Calculus of Variations

Solutions to known and unknown problems

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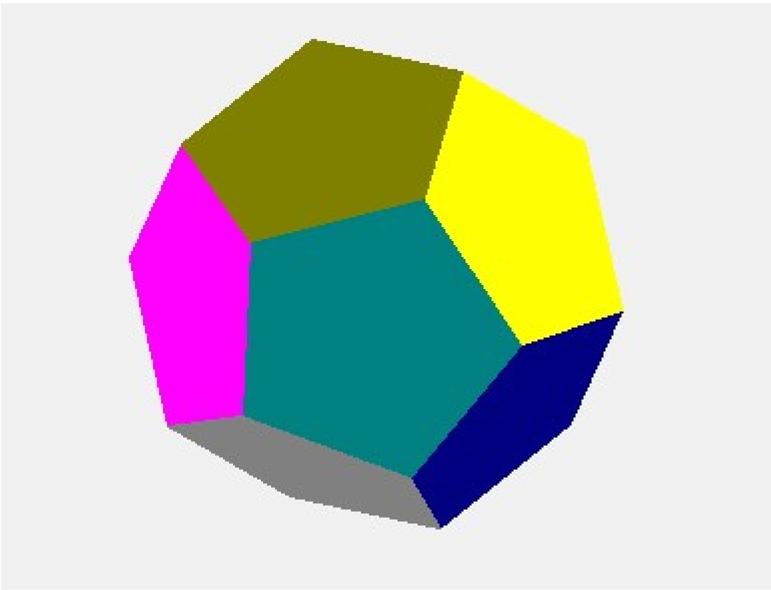

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Acknowledgement

I studied physics and mathematics at the University of Copenhagen from 1965 -1971.
I my second year, one of the subjects that fascinated me, was Professor B. Jessen's hand written notes on calculus, and especially the calculus of variation.
Since at the graduate level I learned, through the Lagrange formalism in analytical mechanics that the principle of least time is actually the foundation of theoretical physics, and the Lagrange equations of motion, are in fact deduced from the calculus of variations.

Of course most of the problems Professor B. Jessen presented, had no analytical solution, but I distinctly remember that he mentioned that the shape of wine barrels, are actually can actually be found as a result of the calculus of variation, that is, designing a barrel which gives the largest volume for a given surface. (i.e. least material to make the barrel).

In the sixties the departments of mathematics and physics, each had one and only one (Danish developed) computer with a magnetic core store of 1 kb. Yes! Graphics was not even spoken of. It was monitored by a teletype, and output was delivered on a printer, with only capital letters.

I 1986 I joined the university again and I took a bachelor degree in computer science. About the same period Borland hired the Dane Anders Hejlsberg to develop Turbo Pascal.
Especially Turbo 5.5 and Turbo 7.0 were great successes. The drawback in the end of the 80'ties, was of course, that it was based on DOS, but it was replaced by Delphi in 1995, based on MS Windows. Delphi was preinstalled with all the well known windows components, but had kept Pascal as the underlying programming language.

Turbo 7.0 had made is possible to solve almost any numerical problem in mathematics and physics, and display the results on a graphics colour screen (although you had to make all component e.g. menus, and mouse control software by hand).

The reason for bringing this up, is that all the numerical solutions of the differential equations, and all the (3D) illustrations in this article are generated by a Turbo 7.0 (70 pages) DOS-program (Mathemat), that I wrote in the early 90'ties. It was a true revelation, to be able to solve any differential equation numerically, draw the orbital motion of satellites, three body problems, you name it. This compared to the years in university, where everything had to be done analytically

After Windows XP, programs using DOS-graphics can no longer run on MS operating systems, so I have kept two machines that run on Win XP. And since Win 98, it is no longer possible to make a screen dump from the graphics screen, so I have also kept a stationary computer that operates under Win 98.
Nevertheless I have for 25 years used that old Turbo 7.0 program, (which has a graphics interface, which look much like Win 95, but handmade), in numerous articles on physics and mathematics. Although obsolete, I have never encountered any mathematical program that could replace i

1. Basic methods for optimization

In elementary calculus, problems of optimization are solved by determination of the stationary points of a real function of one variable i.e. the values, where the function have local max or min. If the function related to the problem is $y=f(x)$, then the stationary points are determined by solving the equation, which comes about, when setting the derivative of $f(x)$ to zero.

(1.1) $$f'(x)=0$$

There is a local max in the point in x_0, if $f'(x_0)=0$ and the variation of signature of $f'(x)$ around x_0 is (+ , 0 , -), that is, $f(x)$ is increasing, flat and decreasing.
The nature of the stationary point can however, also be examined by examining $f''(x)$.
If $f''(x_0)<0$, then there is a max x_0, since it means that $f'(x)$ is decreasing and crosses the x-axis from the positive side to the negative side, i.e. the variation of $f'(x)$ is (+, 0, -) around x_0.
Similarly, there is a local min in x_0 if:

(1.2) $$f'(x_0) = 0 \quad \text{and} \quad f''(x_0) > 0$$

In some connections it is advantageous to rewrite the definition of differentiability.

If f is differentiable in x_0 , then according to the usual definition:

(1.3) $$\frac{f(x_0+h)-f(x_0)}{h} \to f'(x_0) \quad for \;\; h\to 0 \quad \Leftrightarrow$$
$$\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0) \to 0 \, for \quad h\to 0$$

Introducing a so called epsilon function $\varepsilon(h)$, defined by two properties:

$\varepsilon(h)$ is continuous, and $\varepsilon(h)\to 0 \;\; for \quad h\to 0$.

The differentiability of a real function, can then be reformulated as:

$$\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0) = \varepsilon(h) \qquad \Leftrightarrow$$

(1.4) $$\Delta f(h) = f(x_0+h)-f(x_0) = f'(x_0)h + \varepsilon(h)h$$

The last term goes to zero at a higher order than h, so in many cases, we may discard it when h is small. This lead to the formulation:

(1.5) $$\Delta f(h) \approx f(x_0+h)-f(x_0) = f'(x_0)h \qquad \text{when } h \text{ is small.}$$

The expression $\Delta f(h)$ is denoted the *variation* of $f(x)$ around x_0.

If f has an extremum (max/min) i x_0 , then $f'(x_0) = 0$, and consequently the variation of f in x_0 equals zero.

This formulation we shall apply in the following.

1.1 Partial derivatives

For functions of two or more variables e.g. $z = f(x, y)$ one can calculate the so called partial derivatives, where you just differentiate the expression (in the usual manner) after one variable, treating the other variables as constants. Partial derivatives are written with a curved d as ∂.
The two partial derivatives of $z = f(x, y)$ are written:

(1.6) $$\frac{\partial f}{\partial x} \quad and \quad \frac{\partial f}{\partial y}$$

Correspondingly one may write the four second derivatives

(1.7) $$\frac{\partial^2 f}{\partial x^2} \quad , \quad \frac{\partial^2 f}{\partial y^2} \quad , \quad \frac{\partial^2 f}{\partial x \partial y} \quad , \quad \frac{\partial^2 f}{\partial y \partial x}$$

It is an important mathematical theorem that:

(1.8) $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

1.9 Example

If $f(x,y) = xy^2$, then one gets directly:

$$\frac{\partial f}{\partial x} = y^2 \quad , \quad \frac{\partial f}{\partial y} = 2xy \quad , \quad \frac{\partial^2 f}{\partial x^2} = 0 \quad , \quad \frac{\partial^2 f}{\partial y^2} == 2x \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2y$$

The differential is defined in the same manner, as for a function of one variable:

(1.10) $$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

The condition for a function to have an extremum (max/min) in some point x_0 , can then be expressed, as having zero variation in that point.

(1.11) $$\Delta f \approx df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial x} = 0 \ \wedge \ \frac{\partial f}{\partial y} = 0$$

So a function has an extremum, if and only if all the partial derivatives vanish at that point. This applies also to functions of more than two variables.

Without proof, we mention, that one can decide whether a stationary point is a max or a min, by establishing the Jacoby determinant. If the determinant is negative $f(x)$ has a maximum, and if it is positive it is a minimum.
For a function of two variables, the Jacoby determinant is evaluated as shown below.
The generalization to more than two variables is straightforward.

(1.12)
$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

2. The calculus of variations

A real function f of several variables $f(x_1, x_2, x_3,..,x_n)$, is a *mapping* of R^n into R, if there for every $x = (x_1, x_2, x_3, ... x_n)$ is assigned one, and only one real number $y = f(x_1, x_2, x_3,..,x_n)$.

A ***functional*** is a mapping, if there for every subset of functions (of one or more variables), is assigned one real number.

The definite integral, defined for integrable functions, is an example of a functional.
In elementary calculus, the functional I is written with the integral symbol.

(2.1)
$$I(f) = \int_a^b f(x)dx$$

The discipline of mathematics that deals with functionals is called ***functional analysis***.

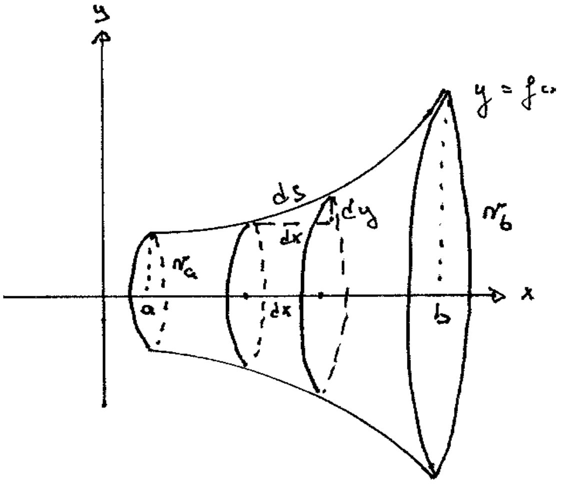
Note:
A mapping, which to a class of functions assigns one and only one function, is called an ***operator.***
A familiar example of an operator is the differential operator $\frac{d}{dx}$, since, for every differentiable function is assigned its differential quotient.

$$f'(x) = \frac{d}{dx} f(x)$$

Other examples are of course any differential equation.

The calculus of variation deals with the task of finding the stationary points (max/min) of a *functional*. That is, to determine the function, where the *functional* has a max or min.

A soap membrane.
A classical example is to determine the shape of a rotational symmetric soap membrane i.e. the curve of intersecting with the *x*-*y* plane, making the surface of the membrane becomes least.



In these kinds of problems, there are usually attached some boundary conditions that the solution must fulfil.
In the case of soap membrane, the revolution membrane must have circular intersection in both ends with radii r_a and r_b , so any solution $y = f(x)$ must conform to:

$$f(a) = r_a \text{ and } f(b) = r_b$$

To establish an functional expression for the intersecting curve of the membrane, we shall look at a small strip with thickness dx at x.

The circumference of the strip is $2\pi f(x) = 2\pi y$. The width of the strip is then (See the figure)

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \sqrt{1 + f'(x)^2}\,dx = \sqrt{1 + y'^2}\,dx \tag{2.3}$$

The contribution from the infinitesimal strip to the area of the membrane is then:

$$dO = 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx \qquad \text{or} \qquad dO = 2\pi y\sqrt{1 + y'^2}\,dx$$

The overall surface can then be found by integrating.

$$O(y) = 2\pi \int_a^b y\sqrt{1 + y'^2}\,dx \tag{2.4}$$

It is quite intentional that we write y instead of $f(x)$, because y is considered an independent variable i.e. the function we want to determine. The task of the calculus of variation is, among all differentiable function to find the function $y = f(x)$, which minimizes the integral.

We shall eventually present the solution to the minimum of the soap membrane, which is one of the few problems that actually have an analytic solution.
But first we shall introduce some consideration of a more general nature, namely to develop methods to determine the stationary points of a functional.

The general solution is due to Leonard Euler, and his theory is called Calculus of Variations.

The Calculus of Variations is one of the strongest and most applicable tools in mathematics, especially when applied to physics, where the equations of motion – including the he general theory of relativity – can be derived from the principle of least time, applying the Euler-Lagrange formalism.
We shall initiate from what the mathematicians somewhat boldly have named: The simplest problem.

2.1 The simplest problem

We consider a function $F = F(x, y, y')$, which is the integrand in the functional, and $y = f(x)$ is a function that is a stationary point of the functional, the function we want to determine.

More specifically, we want to establish the stationary points (max/min) of the functional.

(2.5) $$I(y) = \int_a^b F(y', y, x)dx$$

The reason for this choice of functional is that many problems from the physical and mathematical world can be reduced to this form, of which we have already given an example with the minimization of rotational soap membranes.

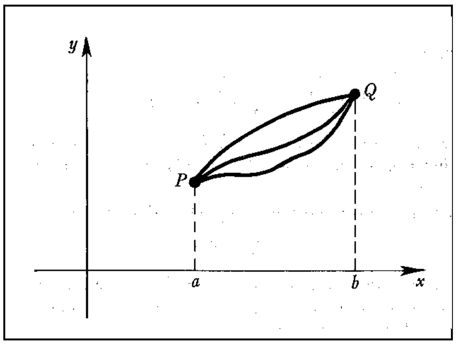
We now consider a small variation δy to the function y. As explained above, the functional $I(y)$ has a stationary point (max/min) if the variation of $I(y)$ vanishes.

(2.6) $$\delta I = I(y + \delta y) - I(y) = 0$$

The very clever trick in the procedure (due to Euler) is the substitution of determining a vanishing variation of a functional to determine a simple stationary point of a real function of one variable.

Accordingly, if δy is a arbitrary variation and ε is a small real number, then $\varepsilon \cdot \delta y$ will be a small variation.
The advantage is of course, that we can view $\varepsilon\delta y$ as a real function of the variable ε and apply the elementary methods from differential calculus, where a function $F(\varepsilon)$ has a stationary point if $F'(\varepsilon) = 0$. Applying this to the functional $I(\varepsilon)$, it means that the variation vanishes if $I'(\varepsilon) = 0$.

Figure showing possible paths from *P to Q*



Formulated more precisely: If

$$I(\varepsilon) = I(y + \varepsilon\, \delta y)$$

has a stationary point at y, then $I(\varepsilon)$ must have a stationary point at 0, so that $I'(0) = 0$ for all variations δy, or

(2.7) $$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0 \text{ independent of } \delta y$$

We shall now demonstrate, that (2.7) leads to a differential equation that y must satisfy.

The calculations that lead to the result are, however not entirely trivial. One should bear in mind that δy is an arbitrary real function, and should be treated as such, contrary to an independent variable as Δx. The variation of $I(y)$ is then written:

(2.7) $$I(\varepsilon) = \int_a^b F(y' + \varepsilon\delta y', y + \varepsilon\delta y, x)dx$$

Then it follows from differentiation with respect to ε.

(2.8) $$\frac{dI}{d\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y}\delta y\right)dx$$

And the condition $\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$ gives then:

(2.9) $$\int_a^b \left(\frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y}\delta y\right)dx = 0$$

To proceed, we shall make the following reasonable assumption that the variation of δy, vanishes at a and b, the end points of the interval, so that $\delta y(a) = \delta y(b) = 0$.
Regarding the first term in the integral, we make a partial integration, according to the formula.

(2.10) $$\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)dx$$

(2.11) $$\int_a^b \frac{\partial F}{\partial y'}\delta y' dx = \left[\frac{\partial F}{\partial y'}\delta y\right]_a^b - \int_a^b \frac{d}{dx}\frac{\partial F}{\partial y'}\delta y dx$$

Because of the condition $\delta y(a) = \delta y(b) = 0$, the first term on the right side of the equation vanishes, and after inserting the second term in (2.9), we obtain.

$$\int_a^b \left(\left(\frac{d}{dx}\frac{\partial F}{\partial y'}\right)\delta y - \frac{\partial F}{\partial y}\delta y\right)dx = 0 \qquad \Leftrightarrow$$

(2.12) $$\int_a^b \left(\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right)\delta y dx = 0$$

If the integral must vanish for *all* variations δy, then the integrand must also vanish, leading to a differential equation of second order in y, known as the Euler-Lagrange equation.

(2.13) $$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

In general, this second order differential equation can not be solved, but we shall look at two simplifying modifications, which makes it is possible to obtain solutions in some cases.

1. F does not depend explicitly of y. (This happens only in rare cases)

$$\frac{\partial F}{\partial y} = 0 \quad \Rightarrow \quad \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

Which reduces the equation to:

(2.14) $$\frac{\partial F}{\partial y'} = c \quad (const)$$

2. F does not depend explicitly of x. Which occur in many known classical problems.

I that case, $\frac{\partial F}{\partial x} = 0$, and in the evaluation of $\frac{d}{dx}\frac{\partial F}{\partial y'}$ we can omit $\frac{\partial F}{\partial x}$.

First we multiply the Euler-Lagrange equation $\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$ by *y'* and subsequently we add and subtract the term $y''\frac{\partial F}{\partial y'}$, resulting in:

(2.15) $$(y'\frac{d}{dx}\frac{\partial F}{\partial y'} + y''\frac{\partial F}{\partial y'}) - (y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}) = 0$$

Since $\frac{\partial F}{\partial x} = 0$, the terms in he last parenthesis are seen to be equal to $\frac{dF}{dx}$, and the terms in the first parenthesis are seen to be equal to $\frac{d}{dx}y'\frac{\partial F}{\partial y'}$.

The Euler-Lagrange equation can then be rewritten in the compact form.

(2.16) $$\frac{d}{dx}(y'\frac{\partial F}{\partial y'} - F) = 0 \quad \Leftrightarrow$$

(2.17) $$y'\frac{\partial F}{\partial y'} - F = Const$$

What we have obtained is to convert a second order differential equation to a first order equation that in some cases can be solved.
Since it usually is this form that is applied, it is often referred to as the Euler-Lagrange equation.

3.1 Least surface for a rotational soap membrane

We demonstrated earlier that the surface that is formed, when a function $y = f(x)$ is rotated around the x- axis is given by the expression (functional $O(y)$)

$$O(y) = 2\pi \int_a^b y\sqrt{1+y'^2}\,dx \tag{3.1}$$

In this case, we therefore have $F(y', y'x) = y\sqrt{1+y'^2}$, and we notice that F does not explicitly depend on x. For this reason we can use the simplified version of the Euler-Lagrange equation.

$$y'\frac{\partial F}{\partial y'} - F = C\,, \quad \text{where} \quad F = y\sqrt{1+y'^2} \tag{3.2}$$

C is a constant, which is to be determined by the boundary conditions.
By inserting F and performing the differentiations, we find:

$$y'\frac{2yy'}{2\sqrt{1+y'^2}} - y\sqrt{1+y'^2} = C \tag{3.3}$$

Multiplying by $\sqrt{1+y'^2}$ and reducing, we arrive at the equation.

$$yy'^2 - y(1+y'^2) = C\sqrt{1+y'^2} \quad \Leftrightarrow \quad y = -C\sqrt{1+y'^2} \tag{3.4}$$

This last equation is a first order differential equation. In principle it can be solved by separation of the variables, and integrating, but the form of the square root invites us to use a substitution, applying the hyperbolic functions:

$$\cosh x = ½(e^x + e^{-x}) \quad \text{and} \quad \sinh x = ½(e^x - e^{-x})$$

It is straightforward to verify that

$$(\cosh x)' = \sinh x \;\; \text{og} \;\; (\sinh x)' = \cosh x \quad \text{and that} \quad \cosh^2 x - \sinh^2 x = 1 \tag{3.5}$$

To get rid of the square root, we put:

$$y' = \sinh(\frac{x-x_0}{y_0})\text{, and it then follows that: } y = y_0\cosh(\frac{x-x_0}{y_0}) + k\,.$$

When inserted in $y = -C\sqrt{1+y'^2}$ it gives:

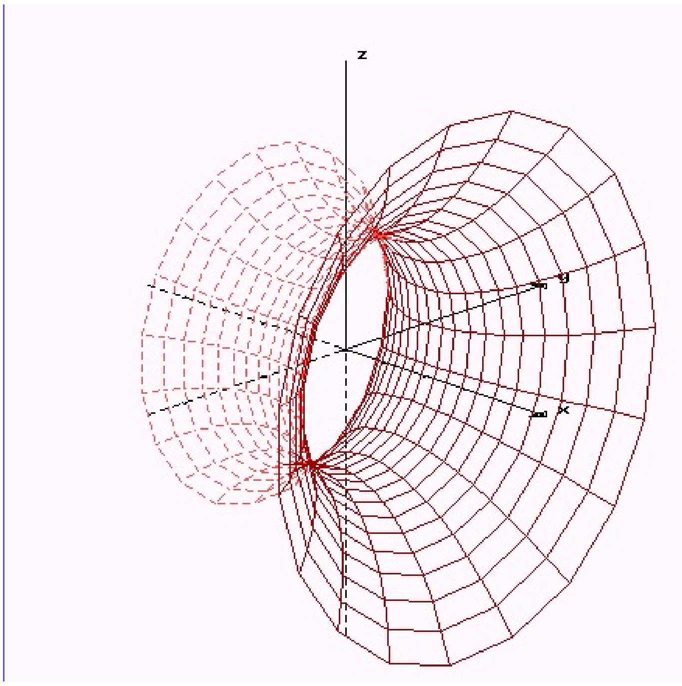
$$y_0\cosh(\frac{x-x_0}{y_0})+k=-C\sqrt{1+\sinh^2(\frac{x-x_0}{y_0})} \Leftrightarrow$$

(3.7) $$y_0\cosh(\frac{x-x_0}{y_0})+k=-C\sqrt{\cosh^2(\frac{x-x_0}{y_0})} \Leftrightarrow$$

$$y_0\cosh(\frac{x-x_0}{y_0})+k=-C\cosh(\frac{x-x_0}{y_0})$$

From the last equation, it follows that

$$y=y_0\cosh(\frac{x-x_0}{y_0})+k$$



is indeed a solution, if and only if

$k = 0$ and $C = -y_0$.

So the solution to the least surface, having the boundary conditions:

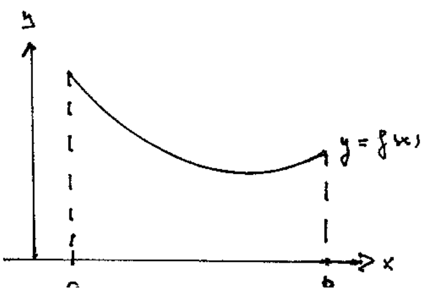
$f(a) = r_a$ and $f(b) = r_b$

is

(3.8) $$y=y_0\cosh(\frac{x-x_0}{y_0})$$

Thus, the curve of intersection with the x-y plane, will be a hyperbolic cosine.
The constants x_0 and y_0 are determined by the boundary conditions e.g. by the radii in the end faces.
To the left, figure (3.9) is shown a (beautiful) computer generated drawing of a rotational hyperbolic soap membrane surface.

3.2 The suspended chain



The problem is in its simplicity to determine the curve that a chain forms, when it is suspended in two points. The preliminary assumptions require only a minimum knowledge of physics.
It is an established fact in physics that a mechanical system, with one or more degrees of freedom i.e. possibility to move freely in one or more directions, will always tend to a position with the lowest potential energy. Because of the inevitable friction, the system will finally find rest in that position.

We shall therefore initiate by setting up an expression for the potential energy of the chain suspended in the two points $(a, f(a))$ *and* $(b, f(b))$, where the chain has a form described by the function $y = f(x)$.

The mass of the chain per unit length is denoted by μ.
As described earlier, the length of an infinitesimal piece *ds* of the curve, corresponding to the increment *dx* is given by the expression:

(3,10) $$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \sqrt{1 + y'^2}\, dx \quad \text{så} \quad ds = \sqrt{1 + y'^2}\, dx$$

The potential energy *dE* of a mass placed at *ds* is, according to the formula $E_{pot} = mgh$.

(3.11) $$dE = \mu g y ds = \mu g y \sqrt{1 + y'^2}\, dx$$

The overall potential energy of the chain is therefore.

(3.12) $$E = \mu g \int_a^b y \sqrt{1 + y'^2}\, dx$$

The problem of determining the form of the chain, having the lowest potential energy, is therefore a variation problem with:

(3.13) $$F(y', y, x) = y \sqrt{1 + y'^2}\, dx$$

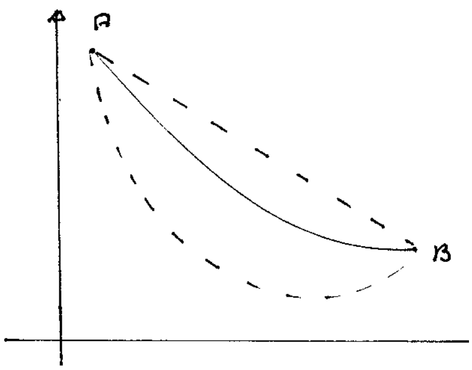
The integrand is, however the same as in the previous problem, (apart from a constant factor), and so is the solution, which we just copy.

(3.14) $$y = y_0 \cosh\left(\frac{x - x_0}{y_0}\right)$$

The curve that a chain suspended in to points forms turns out to be a hyperbolic cosine.

3.3 The Brachistochrone

This is probably the most notorious problem first proposed by Johan Bernoulli, and solved by Euler in his invention of the calculus of variations.



The problem is to determine the trajectory a particle will chose, if it without friction should move from a higher position *A* to a lower position *B* in the shortest possible time.
Instantaneously, one might think that the shortest path (being a straight line) also would be the fastest path, but this is not necessarily the case since, if the trajectory is steeper in the beginning, the particle will gain more speed to traverse the rest of the path.
The solution to the problem is in fact rather surprising.

From kinematics, we know, that $ds = vdt$ (distance = velocity x time)

At the same time $ds = \sqrt{1+y'^2}dx \quad \Rightarrow \quad dt = \dfrac{\sqrt{1+y'^2}}{v}dx$

Conservation of energy in a free fall in the gravitational field gives: ½$mv^2 = mgy \Leftrightarrow \; v = \sqrt{2gy}$
When inserted in the expression for *dt*, it gives:

(3.15) $$dt = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

Thus the exercise is then to determine the minimum value of the functional:

(3.16) $$t_{AB} = \int_a^b \sqrt{\frac{1+y'^2}{2gy}}dx$$

Which is a variation problem with the function:

(3.17) $$F(y', y'x) = \sqrt{\frac{1+y'^2}{2gy}}$$

Since *F* does not depend explicitly on *x*, we shall apply the same version of the Euler-Lagrange equation: $y'\dfrac{\partial F}{\partial y'} - F = C$, Where we insert the expression for *F*. We omit the factor $\sqrt{2g}$.

(3.19) $$y'\frac{2y'}{2\sqrt{y(1+y'^2)}} - \sqrt{\frac{1+y'^2}{y}} = C$$

Multiplying (3.19) by $\sqrt{y(1+y'^2)}$ gives: $y'^2 - (1+y'^2) = C\sqrt{y(1+y'^2)}$, and after some reduction and squaring the equation, it leaves us with the non linear first order differential equation:

(3.20) $$y(1+y'^2) = c \qquad \text{(Where } c \text{ is a new constant)}$$

Since (tan *x*)’ = 1 + $\tan^2 x$, it is tempting to try with the substitution. *y’*= tan *θ,*

Then $$1 + y'^2 = 1 + \tan^2\theta = \frac{1}{\cos^2\theta}.$$

When inserting this, the equation becomes:

(3.21) $$\begin{aligned} &y(1+y'^2) = c \quad \Rightarrow \\ &y = \frac{c}{1+y'^2} = c\cdot\cos^2\theta = \frac{c}{2}(1+\cos 2\theta) \end{aligned}$$

The substitution *y’*= tan *θ,* however only gives one half the parametric of the trajectory.

To obtain $x = x(\theta)$, we shall make the following rewriting:

(3.22) $$\frac{dx}{d\theta} = \frac{dx}{dy}\frac{dy}{d\theta} = -\frac{c}{y'}\sin 2\theta = -c\frac{2\sin\theta\cos\theta}{\tan\theta} = -c2\cos^2\theta = -c(1+\cos 2\theta)$$

Arriving at the sought solution.

(3.23) $$x = -\frac{c}{2}(2\theta - \sin 2\theta) + x_0 \quad and \quad y = \frac{c}{2}(1+\cos 2\theta)$$

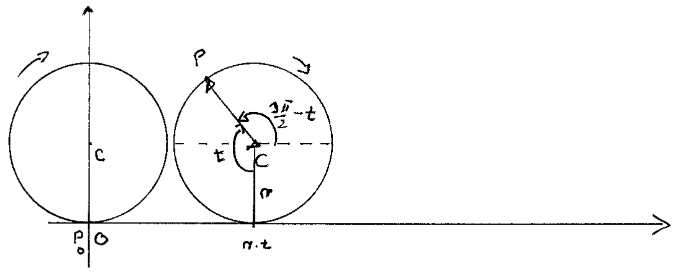
Introducing A=-½c and t = ½θ, we can write it in a more common form.

(3.24) $$x = A(t - \sin t) + x_0 \quad and \quad y = -A(1+\cos t)$$

This we recognize as the parametric of a cycloid, which is called the Brachistochrone.

The mathematical curve of the Brachistochrone is sketched below.
It is the trajectory that a fixed point on a circle rolling on the x-axis follows.

If we choose $x_0 = A(½\pi + 1)$, then a point , which at t = ½π , move until $t = \pi$, will move from the position $(x,y) = (0, A)$ to $(x,y) = (A(\pi - 1), 0)$.

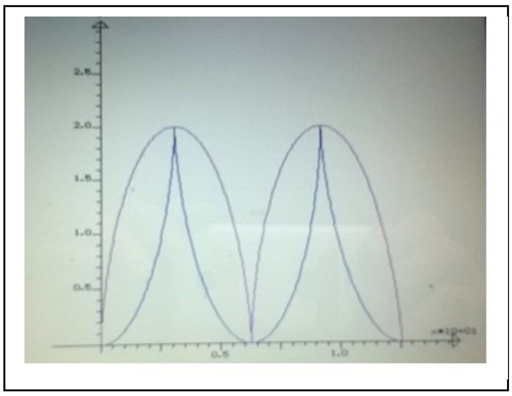


(3.25) **The parametric for the cycloid**

From the figure above, we see: $\vec{OP} = \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \vec{OC} + \vec{CP} = \begin{pmatrix} rt \\ r \end{pmatrix} + \begin{pmatrix} r\cos(\frac{3}{2}\pi - t) \\ r\sin(\frac{3}{2}\pi - t) \end{pmatrix} = \begin{pmatrix} rt - r\sin t \\ r - r\cos t \end{pmatrix}$

And the parametric becomes: $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} rt - r\sin t \\ r - r\cos t \end{pmatrix}$

Below is shown the graph of the mathematical cycloid, and the graph for the solution to the differential equation



4. Extremum with side conditions

Two of the most famous problems that can be solved only by the Calculus of Variations, are to determine the largest area within a closed perimeter and the largest volume within a closed surface. (We know they are a circular disc and a sphere)

To determine the largest volume within a given closed surface area, we initiate by writing the expressions for calculating the volume V and the surface area O, for a body which is the result of rotating a function $y = f(x)$ 360^0 around the x axis.

(4.1) $$V = \int_a^b y^2 dx \qquad and \qquad O = \int_a^b y\sqrt{1+y'^2}\,dx$$

The task is then to find the extremum for the volume V under the side condition O = constant. This is, however not "the simplest problem", but is described as **optimization with a side condition**. In the example stated above, the side condition is that the area of the surface is constant.
Before we enter the calculus of variation with a side condition, we shall introduce the notion of Lagrange multipliers, when finding the extremum of a real function of two or more variables. More precisely, our aim is to determine extremum for a function $z = f(x,y)$ subject to the side condition $g(x,y) = c$ (constant).
If $g(x,y) = c$ can be solved with respect to y, to give $y = h(x)$, then this can be inserted in $z = f(x,y)$ to give: $z = f(x,h(x))$, and the extremum of f can be found by traditional analytic methods i.e. finding the solution of the equation $f'(x,h(x)) = 0$

In the majority of cases, however the equation $g(x,y) = c$ cannot be solved analytically with respect to y, and we are referred to other analytical methods

If f has extremum in x_0 , then the variation of f around x_0 vanishes i.e. the differential of f in x_0 equals 0 for all variations of dx and dy.

(4.2) $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

Since dx and dy are independent, we may conclude that $\frac{\partial f}{\partial x} = 0 \;\; and \;\; \frac{\partial f}{\partial y} = 0$.

But if the extremum is subject to a side condition $g(x,y) = c$, then dx and dy are no longer independent, but bound by the relation

(4.3) $$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$$

In general for two linear equations with to unknowns, where the right side are zero

(4.4) $$\begin{aligned} a_1 x + b_1 y = 0 \\ a_2 x + b_2 y = 0 \end{aligned}$$

We know that the determinant of the system of equations is zero, $D = a_1 b_2 - a_2 b_1 = 0$ and consequently the two pairs of coefficients are proportional to each other.
This is easily seen since, if we put $a_2 = \lambda a_1$ then it implies $b_2 = \lambda b_1$. So $(a_2, b_2) = \lambda(a_1, b_1)$, where λ is a constant.
The set of equations (4.4) are equivalent to the set of equations, fetched from (4.2) and (4.3). Provided we replace the coefficients (a_1, b_1) and (a_2, b_2) with the partial derivatives of f and g, and x with dx , and y with dy.

(4.5)
$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$
$$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$$

Thus we conclude: (There is a tradition of writing $-\lambda$ instead of λ)

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = -\lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) \quad \Leftrightarrow \quad \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) + \lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = 0$$

Or written out

(4.6)
$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \qquad og \qquad \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

But this signifies precisely that the function $f(x,y) + \lambda\, g(x,y)$ has extremun in the point.

We recapitulate the results. The function $f(x,y)$ has extremum in a point, with the side condition $g(x,y) = c$, if and only if the function $f(x,y) + \lambda g(x,y)$ has extremum at that point.

The constant λ is to be determined from the boundary conditions of the problem

λ is denoted a Lagrange *multiplier*. The theory for Lagrange multipliers can easily be extended to functions of more than two variables, since there is always the same number of Lagrange multipliers, as there are side conditions.

4.1 Largest area within a given perimeter

One problem that inherits to ancient time, is the question to determine the largest area within a fixed perimeter. Already the Greek mathematicians knew that the answer is a circle with radius r and perimeter $2\pi r$. A formal proof has, however not been given before discovery of the variation calculus.
We consider an area confined by a function $y = f(x)$ and the x-axis. The function is defined in an interval [a ,b] , where $f(a) = f(b) = 0$.
Since the part of the perimeter on the x-axis with length $b - a$, which is at the x-axis, is the same for all functions, we shall leave it out of the calculations
The length ds of the piece of the function graph that correspond to dx is:

(4.7) $$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})^2}\,dx = \sqrt{1 + y'^2}\,dx \text{ , so } \quad ds = \sqrt{1 + y'^2}\,dx$$

The length s_{ab} of the graph $y= f(x)$, and the area A confined by the graph of $f(x)$ and the x-axis are:

(4.8) $$s_{ab} = \int_a^b \sqrt{1 + y'^2}\,dx \quad \text{and} \quad A = \int_a^b f(x)dx = \int_a^b y dx$$

We seek extremum for the area A under the side condition s_{ab} = konstant. The Lagrange function $F(y', y, x)$ then becomes:

(4.9) $$F(y', y, x) = y + \lambda\sqrt{1 + y'^2}$$

As F does not depend explicitly on x, we apply the reduced form of the Euler Lagrange equations.

(4.10) $$y'\frac{\partial F}{\partial y'} - F = C$$

And doing the differentiations gives:

(4.11) $$y'\frac{\lambda y'}{\sqrt{1 + y'^2}} - \lambda(y + \sqrt{1 + y'^2}) = C$$

Multiplying by $\sqrt{1 + y'^2}$ gives:

(4.12) $$\lambda y'^2 - \lambda y\sqrt{1 + y'^2} - \lambda(1 + y'^2) = C\sqrt{1 + y'^2}$$

After some reduction:

(4.13) $$(y + C)\sqrt{1 + y'^2} = -\lambda$$

This equation can not be solved for all values of λ and C. So we restrict ourselves showing that the solution is part of an arc of a circle.
We know that a first order differential equation has one and only one solution, where $f(x_0) = y_0$.
So, if we find a solution, it is *the solution*.
We therefore try with a circle: $x^2 + y^2 = r^2 \quad \Leftrightarrow \quad y = \sqrt{r^2 - x^2}$ (in the upper half plane)

(4.14) $$(\sqrt{r^2 - x^2} + C)\sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} = -\lambda \quad \Leftrightarrow$$

$$(\sqrt{r^2-x^2}+C)\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}=-\lambda$$

(4.15) $$(\sqrt{r^2-x^2}+C)\frac{r}{\sqrt{r^2-x^2}}=-\lambda$$

From (4.15) we see that the circle arc is actually a solution, provided $C=0$ and $-\lambda = r$.
Thus we have given an analytic answer to the question of the largest area within a perimeter of fixed length.

4.2 Largest volume for a given surface area

We shall now return to the problem of determining the largest volume, within a given surface. The volume and surface area are given by the integrals.

(4.16) $$V(y)=\pi\int_a^b y^2dx \qquad O(y)=2\pi\int_a^b y\sqrt{1+y'^2}\,dx=k$$

When using the theory of extremum with a side condition, the problem can be treated, by applying the Lagrange multipliers.
This corresponds to seek extremum for the functional: $I(y)=V(y)+\lambda O(y)$

(4.17) $$I(y)=\pi\int_a^b y^2dx+\lambda 2\pi\int_a^b y\sqrt{1+y'^2}\,dx=\pi\int_a^b(y^2+2\lambda y\sqrt{1+y'^2})dx=\pi\int_a^b F(y',y,x)dx$$

Where $$F(y',y,x)=y^2+2\lambda y\sqrt{1+y'^2}$$

$F(y',y,x)$ does not depend explicitly on x, so we can use the simplified form of the Euler-Lagrange equations.

(4.18) $$y'\frac{\partial F}{\partial y'}-F=C$$

Inserting the expression for F and, carrying out the differentiations, we get:

(4.19) $$y'\frac{2\lambda y' y}{\sqrt{1+y'^2}}-(y^2+2\lambda y\sqrt{1+y'^2})=C$$

Multiplying by $\sqrt{1+y'^2}$ followed by some reduction, we have

(4.20) $$(y^2+C)\sqrt{1+y'^2}+2\lambda y=0$$

Although this differential equation can in principle be solved by separating the variables, it can not be carried out, at least not for all values of C and λ.

Our presumption is however, that the solution is a sphere, so we make an attempt with the equation of a sphere.

$$x^2 + y^2 = r^2 \Leftrightarrow y = \pm\sqrt{r^2 - x^2}$$

By inserting in (4.11) we are faced with the equation

$$(r^2 - x^2 + C)\sqrt{1 + \frac{x^2}{r^2 - x^2}} + 2\lambda\sqrt{r^2 - x^2} = 0 \tag{4.21}$$

Multiplying by $\sqrt{r^2 - x^2}$ and reduction we find:

$$(r^2 - x^2 + C)\sqrt{r^2} + 2\lambda(r^2 - x^2) = 0 \tag{4.22}$$

From which we conclude that $y = \pm\sqrt{r^2 - x^2}$ is a solution, if and only if C=0 and $\lambda = -\frac{1}{2}r$

We have thus proved that the sphere is the (rotational symmetric) form that has the largest volume for a given surface.

4.3 On the shape of Wine barrels

If we relax on the claim $f(a) = f(b) = 0$, the rotational body is no longer a sphere but a barrel with radii $f(a) = f(b) = r$, at the end faces. This will correspond to the case $C \neq 0$.
However, as mentioned above, the differential equation:

$$(y^2 + C)\sqrt{1 + y'^2} + 2\lambda y = 0 \tag{4.23}$$

does not have a solution, which can be expressed by known real functions.
Solving the equation numerically, however it turns out that the solution reveals a distinct shape of a wine barrel.

Wine barrels have had the familiar shape for many hundred years. Designing a barrel is a compromise between two regards. One is having the largest volume for the least materials.
We know that is the spherical container. On the other hand, spherical containers are certainly not the optimal solution, if the intention is to stack as many barrels as possible in a wine cellar
Furthermore spherical containers are not particular stable. For this reason the most obvious choice would be a cylindrical container, with a circular cross section.

The compromise has been the traditional wine barrel, and it has been verified, that the shape of the barrels that have been used for centuries in Europe, are in fact very close to what you obtain from the calculus of variations.

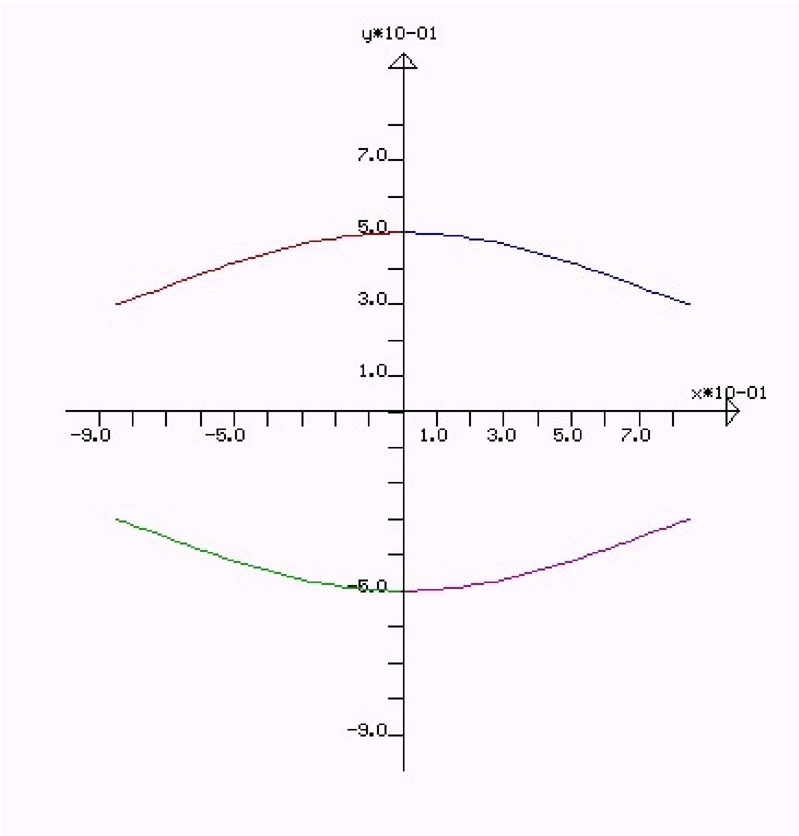
In creating the computer generated solution, we have assumed that the solution is symmetric about the y axis, and furthermore that: The barrel has end points at $-a$ and a, and is symmetric around the y-axis.

$$y'(0) = 0, \quad y(0) = R, \quad y'(a) = \alpha, \quad y(a) = r.$$

One may choose different values of *R* (the maximum radius of the barrel) and *r* (the radii of the end faces), but we shall only show one examples:
When these values are inserted in the differential equation, we get two equations which fix λ og *C*.

(4.24) $$R^2 + C + 2\lambda R = 0 \quad \text{and} \quad (r^2 + C)\sqrt{1+\alpha^2} + 2\lambda r = 0$$

With an appropriate choice of $R, r \text{ and } \alpha$ these equation can be solved, and inserted in the differential equation (4.12) it yields the amazing shape of (an intersection with the $x - y$ plane) of a wine barrel.



4.4 The shape of a hanging water drop

Soap bubbles, water drops in air, drops of quicksilver are all held together by the *surface tension*.

The surface tension for a certain liquid as water, soap water, quick silver is defined as the force acting perpendicular on a (mathematical) cut, divided by the length of the cut

In contrast to the rules for elastic materials, the surface tension does not depend of the size of the surface.
From this follows immediately, that the potential energy, caused by the surface tension is directly proportional to the area of the surface.

Since a physical system will always to seek to a state of minimum potential energy, it then follows that drops of water in a free fall in air will have a spherical shape, since it gives the least surface for a given volume.

Quite differently it relates to a drop of water, which hangs on a leaf or a water tub, or a quick silver drop lying on plane underlay. Here the potential energy consist namely on two contributions.

1. The potential energy from the surface tension, which is least, when the drop has a spherical shape
2. The gravitational potential energy: $E_{pot} = mgh$, which is least, when the drop is smeared out over the largest area.

If the mass gets too large, the surface tension is not able to told the drop together, in accordance with the experience, that one does not observe dew drops larger than a couple of millimeters, and likewise in laboratories in schools (before 1980), the observation that, when quick silver drops assemble, they do not form bigger and bigger bullets, but splash out on the underlay.

A liquid drop with surface S, has as a consequence of the surface tension a potential energy $E_{pot}= \gamma S$, where γ is a material constant.

$$\gamma_{water} = 76\ mN/m, \quad \gamma_{Hg} = 465\ mN/m, \text{ while } \gamma_{alcohol} = 22\ mN/m.$$

The value for alcohol explains, why you seldom observe drops of alcohol.

The shape of water or dew drops is an obvious challenge for the calculus of variation, since the problem may be stated as finding the minimum of E_{pot}(surface tension) + E_{pot}(gravity)

Unfortunately, there is no analytical solution to the problem, but the numerical solution reveals what a hanging water drop, of various size looks like. This can then be compared to enlarged photos of water drops.
At the same time the resulting differential equation, can be used to estimate the maximum size for hanging water drops.

We shall first establish the formulas for the surface S of a rotational symmetric body (around the y axis, since a water drop is rotational symmetric around a vertical axis), and for the potential energy E (caused by the surface tension) of a circular disc, with thickness dy, located in the height y.

$$dS = 2\pi x\sqrt{dx^2 + dy^2} = 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi x\sqrt{1+y'^2}\,dx \tag{4.25}$$

$$dE_{pot\ tension}(S) = \gamma dS = 2\pi\gamma x\sqrt{1+y'^2}\,dx \tag{4.26}$$

$$dV = \pi x^2 dy = \pi x^2 \frac{dy}{dx}dx = \pi x^2 y'\,dx$$

$$dE_{pot\ gravity}(V) = (\rho dV)gy = \rho g\pi x^2 yy'\,dx \tag{4.27}$$

Then we use the Euler-Lagrange equations to seek extremum for the functional:

(4.28) $$2\pi\gamma\int_{-r}^{r} x\sqrt{1+y'^2}\,dx + \pi\rho g\int_{-r}^{r} x^2 yy'dx = \pi\int_{-r}^{r}(2\gamma x\sqrt{1+y'^2} + \rho g x^2 yy')dx$$

Adding the side condition:

(4.29) $$\pi\int_{-r}^{r} x^2 y'dx = V_0$$

We leave out the factor π, and our *F*(*y'*,*y*, *x*) is then expressed by the Lagrange multiplier λ.

(4.30) $$F(y',y,x) = 2\gamma x\sqrt{1+y'^2} + \rho g x^2 yy' + \lambda x^2 y'$$

In this case *F* does explicitly depend on *x*, and we are bound to use the general form of the Euler – Lagrange equations, which is a second order differential equation in *y*.

(4.31) $$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

First we calculate the partial derivatives:

(4.32) $$\frac{\partial F}{\partial y} = \rho g x^2 y' \quad \text{and} \quad \frac{\partial F}{\partial y'} = \frac{2\gamma x y'}{\sqrt{1+y'^2}} + \rho g x^2 y + \lambda x^2$$

And next we calculate the total derivative

(4.33) $$\frac{d}{dx}\frac{\partial F}{\partial y'} = 2\gamma\left(\frac{y'}{\sqrt{1+y'^2}} + x\left(\frac{y''}{\sqrt{1+y'^2}} - \frac{y'2y'y''}{2(1+y'^2)^{\frac{3}{2}}}\right)\right) + \rho g(2xy + x^2 y') + 2\lambda x$$

Plugging them in the Eulers-Lagrange equations followed by some reduction, we have:

(4.34) $$\frac{2\gamma}{\sqrt{1+y'^2}}\left(y' + \frac{xy''}{1+y'^2}\right) + \rho g 2xy + 2\lambda x = 0$$

Finally we isolate *y''*:

(4.35) $$y'' = -(1+y'^2)\left(\sqrt{1+y'^2}\left(\frac{\rho g}{\gamma}y + \frac{\lambda}{\gamma}\right) + \frac{y'}{x}\right)$$

Even for the experienced mathematician the prospect of finding an analytical solution to (4.35) is probably nil.
If we want to create a numerical solution, which initiate in (0,0), we must remove the minus sign, since otherwise the numerical solution will turn downwards in the negative half plane of *y*.

It is not so easy to comment on the value of λ. Below is shown solutions with λ = 50, λ = 100 and λ = 500. The biggest drop has λ = 50.

Doing the numerical solution is not without obstacles however, since the shape has two vertical tangents (where *y'* is undefined), and the expression (4.24) is undefined for $x = 0$.
We are therefore referred to a partition of the solution into four pieces, avoiding the end points, what also appears from the solutions shown below.

The numerical solutions show three water drops having widths 0.35 *mm*, 1.5 *mm* and 2.3 *mm*. The largest drop has a length of 8 *mm*.
If you do numerical experiments with larger drops, you will find no solution for the upper part of the drop.

Looking at the differential equation, one may also understand, why there is an upper limit for the size of the drop.
If we consider a point (a,b), where the curve of intersection goes from being convex to being concave , the condition must be that $y'' < 0$ for $y > b$.
The result is the inequality:

$$\text{(4.36)} \qquad \sqrt{1+y'^2}\left(\frac{\rho g}{\gamma}b+\frac{\lambda}{\gamma}\right)+\frac{y'}{a}<0 \;\Leftrightarrow\; a\left(\frac{\rho g}{\gamma}b+\frac{\lambda}{\gamma}\right)<-\frac{y'}{\sqrt{1+y'^2}}$$

The function $f(y')=\dfrac{y'}{\sqrt{1+y'^2}}$ has its set of values in the interval]-1, 1[,

so the condition can be expressed as:

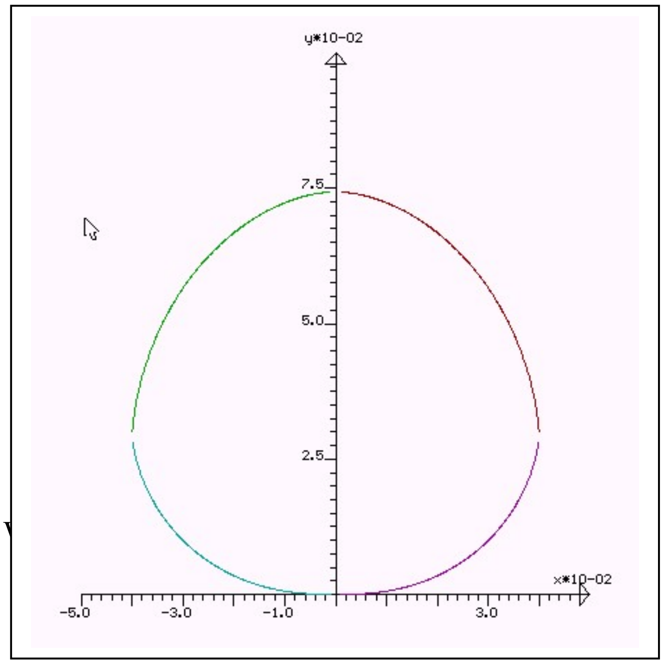
$$a\left(\frac{\rho g}{\gamma}b+\frac{\lambda}{\gamma}\right)<1 \;\Leftrightarrow\; \lambda<\frac{\gamma}{a}-\rho g b\,.$$

If we insert $\gamma = 76$ *dyn/cm,* and $\rho g = 982$ *dyn/cm*^{3},we obtain the inequality $\lambda < \dfrac{76}{a} - 982b$.

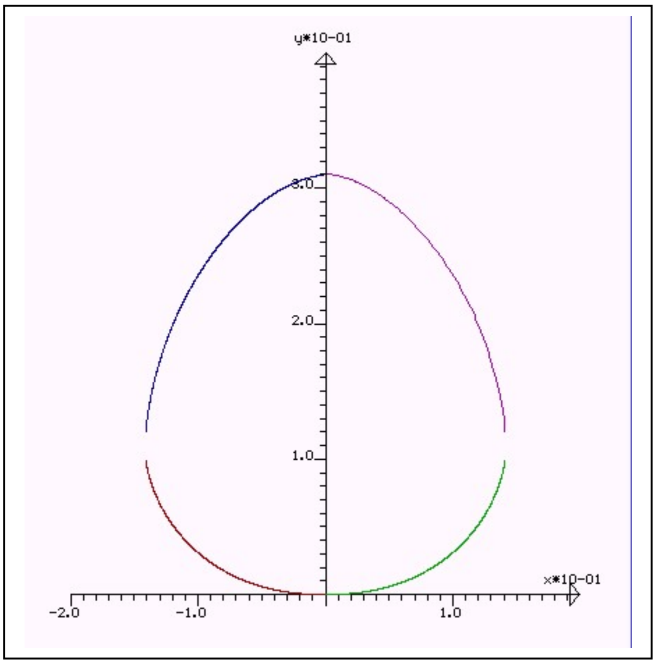
If we estimate $b = \frac{1}{2}a$, then we can calculate for which values of a $\dfrac{76}{a} - 491a > 0$, and we find that $a < 0{,}4$ *cm*. The estimate is then that the width of a drop has a maximum of 4 *mm*, which is in perfect agreement, when trying to solve the differential equation numerically
Choosing an estimate $b = \frac{1}{3}a$ we find a maximum width of 0,5 *cm*.

Below is shown vertical cross sections of water drops corresponding to widths 0.35 *mm*, 1.5 mm and 2.3 mm. The largest drop has a length of 8 *mm*.

Water drop width 0.35 *mm*



Water drop width 1.5 *mm*



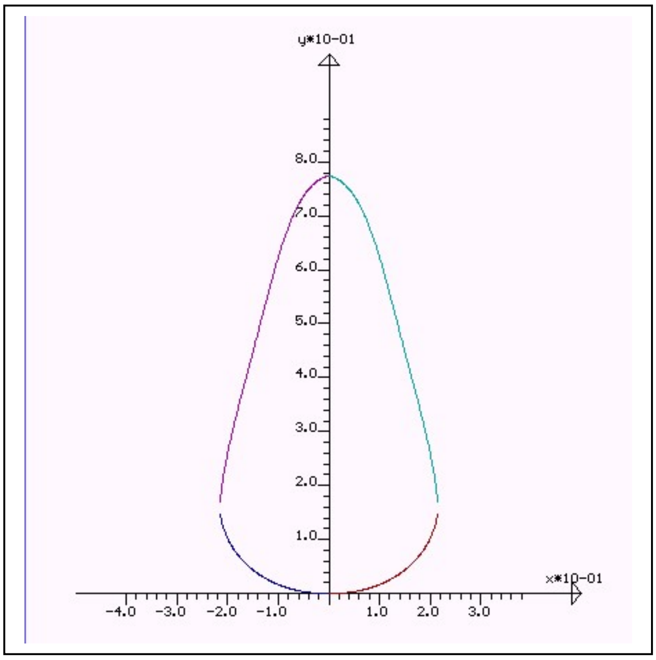


Photo taken from a water drop.
Notice the flagrant resemblance, with
the computer generated solutions