The number of Bricks in a 4-sided pyramid

The number of Oranges in a 3-sided pyramid

This is an article from my home page: www.olewitthansen.dk

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1. The number of quadratic bricks in a 4-sided pyramid

Someone teaching in the 9th grade of grammar school approached me because of a exercise in a text book, which encouraged the students to find the number of quadratic bricks in the *n*'th layer of a 4-sided pyramid built by centicubes, with 1 at the top, 3^2 in the next layer, and so on.
This is very simple, since the answer is $(2n+1)^2$.
But also they were challenged to find the total number of bricks in a pyramid with *n*-layers!

The latter problem was the reason, why the teacher consulted me. The answer is:

$$1^2+3^2+\ldots+(2n+1)^2 \quad n=0,\ 1,\ 2,\ldots$$

But to establish a formula for the sum of squares of the uneven number is far beyond the capabilities of students in the 9th grade, and that holds as well for the teachers, since they are not university educated in Denmark, and the problem is clearly in a university mathematics level.

I was somewhat in doubt myself, but then I remembered, that a long time ago, I wrote an article deriving the recursion formula for the sum of the *q*'th power of the integers from 1 to n.

$$S_q(n)=1^q+2^q+\ldots+n^q$$

The article can be found on my home page, but only in Danish.
It has however a story attached to it, since it was the subject (together with Cauchy's inequality, and Abelian summation!) of the first two lectures the professor gave, when I started at the university in 1964. Even if our mathematical skills entering the university then were far better, than they are today, I felt lost, contemplating to choose a less demanding study.

Applied to the quadratic integers the formula is:

$$S_2(n)=1^2+2^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1)$$

From which we find.

$$S_2(1)=1,\ S_2(2)=5,\ S_2(3)=14,\ S_2(4)=30,\ S_2(5)=55,\ S_2(6)=91,\ S_2(7)=140,\ S_2(8)=204.$$

To obtain a formula for the uneven integers

$$U_2(2n+1)=1^2+3^2+\ldots+(2n+1)^2,$$

One may proceed systematically

$n=1$: $\quad U_2(3)=S_2(3)-2^2=S_2(3)-2^2(1)=S_2(3)-2^2S_2(1)$

$n=2$: $\quad U_2(5)=S_2(5)-(2^2+4^2)=S_2(5)-2^2(1^2+2^2)=S_2(5)-2^2S_2(2)$

$n=3$: $\quad U_2(7)=S_2(7)-2^2-4^2-6^2=S_2(7)-2^2(1^2+2^2+3^2)=S_2(7)-2^2S_2(3)$

Leading to the general formula:

$$
\begin{aligned}
U_2(2n+1) &= S_2(2n+1) - 2^2 S_2(n) \\
&= \tfrac{1}{6}((2n+1)(2n+2)(4n+3) - 4n(n+1)(2n+1)) \\
&= \tfrac{1}{6}((2n+1)(n+1)(2(4n+3) - 4n)) \\
&= \tfrac{1}{6}(2n+1)(n+1)(4n+6)
\end{aligned}
$$

$$U_2(2n+1) = \tfrac{1}{3}(2n+1)(n+1)(2n+3)$$

We check the formula for n = 1, 2, 3, 5.

$$
\begin{aligned}
&n=1: \; U_2(3) = S_2(3) - 2^2 S_2(1) = \tfrac{1}{3}(2+1)(1+1)(2+3) = 10 \\
&n=2: \; U_2(5) = S_2(5) - 2^2 S_2(2) = 55 - 4\cdot 5 = \tfrac{1}{3}(4+1)(2+1)(4+3) = 35 \\
&n=3: \; U_2(7) = S_2(7) - 2^2 S_2(3) = 140 - 4\cdot 14 = \tfrac{1}{3}(6+1)(3+1)(6+3) = 84 \\
&n=4: \; U_2(9) = S_2(9) - 2^2 S_2(4) = 285 - 4\cdot 30 = 165
\end{aligned}
$$

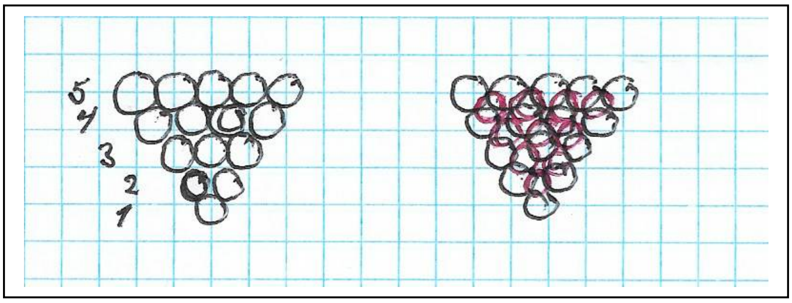
It seems to be correct.

2. The number of oranges in a 3-sided pyramid

A classical problem related to the pyramid problem above is to calculate the number of oranges (balls) in a pyramid having *n* layers. Below is a sketch of triangle composed of identical balls. The number of balls in the triangle is easily seen to be.

$$S_n = 1 + 2 + 3 + \ldots (n-1) + n = \frac{n(n+1)}{2}$$

This according to the formula for an arithmetic series: $S_n = \frac{n}{2}(a_1 + a_n)$, where n is the number of terms and a_1 and a_n are the first and last element respectively. For *n* =5, the result is 15, which is seen to be correct.



Building a pyramid, where the bottom layer has *n* balls in the side of the triangle, the next layer will have *n* – 1, and consequently:

$$S_{n-1} = 1 + 2 + 3 + \ldots (n-1) = \frac{n(n-1)}{2}$$

The pyramid ends, having 1 ball at the top, and we must evaluate the sum:

$$\begin{aligned} S_{pyramid} &= S_n + S_{n-1} + S_{n-2} + \ldots + S_1 \\ &= \frac{n(n+1)}{2} + \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} + \ldots \frac{(1)(2)}{2} = \\ &= \frac{1}{2}(n(n+1) + n(n-1) + (n-1)(n-2) + \ldots + 2) \end{aligned}$$

If this looks a bit distressing, it is vastly simplified using: $n(n+1) = n^2 + n$

$$S_{pyramide} = \frac{1}{2}(n(n+1) + n(n-1) + (n-1)(n-2) + \ldots + 2)$$
$$S_{pyramide} = \tfrac{1}{2}((n^2 + n) + ((n-1)^2 + (n-1)) + ((n-2)^2 + (n-2)) + \ldots + (1+1))$$

It is most easily handled using the summation symbol.

$$2S_{pyramide} = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

The first tem being the sum of the quadratic integers shown above:

$$S_2(n) = 1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6} n(n+1)(2n+1)$$

And the second is the arithmetic series with the sum formula: $S_n = 1 + 2 + 3 + \ldots (n-1) + n = \dfrac{n(n+1)}{2}$

Adding the two formulas, we get the numbers of balls in the pyramid.

$$\begin{aligned} S_{pyramid} &= \tfrac{1}{2}(\tfrac{1}{6} n(n+1)(2n+1) + \tfrac{1}{2} n(n+1)) \\ &= \tfrac{1}{12}(n(n+1)(2n+1+3) \end{aligned}$$

$$S_{pyramid} = \tfrac{1}{6} n(n+1)(n+2)$$