3) \qquad \text{and} \qquad \frac{d\vec{s}}{ds} = \vec{e}\,.$$

From $\vec{v}_1^{\,2} = \vec{v}_1 \cdot \vec{v}_1 = 1$ we deduce

$$\vec{v}_1 \cdot \frac{d\vec{v}_1}{ds} = 0 \qquad \text{so that} \qquad \frac{d\vec{v}_1}{ds} \qquad \text{is perpendicular to } \vec{v}_1 .$$

We then define a *unit vector* $\vec{v}_2$, in correspondence with the definition of curvature for plane curves, by the equation:

(2.1) $$\frac{d\vec{v}_1}{ds} = \kappa \vec{v}_2$$

κ is called the *curvature*. Since $\vec{v}_2$ is also a unit vector:

$$\vec{v}_2 \cdot \vec{v}_2 = 1$$

It follows then, that

$$\vec{v}_2 \cdot \frac{d\vec{v}_2}{ds} = 0 \qquad \text{So} \qquad \frac{d\vec{v}_2}{ds} \qquad \text{is normal to } \vec{v}_2 ,$$

It can therefore be written:

(2.2) $$\frac{d\vec{v}_2}{ds} = \alpha \vec{v}_1 + \beta \vec{v}_3 ,$$

Where $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ is also a unit vector.

$\vec{v}_1 , \vec{v}_2 , \vec{v}_3$ are thus three local unit base vectors, at the point in question.

From $\vec{v}_1 \cdot \vec{v}_2 = 0$, we find:

$$\vec{v}_1 \cdot \frac{d\vec{v}_2}{ds} + \vec{v}_2 \cdot \frac{d\vec{v}_1}{ds} = 0 \qquad \Leftrightarrow$$
$$\vec{v}_1 \cdot (\alpha \vec{v}_1 + \beta \vec{v}_3) + \vec{v}_2 \cdot \kappa \vec{v}_2 = 0$$

$$\vec{v}_1 \cdot \vec{v}_1 = 1 \quad \wedge \quad \vec{v}_1 \cdot \vec{v}_3 = 0 \quad \wedge \quad \vec{v}_2 \cdot \vec{v}_2 = 1 ,$$

So we get: $\alpha + \kappa = 0$. or $\alpha = -\kappa$.

β is called the *torsion*, and is written τ. We shall then deduce the relations between the three local base vectors, their derivatives, the curvature and the torsion.

I: $$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 \quad \Rightarrow \quad \vec{v}_1 \cdot \vec{v}_3 = 0 \quad \Rightarrow \quad \frac{d\vec{v}_1}{ds} \cdot \vec{v}_3 + \vec{v}_1 \cdot \frac{d\vec{v}_3}{ds} = 0$$

II: $$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 \quad \Rightarrow \quad \vec{v}_2 \cdot \vec{v}_3 = 0 \quad \Rightarrow \quad \frac{d\vec{v}_2}{ds} \cdot \vec{v}_3 + \vec{v}_2 \cdot \frac{d\vec{v}_3}{ds} = 0$$

III: $$\vec{v}_3{}^2 = 1 \quad \Rightarrow \quad \vec{v}_3 \cdot \frac{d\vec{v}_3}{ds} = 0$$

I: $\kappa\vec{v}_2 \cdot \vec{v}_3 + \vec{v}_1 \cdot \dfrac{d\vec{v}_3}{ds} = 0 \quad \Rightarrow \quad \vec{v}_1 \cdot \dfrac{d\vec{v}_3}{ds} = 0$

II: $(-\kappa\vec{v}_1 - \tau\vec{v}_3) \cdot \vec{v}_3 + \vec{v}_2 \cdot \dfrac{d\vec{v}_3}{ds} = 0 \quad \Rightarrow \quad -\tau + \vec{v}_2 \cdot \dfrac{d\vec{v}_3}{ds} = 0$

III: From $\vec{v}_3 \cdot \dfrac{d\vec{v}_3}{ds} = 0$ *and* $\vec{v}_1 \cdot \dfrac{d\vec{v}_3}{ds} = 0$ it follows $\vec{v}_3 \perp \dfrac{d\vec{v}_3}{ds}$ and $\vec{v}_1 \perp \dfrac{d\vec{v}_3}{ds}$

This shows that $\dfrac{d\vec{v}_3}{ds} = c\vec{v}_2$ where c is some constant. Inserted in II it gives: $-\tau + c = 0 \quad c = \tau$.
We are then able to express the derivatives of the base vectors by the base vectors themselves
These formulas are called Frenet formulas.

(2.3) $$\frac{d\vec{v}_1}{ds} = \kappa\vec{v}_2 \qquad \frac{d\vec{v}_2}{ds} = -\kappa\vec{v}_1 - t\vec{v}_3 \qquad \frac{d\vec{v}_3}{ds} = \tau\vec{v}_2$$

Example
The spiral curve has a parametric representation

$$\varphi(t) = (x_1, x_2, x_3) = (r\cos t, r\sin t, t),$$

Where $\dot{\varphi}(t) = (-r\sin t, r\cos t, 1)$

$$\dot{s} = \sqrt{\dot{x}_1{}^2 + \dot{x}_2{}^2 + \dot{x}_3{}^2} = \sqrt{r^2 + 1}$$

$$\vec{v}_1 = \frac{1}{\sqrt{r^2+1}}(-r\sin t, r\cos t, 1)$$

$$\frac{d\vec{v}_1}{ds} = \kappa\vec{v}_2 \quad \text{and} \quad \frac{d\vec{v}_1}{ds} = \frac{d\vec{v}_1}{dt}\frac{dt}{ds} = \frac{(-r\cos t, -r\sin t, 0)}{\sqrt{r^2+1}} \quad \Rightarrow$$

$$\vec{v}_2 = \frac{(-r\cos t, -r\sin t, 0)}{r} \quad \wedge \quad \kappa = \frac{r}{\sqrt{r^2+1}}$$

For the cross product of two vectors, $\boldsymbol{a} = (a_1\ a_2,\ a_3)$ and $\boldsymbol{b} = (b_1\ b_2,\ b_3)$ we have the formula:

$$\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

$$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \left(\begin{vmatrix} r\cos t & 1 \\ -r\sin t & 0 \end{vmatrix}, \begin{vmatrix} 1 & -r\sin t \\ 0 & -r\cos t \end{vmatrix}, \begin{vmatrix} -r\sin t & r\cos t \\ -r\cos t & -r\sin t \end{vmatrix} \right)$$

$$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = (r\sin t, -r\cos t, r^2)$$

$$\frac{d\vec{v}_3}{ds} = \tau\vec{v}_2$$

$$\frac{d\vec{v}_3}{ds} = \frac{d\vec{v}_3}{dt}\frac{dt}{ds} = \frac{1}{\sqrt{r^2+1}}(r\cos t, r\sin t, 0)$$

$$\frac{d\vec{v}_3}{ds} = -\frac{r}{\sqrt{r^2+1}}\vec{v}_2 \qquad \Rightarrow \qquad \tau = -\frac{r}{\sqrt{r^2+1}}$$

3. Differentiable surface in 3-dimensional space

In this section, we confine ourselves to surfaces in ordinary three-dimensional space.
Let $\Omega = I_1 \times I_2 \subseteq R^2$ be an open interval where $I_1, I_2 \subseteq R$. A parametric representation of a surface is a mapping of Ω into space.
The two parameter values are traditionally written with upper indices i.e. $(u^1,u^2), (v^1,v^2)$ .Or in the case of a coordinate transformation $(\overline{u}^1,\overline{u}^2), (\overline{v}^1,\overline{v}^2)$. Notice, that the indices are not exponents!

Given a coordinate system $(O,\ \vec{e}_1, \vec{e}_2, \vec{e}_3)$ where O is the origin. $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the orthogonal base vectors.

$$\vec{e}_i \cdot \vec{e}_j = \delta_{i\,j}$$

and

$$\delta_{i\,j} = 1 \; for \; i = j \; and \;\; 0 \; otherwise,$$

where δ_{ij} is the Kronceker symbol.

The parametric representation of a point P is given by:

$$x_1 = x_1(u^1,u^2) \qquad x_2 = x_2(u^1,u^2) \qquad x_3 = x_3(u^1,u^2)$$

If the coordinate functions are differentiable, we may form the partial derivatives

$$\vec{D}_1 = \frac{\partial}{\partial u^1} \quad \text{and} \quad \vec{D}_2 = \frac{\partial}{\partial u^2}$$

(3.1)

$$\vec{D}_1 P = \left(\frac{\partial x_1}{\partial u^1}, \frac{\partial x_2}{\partial u^1}, \frac{\partial x_3}{\partial u^1}\right) \qquad \vec{D}_2 P = \left(\frac{\partial x_1}{\partial u^2}, \frac{\partial x_2}{\partial u^2}, \frac{\partial x_3}{\partial u^2}\right)$$

By a change of parameter, we understand a one to one mapping from (u^1,u^2) into $(\overline{u}^1,\overline{u}^2)$, so that

$$u^1 = u^1(\overline{u}^1,\overline{u}^2) \qquad u^2 = u^2(\overline{u}^1,\overline{u}^2)$$

The functional matrices:

$$\left\{\frac{\partial u^i}{\partial \overline{u}^j}\right\} = \begin{pmatrix} \frac{\partial u^1}{\partial \overline{u}^1} & \frac{\partial u^1}{\partial \overline{u}^2} \\ \frac{\partial u^2}{\partial \overline{u}^1} & \frac{\partial u^2}{\partial \overline{u}^2} \end{pmatrix} \quad \text{and} \quad \left\{\frac{\partial \overline{u}^i}{\partial u^j}\right\} = \begin{pmatrix} \frac{\partial \overline{u}^1}{\partial u^1} & \frac{\partial \overline{u}^1}{\partial u^2} \\ \frac{\partial \overline{u}^2}{\partial u^1} & \frac{\partial \overline{u}^2}{\partial u^2} \end{pmatrix}$$

are inverse to each other, as a consequence of the chain rule.

$$\frac{\partial u^i}{\partial \overline{u}^j}\frac{\partial \overline{u}^j}{\partial \overline{u}^k} = \delta_{ik}$$

For the partial derivative of x^i, we write:

(3.2) $$D_1^{\ i} = \frac{\partial x_i}{\partial u^1} \quad \text{and} \quad D_2^{\ j} = \frac{\partial x_j}{\partial u^2}$$

In the same manner for the transformed coordinates:

(3.3) $$\overline{D}_i^{\ j} = \frac{\partial x^j}{\partial \overline{u}^i} = \sum_{k=1}^{2} \frac{\partial x^j}{\partial u^k}\frac{\partial u^k}{\partial \overline{u}^i} = \frac{\partial x^j}{\partial u^k}\frac{\partial u^k}{\partial \overline{u}^i}$$

In the last expression, we have applied Einstein's summation convention:
Whenever an index appears twice, summation over that index is understood.
We shall consequently use the summation convention in the following.

If $u^i = u^i(t)$ is a parametric representation, then $P(u^1(t), u^2(t))$ is a parametric curve on the surface. The derivative with respect to t along the curve is:

$$\frac{d}{dt}P(u^1(t),u^2(t)) = D_i P(u^1(t),u^2(t))\frac{du^i}{dt}$$

When it is written out in coordinates

$$\frac{d}{dt}x_i(u^1(t),u^2(t)) = \frac{\partial x_i}{\partial u^j}\frac{du^j}{dt}$$ (implied summation over j)

The tangent vectors are

$$\vec{D}_i = \left(\frac{\partial x_1}{\partial u_i}, \frac{\partial x_2}{\partial u_i}, \frac{\partial x_3}{\partial u_i}\right)$$

And the plane through $P(u^1, u^2)$, spanned by the two vectors $\vec{D}_1$ and $\vec{D}_2$ is called the tangent plane in P.
Any parameter curve through P, has its tangent situated in the tangent plane, which also implies that the tangent plane is independent of the particular parametric representation.
Any choice of parametric representation leads to the same tangent plane, and for that reason we can define a local coordinate system in the tangent plane:

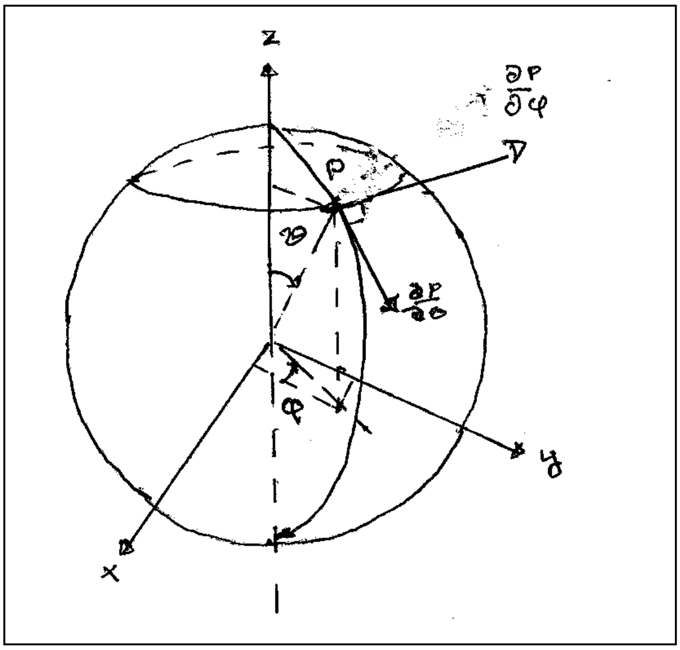
$$(P(u^1,u^2), \vec{D}_1 P(u^1,u^2), \vec{D}_2 P(u^1,u^2))$$

Example
In polar coordinates a sphere has the parametric representation:

$$x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi \quad z = r\cos\theta$$

$$\vec{D}P_\theta = (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta) \quad \vec{D}P_\varphi = (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0)$$

It is obvious that the two tangent vectors are perpendicular to each other, as one may also see from the figure below.



Any vector ***a*** parallel to the tangent plane, is a tangent vector, and can be expanded on the vectors $\vec{D}_1 P(u^1, u^2)$ *and* $\vec{D}_2 P(u^1, u^2)$:

$$\vec{a} = a^i \vec{D}_i P(u^1, u^2)$$

The transformation of the coordinates to conform to the parameters $(\bar{u}^1, \bar{u}^2)$ goes as follows:

$$\vec{a} = a^i \vec{D}_i P(u^1, u^2) = \bar{a}^j \vec{\bar{D}}_j P(\bar{u}^1, \bar{u}^2) = \bar{a}^j \vec{D}_i P(u^1, u^2) \frac{\partial u^i}{\partial \bar{u}^j}$$

It follows then: $a^i = \bar{a}^j \dfrac{\partial u^i}{\partial \bar{u}^j}$

3.1 The metric form

If ***a*** and ***b*** are two tangent vectors, one can form the scalar product ***a·b***.

$$\vec{a} = a^i \vec{D}_i P(u^1, u^2) \quad , \quad \vec{b} = b^j \vec{D}_j P(u^1, u^2)$$
$$\vec{a} \cdot \vec{b} = a^i b^j \vec{D}_i P(u^1, u^2) \cdot \vec{D}_j P(u^1, u^2) = a^i b^j g_{ij}$$

where

(3.4) $$g_{ij} = \vec{D}_i P(u^1, u^2) \cdot \vec{D}_j P(u^1, u^2)$$

g_{ij} is called the metric (fundamental) form. It has four components: g_{11}, g_{12}, g_{21}, g_{22}.
g_{11} and g_{22} are obviously positive, whereas $g_{12} = g_{21}$, because the scalar product is commutative.

By its definition, it is seen, that the metric form can be determined only by measuring lengths, and angles in the surface itself, i.e. the metric form is an invariant for the intrinsic geometry of the surface.

(3.5) $$g = \det\{ g_{ij} \} = g_{11} \cdot g_{22} - g_{12}^2$$

$$g = |\vec{D}_1 P|^2 \cdot |\vec{D}_2 P|^2 - |\vec{D}_1 P \cdot \vec{D}_2 P|^2 = |\vec{D}_1 P \times \vec{D}_2 P|^2 > 0$$

The last equation follows from:

$$|\vec{a} \cdot \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cos v \qquad \text{and} \qquad |\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin v$$

So

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 \cdot |\vec{b}|^2 \sin^2 v = |\vec{a}|^2 \cdot |\vec{b}|^2 (1 - \cos^2 v) =$$
$$|\vec{a}|^2 \cdot |\vec{b}|^2 - |\vec{a}|^2 \cdot |\vec{b}|^2 \cos^2 v = |\vec{a}|^2 \cdot |\vec{b}|^2 - |\vec{a} \cdot \vec{b}|^2$$

The metric form g_{ij} totally determines the measurements of distances on the surface in question. If we have a parametric representation of a curve: $u^i = u^i(t)$ then

$$\left(\frac{ds}{dt}\right)^2 = \left|\frac{d}{dt} P(u^1(t), u^2(t))\right|^2$$
$$= \left(\vec{D}_i P(u^1(t), u^2(t)) \frac{du^i(t)}{dt}\right)^2 = \vec{D}_i P \cdot \vec{D}_j P \frac{du^i(t)}{dt} \frac{du^j(t)}{dt}$$
$$= g_{ij} \frac{du^i(t)}{dt} \frac{du^j(t)}{dt}$$

And consequently

(3.6) $$\frac{ds}{dt} = \sqrt{g_{ij} \frac{du^i(t)}{dt} \frac{du^j(t)}{dt}} \qquad \text{or} \qquad s(t_1, t_2) = \int_{t_1}^{t_2} \sqrt{g_{ij} \frac{du^i(t)}{dt} \frac{du^j(t)}{dt}}\, dt$$

Correspondingly, one may find the angle v between two vectors $\vec{a} = (a^1, a^2)$ *and* $\vec{b} = (b^1, b^2)$ as

(3.7) $$\cos v = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{g_{ij} a^i b^j}{\sqrt{g_{lk} a^l a^k} \sqrt{g_{mn} a^m a^n}}$$

We have chosen different letters for the independent indices in the three expressions, because of the summation convention.
The transformation of the metric form $g_{ij}(u^1, u^2)$ due to a shift in parameters to $(\bar{u}^1, \bar{u}^2)$, may be obtained from the definition

$$g_{ij} = \vec{D}_i P(u^1, u^2) \cdot \vec{D}_j P(u^1, u^2),$$

and the transformation properties of $\vec{D}_i P(u^1, u^2)$.

$$\overline{g}_{ij}(\overline{u}^1, \overline{u}^2) = \vec{\overline{D}}_i P(\overline{u}^1, \overline{u}^2) \cdot \vec{\overline{D}}_j P(\overline{u}^1, \overline{u}^2)$$
$$= \left(\frac{\partial u^k}{\partial \overline{u}^i} \vec{D}_k P(u^1, u^2) \right) \cdot \left(\frac{\partial u^l}{\partial \overline{u}^j} \vec{D}_l P(u^1, u^2) \right)$$

(3.8)
$$\overline{g}_{ij}(\overline{u}^1, \overline{u}^2) = \left(\frac{\partial u^k}{\partial \overline{u}^i} \frac{\partial u^l}{\partial \overline{u}^j} \vec{D}_k P(u^1, u^2) \cdot \vec{D}_l P(u^1, u^2) \right)$$
$$= \frac{\partial u^k}{\partial \overline{u}^i} \frac{\partial u^l}{\partial \overline{u}^j} g_{kl}(u^1, u^2)$$

We recognize that this is same transformation as for the scalar product of two vectors. And it follows, that the scalar product is independent of the choice of parameters (u^1, u^2).

The differential area of a small piece of the surface, can easily be found. If an area dS is spanned by the two vectors ***da*** and ***db***, the area is given by the formula: $dA = |d\vec{a} \times d\vec{b}|$.
Therefore an infinitesimal area of the surface, can be written as:

(3.9)
$$dA = |\vec{D}_i P \times \vec{D}_j P| \, du^1 du^2$$

But we have seen earlier in (3.5) that $|\vec{D}_i P \times \vec{D}_j P|^2 = \det\{g_{ij}\}$, so the formula for the area of the surface is:

(3.10)
$$S = \iint_A \sqrt{\det\{g_{ij}\}} du^1 du^2$$

To measure lengths on the surface, we turn to the "definition" of the metric form.

$$\left(\frac{ds}{dt} \right)^2 = \left| \frac{d}{dt} P(u^1(t), u^2(t) \right|^2$$
$$= \left(\vec{D}_i P(u^1(t), u^2(t) \frac{du^i(t)}{dt} \right)^2 = \vec{D}_i P \cdot \vec{D}_j P \frac{du^i(t)}{dt} \frac{du^j(t)}{dt}$$
$$= g_{ij} \frac{du^i(t)}{dt} \frac{du^j(t)}{dt}$$

We then find:

(3.11)
$$s_{12} = \int_{t_1}^{t_2} \sqrt{g_{ij} \frac{du^i(t)}{dt} \frac{du^j(t)}{dt}} dt$$

3.2 The curvature of a surface

In a regular point $P(u^1,u^2)$ one can define the normal unit vector to the surface as:

(3.12) $$\vec{n}(u^1,u^2)=\frac{\vec{D}_1P(u^1,u^2)\times\vec{D}_2P(u^1,u^2)}{|\vec{D}_1P(u^1,u^2)\times\vec{D}_2P(u^1,u^2)|}$$

If the positive orientation of the surface is from $\vec{D}_1$ to $\vec{D}_2$ then $\vec{n}(u^1,u^2)$ is an outward normal vector.
The first derivatives of $P(u^1, u^2)$ determines the tangent plane, lengths and areas. As it was the case with parameter curves, we need the second derivatives to describe curvature.

(3.13) $$\vec{D}_i\vec{D}_jP(u^1,u^2)=\frac{\partial^2P(u^1,u^2)}{\partial u^i\partial u^j}$$

The second derivatives are vectors in space, and can therefore be expanded on $\vec{D}_1,\vec{D}_2$ *and* $\vec{n}$ (the normal to the tangent plane). For that purpose one has introduced the so called Christoffel symbols Γ^k_{ij} and L_{ij}.
Since $\vec{n}$ is a unit vector $\vec{n}\cdot\vec{n}=1$, then $\vec{n}\cdot\vec{D}_i\vec{n}=0$ which means that $\vec{D}_i\vec{n}$ lies in the tangent plane, and $\vec{D}_i\vec{n}$ can then be expressed as a linear combination of $\vec{D}_1P$ *and* $\vec{D}_2P$ as

$$\vec{D}_i\vec{n}=B_i^j\vec{D}_jP$$

Introducing the symbol B_i^j, (which does not have an independent name).
Since ***n*** is perpendicular to the tangent plane, $\vec{n}\cdot\vec{D}_kP=0$, and one gets by differentiating:

$$\vec{D}_i(\vec{n}\cdot\vec{D}_kP)=\vec{D}_i\vec{n}\cdot\vec{D}_kP+\vec{n}\cdot\vec{D}_i\vec{D}_kP=0$$

$$\vec{D}_i(\vec{n}\cdot\vec{D}_kP)=B_i^j\vec{D}_jP\cdot\vec{D}_kP+\vec{n}\cdot\vec{D}_i\vec{D}_kP=0$$

Since $g_{jk}=\vec{D}_jP\cdot\vec{D}_kP$ by definition we get:

(3.14) $$B_i^jg_{jk}=-\vec{n}\cdot\vec{D}_i\vec{D}_kP=-L_{ik}$$

Introducing the Christoffel symbol L_{ik}
By multiplying with the inverse matrix to $(g_{jk})^{-1}=g^{kl}$ where $g_{jk}g^{kl}=\delta_{jl}$ we find

(3.15) $$B_i^jg_{jk}g^{kl}=-L_{ik}g^{kl}\;\Rightarrow\;B_i^j\delta_{jl}=-L_{ik}g^{kl}\;\Rightarrow\;B_i^l=-L_{ik}g^{kl}$$

Then it follows:

(3.16) $$\vec{D}_i\vec{n} = B_i^j\vec{D}_jP = -L_{ji}g^{kj}\vec{D}_kP$$

Or writing the explicit dependence of the parameters.

(3.16) $$\vec{D}_i\vec{n}(u^1,u^2) = B_i^j\vec{D}_jP(u^1,u^2) = -L_{ji}(u^1,u^2)g^{kj}(u^1,u^2)\vec{D}_kP(u^1,u^2)$$

We are now able to write an expression for the second derivative in the coordinate system $(\vec{D}_1P, \vec{D}_2P, \vec{n})$. Since $\vec{n}\cdot\vec{D}_i\vec{D}_kP = L_{ik}$, L_{ik} is the coordinate along $\vec{n}$.
Introducing the Christoffel symbols Γ_{ij}^k in the equation, we may write:

(3.17) $$\vec{D}_i\vec{D}_jP = \Gamma_{ij}^k\vec{D}_kP + L_{ij}\vec{n}$$

Or writing the explicit dependence of the parameters

(3.17) $$\vec{D}_i\vec{D}_jP(u^1,u^2) = \Gamma_{ij}^k(u^1,u^2)\vec{D}_kP(u^1,u^2) + L_{ij}(u^1,u^2)\vec{n}$$

Let $u^i(s)$ be a natural parametric representation for a curve in the surface.
A natural parameter means that "the speed" $\dfrac{ds}{ds} = 1$ and consequently:

$$\left(\frac{ds}{ds}\right)^2 = g_{ij}(u^1(s),u^2(s))\frac{du^1(s)}{ds}\frac{du^2(s)}{ds} = 1$$

Our objective is to find an expression for the curvature of the parameter curve in *P*, and to this end, we shall apply the Frenet formulas.

(3.18) $$\frac{d\vec{v}_1}{ds} = \kappa\vec{v}_2 \qquad \frac{d\vec{v}_2}{ds} = -\kappa\vec{v}_1 - t\vec{v}_3 \qquad \frac{d\vec{v}_3}{ds} = \tau\vec{v}_2$$

$\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are mutual orthogonal unit vectors. $\vec{v}_1$ is the tangent vector, and $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$.

$$\vec{v}_1(s) = \frac{d}{ds}P(u^1(s),u^2(s)) = \vec{D}_iP(u^1(s),u^2(s))\frac{du^i}{ds}$$

$$\frac{d\vec{v}_1}{ds} = \frac{d}{ds}(\vec{D}_iP(u^1(s),u^2(s))\frac{du^i}{ds}) = \vec{D}_i\vec{D}_jP(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds} + \vec{D}_iP(u^1,u^2)\frac{d^2u^i}{ds^2}$$

$$\vec{D}_i\vec{D}_jP(u^1,u^2) = \Gamma_{ij}^k(u^1,u^2)\vec{D}_kP(u^1,u^2) + L_{ij}(u^1,u^2)\vec{n}$$

(3.19) $$\frac{d\vec{v}_1}{ds} = (\Gamma_{ij}^k(u^1,u^2)\vec{D}_kP(u^1,u^2) + L_{ij}(u^1,u^2)\vec{n})\frac{du^i}{ds}\frac{du^j}{ds} + \vec{D}_iP(u^1,u^2)\frac{d^2u^i}{ds^2}$$

In the last term we have changed the summation index from *i* to *k,* since any letter is applicable, and after rearranged the expression a bit, we find.

$$\frac{d\vec{v}_1}{ds} = (\Gamma^k_{ij}(u^1,u^2)\vec{D}_k P(u^1,u^2) + L_{ij}(u^1,u^2)\vec{n})\frac{du^i}{ds}\frac{du^j}{ds} + \vec{D}_k P(u^1,u^2)\frac{d^2u^k}{ds^2}$$

$$\frac{d\vec{v}_1}{ds} = (\Gamma^k_{ij}(u^1,u^2)\vec{D}_k P(u^1,u^2) + L_{ij}(u^1,u^2)\vec{n})\frac{du^i}{ds}\frac{du^j}{ds} + \vec{D}_k P(u^1,u^2)\frac{d^2u^k}{ds^2}$$

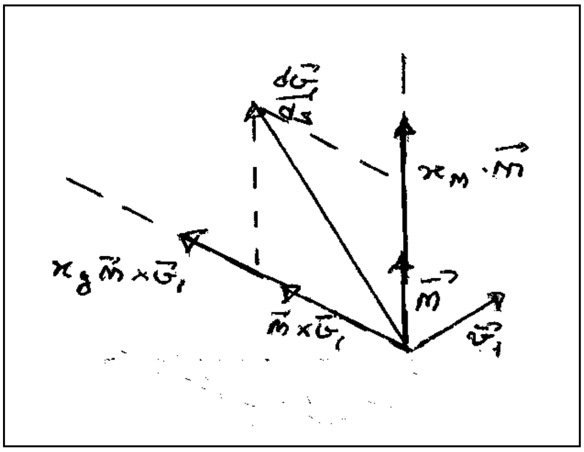
(3.19) $$\frac{d\vec{v}_1}{ds} = (\Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds} + \frac{d^2u^k}{ds^2})\vec{D}_k P(u^1,u^2) + \frac{du^i}{ds}\frac{du^j}{ds}L_{ij}(u^1,u^2)\vec{n}$$

This is perhaps easier to grasp, if we omit (as we shall do in the following) the explicit dependence of the parameters (u^1,u^2).

(3.19) $$\frac{d\vec{v}_1}{ds} = (\frac{d^2u^k}{ds^2} + \Gamma^k_{ij}\frac{du^i}{ds}\frac{du^j}{ds})\vec{D}_k P + \frac{du^i}{ds}\frac{du^j}{ds}L_{ij}\vec{n}$$

Recall that, $\frac{d\vec{v}_1}{ds} = \kappa\vec{v}_2$, so in the expression above, we have resolved the curvature vector in components along the vectors $(\vec{D}_1 P, \vec{D}_2 P, \vec{n})$, the tangent vectors in the plane, and the normal vector to the plane.

Furthermore we have: $\frac{d\vec{v}_1}{ds} \perp \vec{v}_1$ *and* $\vec{n} \perp \vec{v}_1$ which allows us to resolve $\frac{d\vec{v}_1}{ds}$ in a component along $\vec{n}$ and a component along $\vec{n} \times \vec{v}_1$. See the figure.



(3.20) $$\frac{d\vec{v}_1}{ds} = \kappa_g \vec{n} \times \vec{v}_1 + \kappa_n \vec{n}$$

Where according to (3.19)

$$\kappa_n = \frac{du^i}{ds}\frac{du^j}{ds}L_{ij}(u^1,u^2)$$

κ_g is denoted the *geodesic curvature* in *P*, and κ_n is denoted the *normal curvature in P*. They are both signed entities.

If $\vec{a} = a^i \vec{D}_i P$ is a tangent vector in P_0 we find the normal curvature in the direction of $\vec{a}$, by

$$\kappa_n(\vec{a}) = \frac{L_{ij} a^i a^j}{|\vec{a}|^2}$$

Since

$|\vec{a}|^2 = g_{ij} a^i a^j$ We get:

$$\kappa_n(\vec{a}) = \frac{L_{ij} a^i a^j}{g_{kl} a^k a^l} \tag{3.21}$$

The quadratic form $L_{ij} a^i a^j$ is denoted the *second fundamental form*. The first form is the metric form $g_{ij} a^i a^j$

$L_{ij} a^i a^j$ is a bilinear form, and from the theory, we know that it can be brought to diagonal form by an appropriate coordinate transformation. If the transformed coordinate system has the orthogonal tangent base vectors $(\vec{e}_1, \vec{e}_2)$, then for a vector $\vec{a} = a^i \vec{e}_i$ we have:

$$\kappa_n(\vec{a}) |\vec{a}|^2 = \kappa_1 (a^1)^2 + \kappa_2 (a^2)^2$$

κ_i is obviously the normal curvature in the direction of $\vec{e}_i$. κ_1 *and* κ_2 are called the main normal curvatures. Furthermore we have

$$\vec{a} \cdot \vec{e}_1 = |\vec{a}| \cos(\vec{a}, \vec{e}_1) = a^i \vec{e}_i \cdot \vec{e}_1 = a^1$$

$$\begin{aligned}\sin^2(\vec{a}, \vec{e}_1) &= 1 - \cos^2(\vec{a}, \vec{e}_1) \\ &= 1 - \frac{(a^1)^2}{|\vec{a}|^2} = \frac{(a^1)^2 + (a^2)^2 - (a^1)^2}{|\vec{a}|^2} \\ &= \frac{(a^2)^2}{|\vec{a}|^2}\end{aligned}$$

We therefore get Euler's formula.

$$\kappa_n(\vec{a}) = \kappa_1 \cos^2(\vec{a}, \vec{e}_1) + \kappa_2 \sin^2(\vec{a}, \vec{e}_1) \tag{3.22}$$

κ_1 *and* κ_2 are bound to be the maximum and minimum normal curvatures in *P*.

This follows easily, if we consider the function: $f(x) = a\cos^2 x + b\sin^2 x$

$f'(x) = 2a\cos x(-\sin x) + 2b\sin x\cos x \quad f'(x) = 0 \iff \cos x = 0 \vee \sin x = 0$

Since *f* is periodic, it must be a *max* and a *min*.

The two unit vectors $(\vec{e}_1, \vec{e}_2)$,

$$\vec{e}_1 = e_1^i \vec{D}_i P \quad and \quad \vec{e}_2 = e_2^i \vec{D}_i P$$

corresponding to the directions of the main normal curvature, are also the vectors that diagonalize L_{ij} so, written on matrix form.

$$\underline{\underline{e}}' \, \underline{\underline{L}} \, \underline{\underline{e}} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

Since $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ and $\vec{e}_i \cdot \vec{e}_j = g_{ij} e_1^i e_2^j$ we find

$$\underline{\underline{e}}' \, \underline{\underline{g}} \, \underline{\underline{e}} = \underline{\underline{E}}$$

The two main curvatures are roots in the eigenvalue determinant

$$\det(\underline{\underline{e}}' \, \underline{\underline{L}} \, \underline{\underline{e}} - \lambda \, \underline{\underline{e}}' \, \underline{\underline{g}} \, \underline{\underline{e}}) \qquad \text{but} \qquad \underline{\underline{e}}' \, \underline{\underline{L}} \, \underline{\underline{e}} - \lambda \, \underline{\underline{e}}' \, \underline{\underline{g}} \, \underline{\underline{e}} = \underline{\underline{e}}'(\underline{\underline{L}} - \lambda \, \underline{\underline{g}}) \, \underline{\underline{e}}$$

Since both $\underline{\underline{e}}$ and $\underline{\underline{e}}'$ are regular, the two main curvatures are also roots in

(3.22) $$\det(\underline{\underline{L}} - \lambda \, \underline{\underline{g}}) = 0$$

Writing the determinant out:

$$\begin{vmatrix} L_{11} - \lambda g_{11} & L_{12} - \lambda g_{12} \\ L_{21} - \lambda g_{21} & L_{22} - \lambda g_{22} \end{vmatrix} = 0$$

The determinant is a 2. degree polynomial in λ, and it leads to the quadratic equation

$$a\lambda^2 + b\lambda + c = 0 .$$

It is easy to verify, when evaluating the determinant, that

$$a = g_{11} g_{22} - g_{12}{}^2 = g = \det\{g_{ij}\}$$
$$c = L_{11} L_{22} - L_{12}{}^2 = L = \det\{L_{ij}\}$$
$$b = g_{11} L_{22} - 2 g_{12} L_{12} + g_{22} L_{11}$$

Since we know that in the reduced quadratic equation the sum of the roots are equal to the coefficient of λ, and the product of the roots is equal to the last term, we get:

$$\kappa_1 + \kappa_2 = \frac{b}{a} \quad and \quad \kappa_1 \cdot \kappa_2 = \frac{c}{a}$$

From which we obtain:

(3.23) $$\kappa_1 + \kappa_2 = \frac{g_{11}L_{22} - 2g_{12}L_{12} + g_{22}L_{11}}{g} \quad and \quad \kappa_1 \cdot \kappa_2 = \frac{L}{g}$$

For the average curvature $H = \frac{\kappa_g + \kappa_n}{2}$ and for the Gaussian curvature: $K = \kappa_g \cdot \kappa_n$, we thus have the expressions:

(3.24) $$H = \frac{g_{11}L_{22} - 2g_{12}L_{12} + g_{22}L_{11}}{2g} \quad and \quad K = \frac{L}{g}$$

3.3 The intrinsic geometry of a surface

The intrinsic geometric properties of a surface are the geometric properties that can be established by measurements in the surface itself, without referring to the outer space, where the surface is situated.
Which, by the way are the same geometric properties which are conserved by an isometric mapping.
We assume, that the two dimensional surface has a parametric description

$$P(u^1, u^2) = (x_1(u^1, u^2), x_2(u^1, u^2), x_3(u^1, u^2))$$

The metric form is defined by:

$$g_{ij} = \vec{D}_i P(u^1, u^2) \cdot \vec{D}_j P(u^1, u^2)$$

where $\vec{D}_i$ is the partial derivative with respect to u^i.
$\vec{D}_i$ is a vector in ordinary three dimensional space. It then follows from differentiating:

$$\frac{\partial g_{ij}}{\partial u^k} = \vec{D}_k \vec{D}_i P(u^1, u^2) \cdot \vec{D}_j P(u^1, u^2) + \vec{D}_i P(u^1, u^2) \cdot \vec{D}_k \vec{D}_j P(u^1, u^2)$$

We introduce the Christoffel symbols of the first kind by:

$$\Gamma_{ij,k} = \vec{D}_i \vec{D}_j P(u^1, u^2) \cdot \vec{D}_k P(u^1, u^2)$$

Obviously $\Gamma_{ij,k} = \Gamma_{ji,k}$ since the order of partial differentiation can be swapped. We then obtain:

$$\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} = (\Gamma_{ij,k} + \Gamma_{ik,j}) + (\Gamma_{jk,i} + \Gamma_{ji,k}) - (\Gamma_{ki,j} + \Gamma_{kj,i})$$

When permuting the first two indices in two of the terms applying $\Gamma_{ij,k} = \Gamma_{ji,k}$ we get:

$$(\Gamma_{ij,k}+\Gamma_{ik,j})+(\Gamma_{kj,i}+\Gamma_{ji,k})-(\Gamma_{ki,j}+\Gamma_{kj,i})=2\Gamma_{ij,k}$$

thus

(3.25) $$\Gamma_{ij,k}=\tfrac{1}{2}(\frac{\partial g_{jk}}{\partial u^i}+\frac{\partial g_{ki}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k})$$

From the definition

$$\Gamma_{ij,k}=\vec{D}_i\vec{D}_jP(u^1,u^2)\cdot\vec{D}_kP(u^1,u^2)$$

And from (3.17)

$$\vec{D}_i\vec{D}_jP(u^1,u^2)=\Gamma_{ij}^k(u^1,u^2)\vec{D}_kP(u^1,u^2)+L_{ij}(u^1,u^2)\vec{n}$$

We obtain

$$\Gamma_{ij,l}=\vec{D}_i\vec{D}_jP(u^1,u^2)\cdot\vec{D}_lP(u^1,u^2)=\Gamma_{ij}^k\vec{D}_kP(u^1,u^2)\cdot\vec{D}_lP(u^1,u^2)$$

Since

$$\vec{D}_lP(u^1,u^2)\cdot\vec{n}=0\,.$$

We therefore have

$$\Gamma_{ij,l}=\Gamma_{ij}^k g_{kl}$$

By multiplying both sides wit the inverse matrix to g_{kl} , g^{lm} where $g_{kl}\cdot g^{lm}=\delta_{km}$ (and renaming some indices), we find:

(3.26) $$\Gamma_{ij}^k(u^1,u^2)=\tfrac{1}{2}g^{kl}(\frac{\partial g_{kl}}{\partial u^i}+\frac{\partial g_{il}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k})$$

3.4 Expressing the geometric curvature

We are now ready to show that the geodesic curvature of the surface is an intrinsic geometric property. The proof is established by finding an expression for κ_g, from which the assertion can be confirmed immediately. From (3.17) we have

(3.27) $$\frac{d\vec{v}_1}{ds}=\kappa_g\vec{n}\times\vec{v}_1+\kappa_n\vec{n}$$

Where $\vec{v}_1$ is the unit tangent vector, and $\vec{n}$ is the unit normal vector to the tangent plane.
By forming the scalar product with $\vec{n}\times\vec{v}_1$ on both sides, we get:

$$\frac{d\vec{v}_1}{ds}\cdot\vec{n}\times\vec{v}_1=\kappa_g$$

Since $\vec{n}\times\vec{v}_1$ is a unit vector, and $\vec{n}\times\vec{v}_1\perp\vec{n}$. From (3.19)

$$\frac{d\vec{v}_1}{ds}=(\frac{d^2u^k}{ds^2}+\Gamma_{ij}^k(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds})\vec{D}_kP(u^1,u^2)+\frac{du^i}{ds}\frac{du^j}{ds}L_{ij}(u^1,u^2)\vec{n}$$

And also $\vec{v}_1 = \vec{D}_i P(u^1,u^2)\dfrac{du^i}{ds}$

$$\kappa_g = \frac{d\vec{v}_1}{ds}\cdot \vec{n}\times\vec{v}_1 = ((\frac{d^2u^k}{ds^2}+\Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds})\vec{D}_k P(u^1,u^2)+\frac{du^i}{ds}\frac{du^j}{ds}L_{ij}(u^1,u^2)\vec{n})\cdot\vec{n}\times\vec{D}_i P(u^1,u^2)\frac{du^i}{ds}$$

$$\kappa_g = \frac{d\vec{v}_1}{ds}\cdot \vec{n}\times\vec{v}_1 = ((\frac{d^2u^k}{ds^2}+\Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds})\vec{D}_k P(u^1,u^2)+\frac{du^i}{ds}\frac{du^j}{ds}L_{ij}(u^1,u^2)\vec{n})\cdot(\vec{n}\times\vec{D}_i P(u^1,u^2))\frac{du^i}{ds}$$

$$\kappa_g = \frac{d\vec{v}_1}{ds}\cdot \vec{n}\times\vec{v}_1 = ((\frac{d^2u^k}{ds^2}+\Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds})\vec{D}_k P(u^1,u^2))\cdot\vec{n}\times\vec{D}_i P(u^1,u^2)\frac{du^i}{ds}$$

Since

$$(\frac{du^i}{ds}\frac{du^j}{ds}L_{ij}(u^1,u^2)\vec{n})\cdot\vec{n}\times\vec{D}_i P(u^1,u^2)=0$$

$$\vec{D}_k P(u^1,u^2)\cdot(\vec{n}\times\vec{D}_i P(u^1,u^2)) = (\vec{D}_i P(u^1,u^2)\times\vec{D}_k P(u^1,u^2))\cdot\vec{n}$$

We define a new symbol:

(3.28) $$T_{ik} = \vec{D}_i P(u^1,u^2)\times\vec{D}_k P(u^1,u^2)\cdot\vec{n}$$

$$\kappa_g = \frac{d\vec{v}_1}{ds}\cdot\vec{n}\times\vec{v}_1 = ((\frac{d^2u^k}{ds^2}+\Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds})T_{ik}\frac{du^i}{ds}$$

The normal vector is defined as: $\vec{n} = \dfrac{\vec{D}_i P(u^1,u^2)\times\vec{D}_k P(u^1,u^2)}{|\vec{D}_i P(u^1,u^2)\times\vec{D}_k P(u^1,u^2)|}$ so

$$T_{ik} = \frac{|\vec{D}_i P(u^1,u^2)\times\vec{D}_k P(u^1,u^2)|^2}{|\vec{D}_i P(u^1,u^2)\times\vec{D}_k P(u^1,u^2)|}$$

Further more:

$$|\vec{D}_1 P\times\vec{D}_2 P|^2 = |\vec{D}_1 P|^2\cdot|\vec{D}_2 P|^2 - |\vec{D}_1 P\cdot\vec{D}_2 P|^2 = g_{11}g_{22}-g_{12}{}^2 = \det\{g_{ij}\} = g$$

$$T_{11}=T_{22}=0 \text{ Since } \vec{D}_1 P(u^1,u^2)\times\vec{D}_1 P(u^1,u^2) = \vec{D}_2 P(u^1,u^2)\times\vec{D}_2 P(u^{21},u^2)=0$$

$$T_{12}=\sqrt{g} \qquad T_{21}=-\sqrt{g} \ .$$

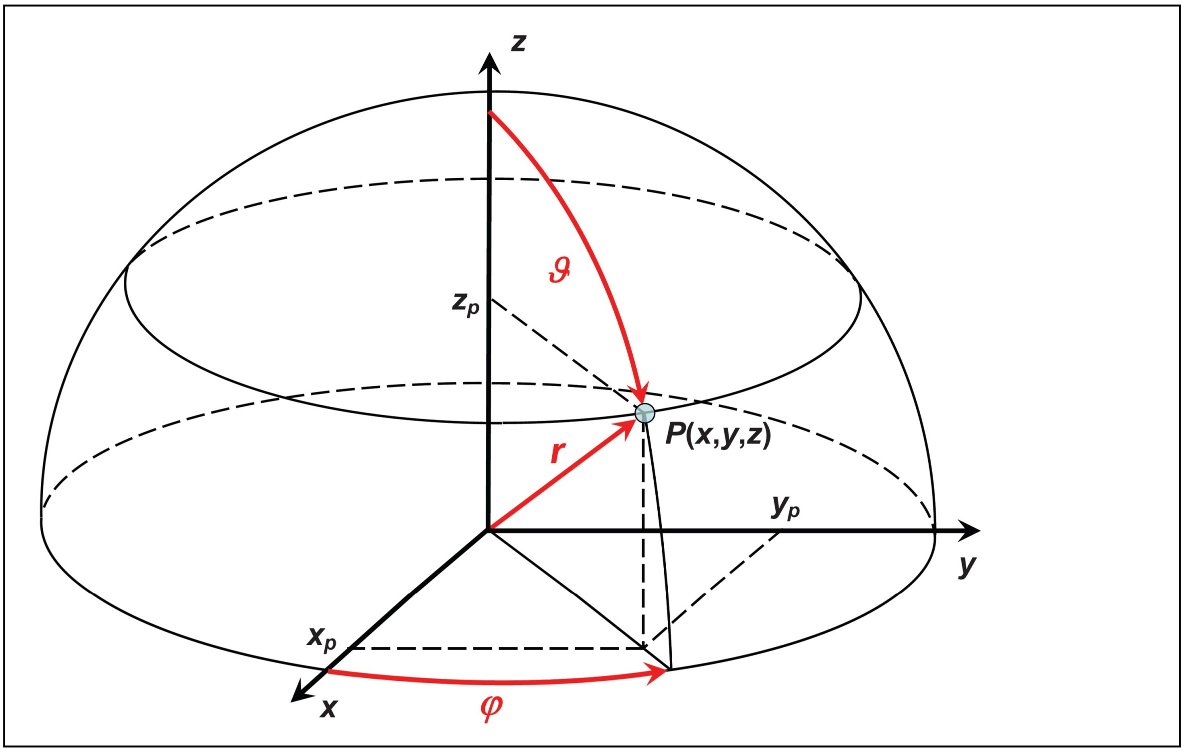
We then arrive at the formula:

(3.29) $$\kappa_g = \frac{du^i}{ds}(\frac{d^2u^k}{ds^2}+\Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds})T_{ik}\,.$$

Which demonstrates that the geodesic curvature can be expressed by the intrinsic properties of the surface only.

3.5 Example. Geodesics on a sphere

As mentioned in the beginning, the examples you may come up with are scarce. However, we shall try to prove that the geodesic on a sphere is a great circle. Even if this should be a rather simple task, it offers some difficulties. The polar coordinates are shown on the figure below.



For the coordinates of $P(x,y,z)$ we have: $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = \cos\theta$.

For the tangent vectors we find:

$$\vec{v}_\theta = \frac{\partial P}{\partial\theta} = (r\cos\theta\cos\varphi\,, r\cos\theta\sin\varphi\,, -r\sin\theta)$$
$$\vec{v}_\varphi = \frac{\partial P}{\partial\varphi} = (-r\sin\theta\sin\varphi\,, r\sin\theta\cos\varphi\,, 0)$$

It is easy to verify that $\vec{v}_\theta \cdot \vec{v}_\varphi = 0$, so they are orthogonal vectors.

One could proceed to find the metric form, and thereby writing the square of the distance element:

$$\frac{ds}{dt}^2 = g_{ij}\frac{du^i(t)}{dt}\frac{du^j(t)}{dt},$$

But the calculations become rather voluminous, so instead we take the distance element directly from the drawing.

To come from P to $P + dP$, we first move horizontally the angle $d\varphi$ along a circle with radius $r\sin\theta$, then we move perpendicular up the circle $rd\varphi$.
The three pieces, $r\sin\theta d\varphi$, $rd\varphi$ and ds, are the sides in a right angle triangle, and we therefore have:

$$ds^2 = (r\sin\theta d\varphi)^2 + (rd\theta)^2$$

We rewrite this a bit as:

$$ds^2 = r^2(1+\sin^2\theta(\frac{d\varphi}{d\theta})^2)d\theta^2$$

For convenience we write $\frac{d\varphi}{d\theta}$ as $\dot{\varphi}$ so our distance element writes:

$$ds^2 = r^2(1+\sin^2\theta\,\dot{\varphi}^2\;d\theta^2)$$

The metric form is then:

$$g_{11} = r^2\sin^2\theta\;,\; g_{22} = r^2\;, g_{12} = g_{21} = 0$$

In the following we put $r = 1$, and seek minimum for the Lagrangian:

$$s = \int Ld\theta = \int\sqrt{1+\sin^2\theta\,\dot{\varphi}^2}\;d\theta$$

The Euler Lagrange equations are

$$\frac{\partial L}{\partial\varphi} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} = 0 \quad \text{And} \quad \frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0$$

$$\frac{\partial L}{\partial\varphi} = 0 \;\Rightarrow\; \frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} = 0 \;\Rightarrow\; \frac{\partial L}{\partial\dot{\varphi}} = \frac{\sin^2\theta\,\dot{\varphi}}{\sqrt{1+\sin^2\theta\,\dot{\varphi}^2}} = c \text{ (Constant)}$$

$$\frac{\sin^2\theta\,\dot{\varphi}}{\sqrt{1+\sin^2\theta\,\dot{\varphi}^2}} = c \quad\Rightarrow$$

$$\sin^4\theta\,\dot{\varphi}^2 = (1+\sin^2\theta\,\dot{\varphi}^2)c^2 \quad\Leftrightarrow$$

$$\dot{\varphi}^2(\sin^4\theta - c^2\sin^2\theta) = c^2 \quad\Leftrightarrow$$

$$\dot{\varphi}^2\sin^2\theta(\sin^2\theta - c^2) = c^2 \quad\Rightarrow$$

$$\dot{\varphi} = \frac{d\varphi}{d\theta} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

$$d\varphi = \frac{cd\theta}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

To integrate this differential, we make the substitution:

$$z = \cot\theta\,;\ dz = -\frac{1}{\sin^2\theta}d\theta \quad \Leftrightarrow \quad d\theta = -\sin^2\theta dz$$

Then we get.

$$d\varphi = \frac{c d\theta}{\sin\theta\sqrt{\sin^2\theta - c^2}} \qquad \Leftrightarrow$$

$$d\varphi = -\frac{c\sin^2\theta dz}{\sin\theta\sqrt{\sin^2\theta - c^2}} = \frac{cdz}{\sqrt{1-\dfrac{c^2}{\sin^2\theta}}}$$

$$d\varphi = \frac{cdz}{\sqrt{1-\dfrac{c^2(\cos^2\theta+\sin^2\theta)}{\sin^2\theta}}} = \frac{cdz}{\sqrt{1-c^2(1+z^2)}}$$

Setting $a = 1-c^2$, we get:

$$d\varphi == \frac{cdz}{\sqrt{1-c^2(1+z^2)}} = \frac{cdz}{\sqrt{a^2-c^2z^2)}} = \frac{cdz}{a\sqrt{1-(\dfrac{cz}{a})^2}}$$

The last term can be integrated to

$$\cos^{-1}(\frac{c}{a}z)\,,$$

and we find

$$\varphi-\varphi_0 = \cos^{-1}(\frac{c}{a}z) \quad \Leftrightarrow \quad \cos(\varphi-\varphi_0) = \frac{c}{a}z \quad \Leftrightarrow$$

$$\cos(\varphi-\varphi_0) = \frac{c}{a}\cot\theta$$

The last equation is (allegedly) the equation for a great circle on a sphere.

So we have finally proved (with some effort) that the shortest path between to points on a sphere is a great circle. A great circle is a geodesic in differential geometry.

3.6 Example. Gaussian and average curvature

Our aim is now to find the Gaussian curvature *K*, the mean curvature *H*, and the two main normal curvatures k_1 and k_2, for a parabola, and a hyperbola.
We have the formulas

(3.30) $$K = \frac{L}{g} \qquad \text{and} \qquad H = \frac{g_{11}L_{22} - 2g_{12}L_{12} + L_{11}g_{22}}{2g}$$

$$g = \det\{g_{ij}\} \qquad\qquad g_{ij} = \vec{D}_i P\cdot\vec{D}_j P$$

$$\vec{n} = \frac{\vec{D}_1 P \times \vec{D}_2 P}{|\vec{D}_1 P \times \vec{D}_2 P|}$$

(3.31) $\qquad L = \det\{L_{ij}\} \qquad\qquad L_{ij} = \vec{n} \cdot \vec{D}_i \vec{D}_j P$

The curvatures k_1 and k_2 satisfy the relations:

$$K = k_1 k_2 \qquad \text{and} \qquad H = \frac{k_1 + k_2}{2}$$

Thus k_1 and k_2 are roots in the quadratic equation

$$\lambda^2 + 2H\lambda + K = 0$$

The parabola

A *parabola* can be represented by

$$P(x, y, z) = (x, y, \tfrac{1}{2}(x^2 + y^2))$$

$$\vec{D}_1 P = \frac{\partial P}{\partial x} = (1,0,x) \qquad \vec{D}_2 P = \frac{\partial P}{\partial y} = (0,1,y)$$

$$g_{ij} = \vec{D}_i P \cdot \vec{D}_j P$$

$$g_{11} = 1 + x^2 \quad ; \quad g_{12} = g_{21} = xy \quad ; \quad g_{22} = 1 + y^2$$

$$\vec{n} = \frac{\vec{D}_1 P \times \vec{D}_2 P}{|\vec{D}_1 P \times \vec{D}_2 P|} = \left(\frac{\begin{vmatrix} 0 & x \\ 1 & y \end{vmatrix}}{|\vec{n}|}, \frac{\begin{vmatrix} x & 1 \\ y & 0 \end{vmatrix}}{|\vec{n}|}, \frac{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{|\vec{n}|} \right) = \frac{(-x,-y,1)}{\sqrt{1 + x^2 + y^2}}$$

$$\vec{D}_1 \vec{D}_1 P = \frac{\partial^2 P}{\partial x^2} = (0,0,1) \qquad \vec{D}_1 \vec{D}_2 P = \frac{\partial^2 P}{\partial x \partial y} = (0,0,0) \qquad \vec{D}_2 \vec{D}_2 P = \frac{\partial^2 P}{\partial y^2} = (0,0,1)$$

$$g = \det\{g_{ij}\} = (1 + x^2)(1 + y^2) - x^2 y^2 = 1 + x^2 + y^2$$

$$L_{11} = \frac{1}{\sqrt{1 + x^2 + y^2}} \quad ; \quad L_{12} = L_{21} = 0 \quad ; \quad L_{22} = \frac{1}{\sqrt{1 + x^2 + y^2}}$$

$$L = \det\{L_{ij}\} = L_{11} L_{22} = \frac{1}{1 + x^2 + y^2}$$

$$K = \frac{L}{g} = \frac{1}{(1+x^2+y^2)^2}$$

$$H = \frac{g_{11}L_{22} - 2g_{12}L_{12} + L_{11}g_{22}}{2g} = \frac{(1+x^2)\frac{1}{\sqrt{1+x^2+y^2}} + (1+y^2)\frac{1}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2}$$

$$H = \frac{2+x^2+y^2}{(1+x^2+y^2)^{\frac{3}{2}}}$$

The two main curvatures k_1 and k_2 are roots in the quadratic equation:

$$\lambda^2 + 2H\lambda + K = 0$$

Which has the solutions

$$k = H \pm \sqrt{H^2 - K}$$

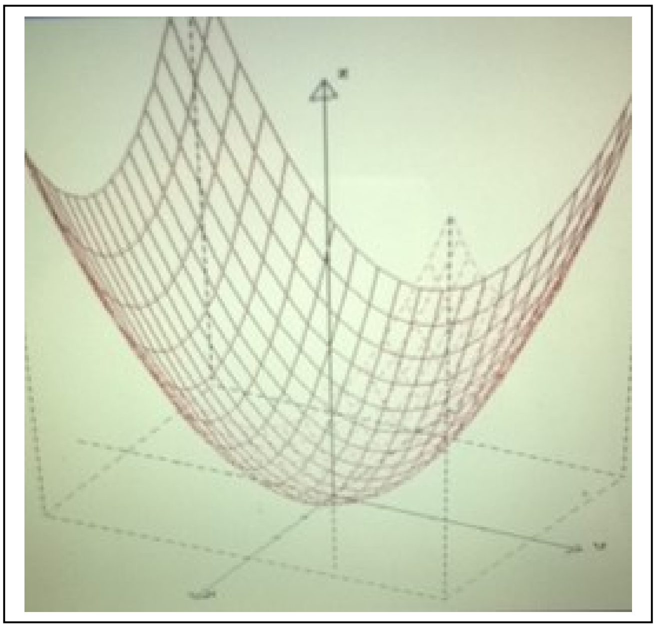
And they therefore satisfy the relations.

$$K = k_1 k_2 \qquad \text{and} \qquad H = \frac{k_1 + k_2}{2}$$

Rather than inserting the expression for H and K in the solution formula, it is easily verified, that

$$k_1 = \frac{1}{(x^2+y^2+1)^{\frac{3}{2}}} \qquad \textit{and} \qquad k_2 = \frac{1}{(x^2+y^2+1)^{\frac{1}{2}}}$$

satisfy the conditions. Below is shown the shape of a parabola.



The hyperbola
Next we shall consider a hyperbola, which may have an equation:

$$P(x,y,z)=(x,y,\tfrac{1}{2}(x^2-y^2))$$

Here we find, quite similar to the case of the parabola.

$$\vec{D}_1P=\frac{\partial P}{\partial x}=(1,0,x) \qquad \vec{D}_2P=\frac{\partial P}{\partial y}=(0,1,-y)$$

$$g_{i\,j}=\vec{D}_iP\cdot\vec{D}_jP$$

$$g_{11}=1+x^2 \quad ; \quad g_{1\,2}=g_{2\,1}=-xy \quad ; \quad g_{22}=1+y^2$$

$$\vec{n}=\frac{\vec{D}_1P\times\vec{D}_2P}{|\vec{D}_1P\times\vec{D}_2P|}=\left(\frac{\begin{vmatrix}0 & x\\ 1 & -y\end{vmatrix}}{|\vec{n}|},\frac{\begin{vmatrix}x & 1\\ -y & 0\end{vmatrix}}{|\vec{n}|},\frac{\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}}{|\vec{n}|}\right)=\frac{(-x,+y,1)}{\sqrt{1+x^2+y^2}}$$

$$\vec{D}_1\vec{D}_1P=\frac{\partial^2P}{\partial x^2}=(0,0,1) \qquad \vec{D}_1\vec{D}_2P=\frac{\partial^2P}{\partial x\partial y}=(0,0,0) \qquad \vec{D}_2\vec{D}_2P=\frac{\partial^2P}{\partial y^2}=(0,0,-1)$$

$$g=\det\{g_{i\,j}\}=(1+x^2)(1+y^2)-x^2y^2=1+x^2+y^2$$

$$L_{11}=\frac{1}{\sqrt{1+x^2+y^2}} \quad ; \quad L_{12}=L_{21}=0 \quad ; \quad L_{22}=\frac{-1}{\sqrt{1+x^2+y^2}}$$

$$L=\det\{L_{i\,j}\}=L_{11}L_{22}=\frac{-1}{1+x^2+y^2}$$

$$K=\frac{L}{g}=-\frac{1}{(1+x^2+y^2)^2}$$

$$H=\frac{g_{11}L_{22}-2g_{12}L_{12}+L_{11}g_{22}}{2g}=\frac{-(1+x^2)\frac{1}{\sqrt{1+x^2+y^2}}+(1+y^2)\frac{1}{\sqrt{1+x^2+y^2}}}{2(1+x^2+y^2)}$$

$$H=\frac{y^2-x^2}{2(1+x^2+y^2)^{\frac{3}{2}}}$$

Since $K = k_1 k_2$ and $H = \dfrac{k_1 + k_2}{2}$ k_1 and k_2 are solutions to the quadratic equation:

$$k^2 - 2Hk + K = 0$$

Which has the solution: $k = H \pm \sqrt{H^2 - K}$

$$k = \frac{y^2 - x^2}{2(1+x^2+y^2)^{\frac{3}{2}}} \pm \sqrt{\left(\frac{y^2 - x^2}{2(1+x^2+y^2)^3}\right)^2 + \frac{1}{(1+x^2+y^2)^2}}$$

$$k = \frac{y^2 - x^2}{2(1+x^2+y^2)^{\frac{3}{2}}} \pm \frac{1}{2(1+x^2+y^2)^{\frac{3}{2}}} \sqrt{(y^2 - x^2)^2 + 4(1+x^2+y^2)}$$

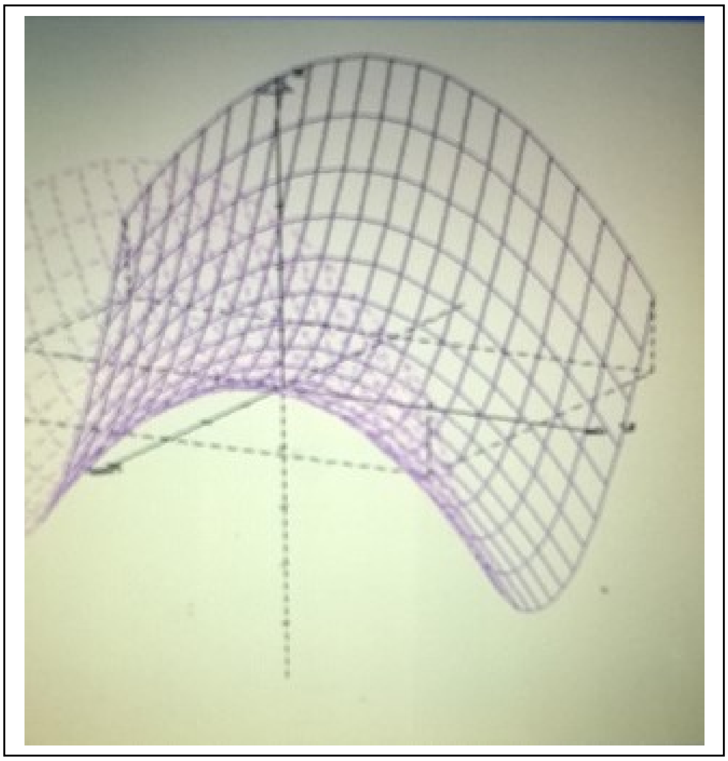
$$k = \frac{1}{2(1+x^2+y^2)^{\frac{3}{2}}} (y^2 - x^2 \pm \sqrt{(y^2 - x^2)^2 + 4(1+x^2+y^2)})$$

Hence

$$k_1 = \frac{1}{2(1+x^2+y^2)^{\frac{3}{2}}} (y^2 - x^2 + \sqrt{(y^2 - x^2)^2 + 4(1+x^2+y^2)})$$

$$k_2 = \frac{1}{2(1+x^2+y^2)^{\frac{3}{2}}} (y^2 - x^2 - \sqrt{(y^2 - x^2)^2 + 4(1+x^2+y^2)})$$

Below is shown a computer drawing of the general shape of a hyperbola



3.7 Covariant derivative

Any vector $\vec{a}$ which is a linear combination of $\vec{D}_1P(u^1,u^2)$ *and* $\vec{D}_2P(u^1,u^2)$ is a tangent vector in the point *P*, since it lies in the tangent plane in *P*.

$$\vec{a} = a^i\,\vec{D}_iP(u^1,u^2) \quad and$$
$$\vec{a}\cdot\vec{D}_jP(u^1,u^2) = a^i\,\vec{D}_iP(u^1,u^2)\cdot\vec{D}_jP(u^1,u^2)$$
$$\vec{a} = a^i g_{ij}$$

If $\vec{a}(t)$ is a vector corresponding to a parameter curve: $\vec{a}(t) = a^i(t)\,\vec{D}_iP(u^1(t),u^2(t))$, one may form the derivative of this vector.

The projection of $\dfrac{d\vec{a}(t)}{dt}$ on the tangent plane is called the *covariant derivative* of $\vec{a}(t)$ at that point.

We shall now develop a formula for the covariant derivative, and afterwards clarify its geometrical significance.

$$\frac{d\vec{a}}{dt} = \frac{da^i}{dt}\vec{D}_iP(u^1(t),u^2(t)) + a^i(t)\vec{D}_j\vec{D}_iP(u^1(t),u^2(t))\frac{du^j}{dt}$$

According to (3.17)

$$\vec{D}_i\vec{D}_jP(u^1,u^2) = \Gamma^k_{ij}(u^1,u^2)\vec{D}_kP(u^1,u^2) + L_{ij}(u^1,u^2)\vec{n}$$

Then we get, after a minor renaming of indices.

$$\frac{d\vec{a}}{dt} = \frac{da^k}{dt}\vec{D}_kP(u^1(t),u^2(t)) + a^i(t)\frac{du^j}{dt}(\Gamma^k_{ij}(u^1(t),u^2(t))\vec{D}_kP(u^1(t),u^2(t)) + L_{ij}(u^1(t),u^2(t))\vec{n})$$

We are only interested in the components in the tangent plane, which is the covariant derivative, so we omit the component along the normal vector. The covariant derivative is written with the symbol δ, instead of d or ∂.

(3.30) $$\frac{\delta\vec{a}(t)}{\delta t} = (\frac{da^k(t)}{dt} + a^i(t)\frac{du^j}{dt}\Gamma^k_{ij}(u^1(t),u^2(t)))\vec{D}_kP(u^1(t),u^2(t))$$

Or without specific reference to the parameters $(u^1(t),u^2(t))$,

(3.30) $$\frac{\delta\vec{a}}{\delta t} = (\frac{da^k}{dt} + a^i\frac{du^j}{dt}\Gamma^k_{ij})\vec{D}_kP$$

From (3.19) we have

$$\frac{d\vec{v}_1}{ds} = (\Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds} + \frac{d^2u^k}{ds^2})\vec{D}_k P(u^1,u^2) + \frac{du^i}{ds}\frac{du^j}{ds}L_{ij}(u^1,u^2)\vec{n}$$

Omitting the last term we obtain the covariant derivative of $\vec{v}_1$

(3.31) $$\boxed{\frac{\delta\vec{v}_1}{\delta s} = (\frac{d^2u^k}{ds^2} + \frac{du^i}{ds}\frac{du^j}{ds}\Gamma^k_{ij})\vec{D}_k P}$$

Setting $\frac{\delta\vec{v}_1}{\delta s} = 0$ we find

(3.32) $$(\frac{d^2u^k}{ds^2} + \frac{du^i}{ds}\frac{du^j}{ds}\Gamma^k_{ij})\vec{D}_k P = 0$$

Which expresses that the tangential component of $\frac{d\vec{v}_1}{ds}$ is zero.
If you move along a straight line in a plane surface in ordinary three dimensional space following the direction of a constant vector $\vec{a}$, then the derivative of that vector is 0.

The shortest path between two points is also a straight line along a constant vector.

This suggests that in an arbitrary surface the shortest path between two points is along a curve, where the covariant derivative is zero. Comparing to the situation in ordinary space, then moving along a path where the covariant derivative is zero is called a parallel displacement.

For the geodesic curvature we found the expression:

(3.34) $$\kappa_g = \frac{du^i}{ds}(\frac{d^2u^k}{ds^2} + \Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds})T_{ik}$$

The differential equation for a curve, where the geodesic curvature is identical zero is then:

(3.35) $$(\frac{d^2u^k}{ds^2} + \Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds}) = 0$$

What we see, when comparing with (3.32), is that a zero covariant derivative, and a zero geodesic curvature are two sides of the same coin.

We shall now show more directly, that a geodesic curve joining to points P_1 and P_2 is in fact the shortest path between the two points.

If $(\varphi^1,\varphi^2) = (\varphi^1(t),\varphi^2(t))$ is a curve running the surface, then the square of the distance element is:

$$ds^2 = g_{ij}(\varphi^i(t)\varphi^j(t))\dot{\varphi}^i(t)\dot{\varphi}^j(t)$$

where as usual $\dot{\varphi}^i(t) = \frac{d\varphi^i(t)}{dt}$

The distance between to points, may then be calculated as:

(3.35)
$$s_{12} = \int_{t_1}^{t_2} \sqrt{g_{ij}(\varphi^i(t)\varphi^j(t))\,\dot{\varphi}^i(t)\dot{\varphi}^j(t)}dt$$

To derive a differential equations for $\varphi^i(t)$, which gives the minimum value for s_{12} we use the classical Lagrangian approach.

$\delta\int Ldt = 0$ where $L = L(x, y, \dot{y}, t)$ and it has the solution:

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial \dot{y}} = 0$$

If we have two variables, $(x, y) = (u^1, u^2) = (\varphi^1(t), \varphi^2(t))$ the equations become:

(3.36)
$$\frac{\partial L}{\partial u^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}^i} = 0$$

Notice that $g_{ij}(u^i, u^j)$ does not depends explicitly on *t*.

If we furthermore choose *t* as a *natural parameter,* we have

$$\frac{ds}{dt} = \frac{ds}{ds} = \sqrt{g_{ij}(u^i, u^j)\,\dot{\varphi}^i(t)\dot{\varphi}^j(t)} = 1$$

We may then drop the square root in the calculations, i.e.

$$\frac{d}{dt}\sqrt{g_{ij}(u^i, u^j)\,\dot{\varphi}^i(t)\dot{\varphi}^j(t)} =$$
$$\frac{1}{2\sqrt{g_{ij}(u^i, u^j)\,\dot{\varphi}^i(t)\dot{\varphi}^j(t)}}\frac{d}{dt}g_{ij}(u^i, u^j)\,\dot{\varphi}^i(t)\dot{\varphi}^j(t) =$$
$$\tfrac{1}{2}\frac{d}{dt}(g_{ij}(u^i, u^j)\,\dot{\varphi}^i(t)\dot{\varphi}^j(t))$$

Thus we find:
$$\frac{\partial L}{\partial u^k} = \frac{\partial g_{ij}}{\partial u^k}\dot{\varphi}^i\dot{\varphi}^j$$
and
$$\frac{\partial L}{\partial \dot{\varphi}^k} = g_{ki}\dot{\varphi}^j + g_{ik}\dot{\varphi}^i = 2g_{ik}\dot{\varphi}^i \qquad \text{since} \qquad g_{ik} = g_{ki}$$

The Lagrangian equation becomes:

(3.37) $$\frac{\partial g_{ij}}{\partial u^k}\dot{\varphi}^i\dot{\varphi}^j - 2\frac{d}{dt}g_{ik}\dot{\varphi}^i = 0$$

Where

$$\frac{d}{dt}g_{ik}\dot{\varphi}^i = \frac{\partial g_{ik}}{\partial u^j}\dot{\varphi}^j\dot{\varphi}^i + g_{ik}\ddot{\varphi}^i$$

and

$$\frac{\partial g_{ik}}{\partial u^j} = \tfrac{1}{2}(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i})$$

Putting it all together we get

$$\frac{\partial L}{\partial u^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}^i} = 0 \qquad \Leftrightarrow$$

$$\frac{\partial g_{ij}}{\partial u^k}\dot{\varphi}^i\dot{\varphi}^j - (\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i})\dot{\varphi}^j\dot{\varphi}^i - 2g_{ik}\ddot{\varphi}^i = 0$$

Multiplying by ½ , changing sign and rearranging the terms we finally get:

(3.38) $$g_{ik}\ddot{\varphi}^i + \tfrac{1}{2}(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k})\dot{\varphi}^j\dot{\varphi}^i = 0 \qquad \Leftrightarrow$$

$$g_{ik}\ddot{\varphi}^i + \Gamma_{ij,k}\dot{\varphi}^j\dot{\varphi}^i = 0$$

and by multiplying with the inverse matrix to g_{ik} , g^{kl} .

$$g_{ik}g^{kl}\ddot{\varphi}^i + g^{kl}\Gamma_{ij,k}\dot{\varphi}^j\dot{\varphi}^i = 0$$

we arrive at the equation

(3.39) $$\ddot{\varphi}^l + \Gamma^l_{ij}\dot{\varphi}^j\dot{\varphi}^i = 0$$

or renaming *l* to *k*

(3.39) $$\ddot{\varphi}^k + \Gamma^k_{ij}\dot{\varphi}^j\dot{\varphi}^i = 0$$

This we compare with the geodesic equation,

(3.40) $$(\frac{d^2u^k}{ds^2} + \Gamma^k_{ij}(u^1,u^2)\frac{du^i}{ds}\frac{du^j}{ds}) = 0$$

We realize it is the same differential equation.

3.8 Teorema egregium

This theorem is due to Gauss. It is deep and it is rather incomprehensible to understand the significance.
To this end, we shall first consider the third derivative of the parametric representation of a surface.
From the formulas (3.17) and (3.20) we have

$$\vec{D}_i\vec{D}_j P = \Gamma^k_{ij}\vec{D}_k P + L_{ij}\vec{n}$$
$$\vec{D}_i\vec{n} = B_i^j\vec{D}_j P = -L_{ji}\,g^{kj}\vec{D}_k P$$

Changing the summation indices, we find:

$$\vec{D}_j\vec{D}_k P = \Gamma^l_{jk}\vec{D}_l P + L_{jk}\vec{n}$$
$$\vec{D}_i\vec{n} = B_i^j\vec{D}_j P = -g^{mn}L_{ni}\vec{D}_m P$$

From these formulas we get:

$$\vec{D}_i\vec{D}_j\vec{D}_k P = \frac{\partial\Gamma^l_{jk}}{\partial u^i}\vec{D}_l P + \Gamma^l_{jk}\vec{D}_i\vec{D}_l P + \frac{\partial L_{jk}}{\partial u^i}\vec{n} + L_{jk}\vec{D}_i\vec{n}$$

$$\vec{D}_i\vec{D}_j\vec{D}_k P = \frac{\partial\Gamma^m_{jk}}{\partial u^i}\vec{D}_m P + \Gamma^l_{jk}(\Gamma^m_{il}\vec{D}_m P + L_{il}\vec{n}) + \frac{\partial L_{jk}}{\partial u^i}\vec{n} - L_{jk}g^{mn}L_{ni}\vec{D}_m P$$

$$\vec{D}_i\vec{D}_j\vec{D}_k P = (\frac{\partial\Gamma^m_{jk}}{\partial u^i}\vec{D}_m P + \Gamma^l_{jk}\Gamma^m_{il} - L_{jk}g^{mn}L_{ni})\vec{D}_m P + (\Gamma^l_{jk}L_{il} + \frac{\partial L_{jk}}{\partial u^i})\vec{n}$$

Since the sequence of differentiating is immaterial, we have $\vec{D}_i\vec{D}_j\vec{D}_k P = \vec{D}_j\vec{D}_i\vec{D}_k P$.
That is in fact the same vector. Putting the two expressions together, and doing at bit of manipulation with the indices,
One arrives at the equation:

(3.34) $$\frac{\partial\Gamma^m_{jk}}{\partial u^i} - \frac{\partial\Gamma^m_{ik}}{\partial u^j} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^l_{jl} - L_{jk}g^{mn}L_{ni} + L_{ik}g^{mn}L_{nj} = 0$$

Which is called Gauss' equation, and

(3.35) $$\frac{\partial L_{jk}}{\partial u^i} - \frac{\partial L_{ik}}{\partial u^j} + \Gamma^l_{jk}L_{il} - \Gamma^l_{ik}L_{jl} = 0$$

Which is called the Marnardi and Cadazzi equation.

We then introduce the entity: R^l_{ijk} by

(3.36) $$R^l_{\ i\,j\,k} = \frac{\partial \Gamma^l_{i\,k}}{\partial u^j} - \frac{\partial \Gamma^l_{i\,j}}{\partial u^k} + \Gamma^m_{i\,k}\Gamma^l_{j\,m} - \Gamma^m_{i\,j}\Gamma^l_{k\,m}$$

It is called the Riemann tensor or the curvature tensor. Gauss' equation can then be written

(3.37) $$L_{jk} g^{mn} L_{ni} + L_{ik} g^{mn} L_{nj} = R^l_{\ ijk}$$

One should notice that the curvature tensor is expressed only by the Christoffel symbols and its derivatives, and is therefore an intrinsic geometric entity.

Gauss' *teorema egregium* then expresses that the Gaussian curvature is an intrinsic geometric entity.

From (3.24) we have $K = \dfrac{L}{g}$ where $L = \det\{L_{i\,j}\}$ and $g = \det\{g_{i\,j}\}$
From Gauss' equation we get by multiplying by g_{mp} and summation over m.

$$L_{jk} L_{pi} + L_{ik} L_{p\,j} = g_{m\,p} R^m_{\ k\,i\,j}$$

From which

$$L = L_{11} L_{22} + L^2_{i12} = g_{m2} R^m_{\ 121}$$

From which we finally get

$$K = \frac{1}{g} g_{m2} R^m_{121}$$

This finishes the proof, that the Gaussian curvature is an intrinsic geometric entity.

References:
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