The sum $S_q(n)$ of integer powers

$$S_q(n) = \sum_{k=1}^{n} k^q$$

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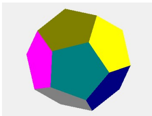

Ole Witt-Hansen 2005

Sums of integer powers

The wish to be able to find the sums of integers raised to an integer power appears in various circumstances. For example the classical problem of computing the number of balls in a three sided pyramid with n layers is:

$$\sum_{k=1}^{n}\frac{k(k+1)}{2}=\frac{1}{2}\sum_{k=1}^{n}k^2+\frac{1}{2}\sum_{k=1}^{n}k=\frac{1}{2}\sum_{k=1}^{n}k^2+\frac{n(n+1)}{4}$$

The last sum is the well known formula for algebraic series, whereas the first sum of the quadratic numbers from 1 to n: $\sum_{k=1}^{n}k^2$ is less simple to accomplish.

It turns out, (to my knowledge) that it is only possible to find a recursion formula for the sums:

$$S_n(q)=1^q+2^q+3^q+...+n^q \quad ,$$

where $q = 1,2,3\ldots.,$ and $q = 2$ corresponds to the sum above.

In the first university lecture I joined in mathematics in 1964, the professor used the first hour to introduce the use of summation and product symbols, illustrating this by deriving among other examples the Cauchy-Swartz inequality.

But already in the second lecture he proved a recursion formula for sums of integer powers. (Much has certainly changed in the study of mathematics since 1964, but listening to this first two hour lecture, I began to wonder, whether I had chosen the right line of study).

I shall try to recall the Professors proof, but in my own formulation

$$S_n(q)=1^q+2^q+3^q+...+n^q=\sum_{k=1}^{n}k^q=\sum_{k=0}^{n-1}(k+1)^q=1+\sum_{k=1}^{n-1}(k+1)^q$$

The trick is to apply the binomial formula on each of the terms $(k+1)^q$ in the sum, taking the term with the highest power of q outside the summation symbol.

$$S_q(n)=1+\sum_{k=1}^{n-1}\sum_{j=0}^{q}\binom{q}{j}k^j$$

$$S_q(n)=1+\sum_{k=1}^{n-1}\binom{q}{q}k^q+\sum_{k=1}^{n-1}\sum_{j=0}^{q-1}\binom{q}{j}k^j=$$

$$1+S_q(n-1)+\sum_{k=1}^{n-1}\sum_{j=0}^{q-1}\binom{q}{j}k^j \quad\Rightarrow\quad S_q(n)-S_q(n-1)-1=\sum_{k=1}^{n-1}\sum_{j=0}^{q-1}\binom{q}{j}k^j$$

Since $S_q(n)-S_q(n-1)=n^q$, we then get, if we alternate the order of summation in the last term.

$$n^q - 1 = \sum_{j=0}^{q-1} \binom{q}{j} \sum_{k=1}^{n-1} k^j \iff n^q - 1 = \sum_{j=0}^{q-1} \binom{q}{j} S_j(n-1)$$

Where we have noticed that the last sum is equal to $S_j(n-1)$.

In this formula, we replace *n* by *n* +1, *q* by *q* +1, and extract the last term from the summation.

$$(n+1)^{q+1} - 1 = \sum_{j=0}^{q} \binom{q+1}{j} S_j(n) \iff \binom{q+1}{q} S_q(n) + \sum_{j=0}^{q-1} \binom{q+1}{j} S_j(n)$$

Since $\binom{q+1}{q} = q+1$, we finally obtain the recursion formula by isolating $S_q(n)$.

$$(q+1)S_q(n) = (n+1)^{q+1} - \sum_{j=0}^{q-1} \binom{q+1}{j} S_j(n) - 1$$

So if the integer power sums: $S_0(n)$, $S_1(n)$, $S_2(n)$..... $S_{q-1}(n)$ are already calculated, we may find the formula for $S_q(n)$.

We immediately see that $S_0(n) = 1+1+1+..+1 = n$, and

$$2S_1(n) = (n+1)^2 - 1 + S_0(n) = (n+1)^2 - 1 - n = n(n+1) \iff S_1(n) = \frac{n(n+1)}{2}$$

As it is the sum of the well known algebraic series: 1+2+3+…+*n*

For $S_2(n)$, we then get:

$$3S_2(n) = (n+1)^3 - S_0(n) - 3S_1(n) - 1 = (n+1)^3 - n - \frac{3n(n+1)}{2} - 1 =$$
$$(n+1)(n^2 + 1 + 2n - 1 - \frac{3}{2}n) = (n+1)(n^2 + \frac{1}{2}n) = \frac{1}{2}n(n+1)(2n+1)$$
$$S_2(n) = \frac{1}{6}n(n+1)(2n+1)$$

This formula for $S_2(n)$ is then used to derive the formula for the number of balls in a three sided pyramid having *n* layers:

$$
\begin{aligned}
S_n(balls\ \ in\ \ pyramid) &= \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k \\
&= \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{n(n+1)}{4} = \frac{1}{12} n(n+1)(2n+1) + \frac{n(n+1)}{4} \\
&= \frac{1}{4} n(n+1)(\frac{1}{3}(2n+1)+1) = \frac{1}{12} n(n+1)(2n+4) \\
&= \frac{1}{6} n(n+1)(n+2)
\end{aligned}
$$

If we continue with some ”elementary” but also somewhat cumbersome calculations we find:

$$
\begin{aligned}
S_3(n) &= \frac{1}{4} n^2 (n+1)^2 \\
S_4(n) &= \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)
\end{aligned}
$$

Notice that: $S_3(n) = S_1(n)^2$, but the explanation is not that simple.

The problem of determining the number of balls in a 3-sided pyramid is an old one, and to stack up balls (or oranges) in this manner is called Kepler’s balls, because Kepler, Newton, Gauss and others tried to prove that it was the most efficient (least volume) to stack balls.
None of them succeeded, however, but around 2003 someone claimed, that they have proved the assertion. (But how?)
One may of course determine the ratio between the volume of the balls and the volume of the pyramid and I have found it to be 0.74. On the other hand if the balls are stacked right above each other the ration decreases to 0.52.