Advanced group theory
U(2), U(3), SU(2) and SU(3)

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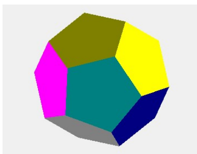

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Contents

Fundamentals of Group Theory

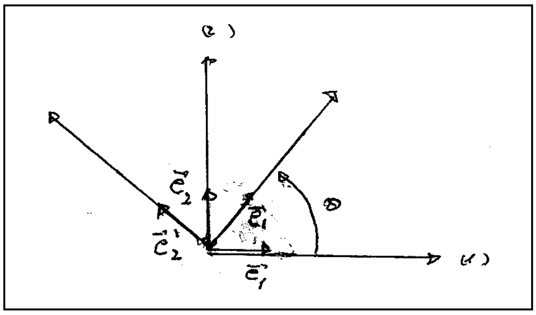
We shall assume that you are familiar with elementary group theory, complex numbers, and linear algebra and its representation by matrices. Otherwise, you may look it up in my home-page www.olewitthansen.dk: Elementary Group theory, The complex number system, and Eigenvalue problems in linear algebra.

An algebraic structure $(G,*)$ is called a group, if:

1. The composition * is associative.
2. There exists a neutral element *e*.
3. Every element in $(G,*)$ has an inverse element.

The theory of particles, especially the standard model is highly based on symmetries and advanced Group Theory.

1. Rotations of the Cartesian coordinate system.



First we shall consider an ordinary coordinate system in the plane, which is rotated an angle θ.
The base vectors $(\vec{e}_1,\vec{e}_2)$ are rotated by an angle θ into $(\vec{e}_1',\vec{e}_2')$. From the figure we see, that:

$\vec{e}_1' = \vec{e}_1\cos\theta + \vec{e}_2\sin\theta$ *and* $\vec{e}_2' = -\vec{e}_1\sin\theta + \vec{e}_2\cos\theta$

When written in matrix form.

$$(\vec{e}_1',\vec{e}_2') = (\vec{e}_1,\vec{e}_2)\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$$

Then we consider a rotation in space around the *z*-axis. The base vectors are $(\vec{e}_1,\vec{e}_2)$
If $P\ (x_1,x_1,x_1)$ is a point in space and $P'(x'_1,x'_1,x'_1)$ is the point after rotation, then we may write the two equations, since the x_3 and $\vec{e}_3$ are unchanged

$$\vec{OP} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 \quad \text{and} \quad \vec{OP} = x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3\vec{e}_3$$

If we take the scalar product of the first equation with $\vec{e}_1'$ and $\vec{e}_2'$ respectively, we have.

$$x_1' = \vec{OP}\cdot\vec{e}_1' = x_1\vec{e}_1\cdot\vec{e}_1' + x_2\vec{e}_2\cdot\vec{e}_1' + x_3\vec{e}_3\cdot\vec{e}_1' = x_1\cos\theta + x_2\sin\theta + 0$$
$$x_2' = \vec{OP}\cdot\vec{e}_2' = x_1\vec{e}_1\cdot\vec{e}_2' + x_2\vec{e}_2\cdot\vec{e}_2' + x_3\vec{e}_3\cdot\vec{e}_2' = -x_1\sin\theta + x_2\cos\theta + 0$$
$$x_3' = \vec{OP}\cdot\vec{e}_3' = \vec{OP}\cdot\vec{e}_3 = x_1\vec{e}_1\cdot\vec{e}_3 + x_2\vec{e}_2\cdot\vec{e}_3 + x_3\vec{e}_3\cdot\vec{e}_3 = x_3$$

In 3-dimensional space, this transformation corresponds to a rotation around the *z*-axis, an angle θ. The transformation matrix then becomes:

(1.1) $$\begin{pmatrix}x' \\ y' \\ z'\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x \\ y \\ z\end{pmatrix}$$ (Rotation about the z-axis)

For the rotations around the y-axis or the *x*-axis, we have quite similar expressions

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{(Rotation about the y-axis)}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{(Rotation about the } x\text{-axis)}$$

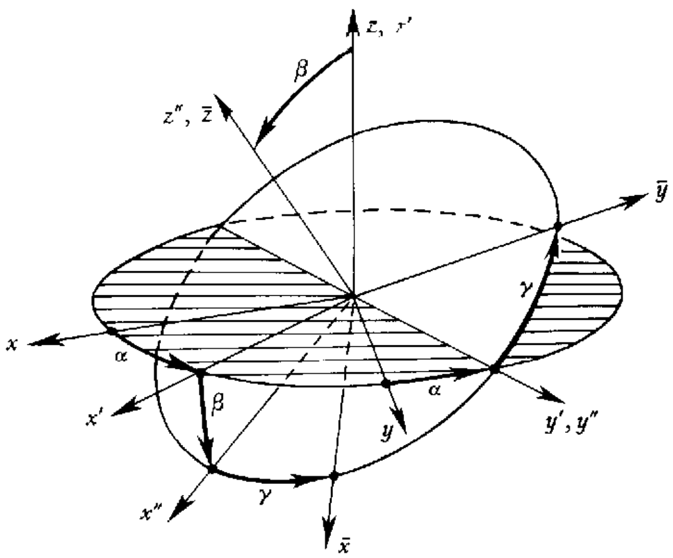
To bring the coordinate system into an arbitrary angular position, we need three rotations. They are traditionally chosen as a rotation around the *z*-axis an angle α, followed by a rotation around the new *y*-axis (*y*') an angle β, and finally an angle γ around the new *z*-axis z''. The overall rotation is found by multiplying the three matrices. The angles α, β, γ are called the Euler angles. They are illustrated below:

$$R(\alpha,\beta,\gamma) = R_z''(\gamma)\, R_y'(\beta)\, R_z(\alpha) =$$

$$\begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$(1.2) \quad \begin{pmatrix} \cos\beta\cos\alpha\cos\gamma - \sin\alpha\sin\gamma & \cos\beta\sin\alpha\cos\gamma + \cos\alpha\sin\gamma & -\sin\beta\cos\gamma \\ -\cos\beta\cos\alpha\sin\gamma - \sin\alpha\cos\gamma & -\cos\beta\sin\alpha\sin\gamma + \cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix}$$

Showing the Euler angles.



In general a rotation in a *n*-dimensional space is a linear transformation given by the equation.

(1.3) $$x_j' = \sum_{i=1}^{n} R_{ij} x_i$$

We shall occasionally apply Einstein's summation convention, that is, whenever an index appears twice in an expression summation over that index is understood.
So we write (1,3) as

(1.4) $$x_j' = R_{ij} x_i$$

Which is the same as the matrix equation $\underline{\underline{x}}' = \underline{\underline{Rx}}$.

The transposed matrix $\underline{\underline{R}}^T$ to a matrix $\underline{\underline{R}}$ is a matrix where the rows and columns change role, such that $R^T{}_{ij} = R_{ji}$. The transposed to a column vector is a row vector with the same elements.

The scalar products of two vectors: $\vec{x} \cdot \vec{y} = \underline{\underline{x}}^T \underline{\underline{y}} = \underline{\underline{x}}\, \underline{\underline{y}}^T$.

The length of a vector and the scalar product is conserved by a rotation. This implies:

(1.5) $$\underline{\underline{x}}'^T \underline{\underline{x}}' = (\underline{\underline{R}}\,\underline{\underline{x}})^T \underline{\underline{R}}\,\underline{\underline{x}} = \underline{\underline{x}}^T \underline{\underline{R}}^T \underline{\underline{R}}\,\underline{\underline{x}} = \underline{\underline{x}}^T \underline{\underline{x}} \quad \Rightarrow \quad \underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{E}}$$

A matrix $\underline{\underline{M}}$, where $\underline{\underline{M}}^T \underline{\underline{M}} = \underline{\underline{E}}$ is called *unitary*. The determinant of a unitary matrix is 1 or -1
We know from linear algebra that the determinant of the product matrix is the product of the determinant of each matrices and that determinant of a matrix and the transposed matrix are the same. The latter follows from the fact that each term in the evaluation of the determinant correspond to a term in the evaluation of the determinant of the transposed matrix, having the same factors, but in a different positions.

(1.6) $$det\,(\underline{\underline{R}}^T \underline{\underline{R}}) = det\,(\underline{\underline{R}}^T)\, det\,(\underline{\underline{R}}) = det\,(\underline{\underline{R}})\, det\,(\underline{\underline{R}}) = det\,(\underline{\underline{R}})^2 = det\,(\underline{\underline{E}})^2 = 1$$

So *det* ($\underline{\underline{R}}$)=+/- 1. When *det* ($\underline{\underline{R}}$) =1 it is called a proper rotation.
From the equation $\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{E}}$ we infer that $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$.
Also the product of two unitary matrices is a unitary matrix, since

(1.7) $$\det(\underline{\underline{R}}_1 \underline{\underline{R}}_2) = \det(\underline{\underline{R}}_1)\det(\underline{\underline{R}}_2) = 1$$

Also we know from linear algebra that the transposed matrix of a product is the inverse product of the transposed matrices. Furthermore:

(1.8) $$(\underline{\underline{R}}_1 \underline{\underline{R}}_2)^T = \underline{\underline{R}}_2^T\, \underline{\underline{R}}_1^T = \underline{\underline{R}}_2^{-1}\, \underline{\underline{R}}_1^{-1} = (\underline{\underline{R}}_1 \underline{\underline{R}}_2)^{-1}$$

Complex conjugation is denoted with an asterisk *. $(a + ib)^* = (a - ib)$, where i is the complex unit $i^2 = -1$.

However, working with complex numbers the scalar product of two vectors $u_1 \cdot u_2$ must be replaced by $u_1^* \cdot u_2$ (since otherwise the length of a vector would not be a real positive number).

Also working with complex numbers, the transpose of a matrix, must be replaced by its Hermitian conjugate $\underline{\underline{A}}^{\#}$, which is the transposed complex conjugated matrix.

Real unitary matrices form a group under matrix multiplication. This group is called the special unitary group and is denoted *SO*(3).
Unitary matrices with determinant -1 also preserve scalar product. This comes about by inverting of the coordinate axes combined with a proper rotation.
rotation using the Euler angles

2. The group *SU*(2)

Using complex numbers a matrix *U* is unitary if

(2.1) $$\underline{\underline{U}}\,\underline{\underline{U}}^{\#} = \underline{\underline{U}}^{\#}\,\underline{\underline{U}} = \underline{\underline{E}}$$

We use the symbol # for a Hermitian matrix, that is the complex conjugate (*) and transposed *T* matrix. The product of two Hermitian matrices form a Hermitian matrix, and the inverse matrix to two Hermitian matrices is also Hermitian.
We may write $det(\underline{\underline{U}}) = e^{in\alpha}$. Where α is real, in accordance with the fact that

(2.2) $$det(\underline{\underline{U}}\,\underline{\underline{U}}^{\#}) = det(\underline{\underline{U}})\,det(\underline{\underline{U}}^{\#}) = e^{in\alpha}\,e^{-in\alpha} = 1$$

The special unitary group *SU*(2) is the group of all 2x2 unitary matrices with determinant = 1. They form a group, since the matrix product of two matrices having determinant = 1 is a matrix with determinant = 1, which follows from the product rule of determinants.
SU(2) is a subgroup of *U*(2) the group of 2x2 matrices.
An element of *U*(2) can be written as a "phase factor" $e^{i\alpha}$ and an element of *SU*(2).
The group *SU*(2), is related in a remarkable way to the rotation group *SO*(3). This is a central point in the electroweak sector of the Standard Model. An element of *U*(2) can formally be written as

(2.3) $$\underline{\underline{U}} = \exp(i\underline{\underline{H}})$$

Where *H* is a Hermitian matrix, and where the exponential function is given by its Taylor expansion, which insures that the functional definition of the exponential function $e^x e^y = e^{x+y}$ is still valid. A general expression for a 2x2 Hermitian matrix can be written as.

(2.3) $$\underline{\underline{H}} = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix}$$

Where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real numbers. This enables us to write.

(2.4) $$\underline{\underline{H}} = \alpha_0\underline{\underline{E}} + \alpha_k\sigma^k$$

where the index *k* runs from 1 to 3, and σ^k are the so called Pauli spin matrices:

(2.5) $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

These matrices satisfy

(2.6) $(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = (1) \quad \sigma^j\sigma^k + \sigma^k\sigma^j = 2\delta_{jk}(1) \quad [\sigma^1, \sigma^2] = \sigma^1\sigma^2 - \sigma^2\sigma^1 = 2i\sigma^3$, ect.

We change the notation, so we write a matrix as a capital letter in a parenthesis, instead of a double underscored capital letter.

Since the unit matrix commutes with all matrices, a general member of $U(2)$ can be written as:

(2.7) $(U) = \exp i(\alpha_0(1) + \alpha_k\sigma^k) = \exp(i\alpha_0)\exp(i\alpha_k\sigma^k)$

Omitting the phase factor, the elements of $SU(2)$ are of the form:

(2.8) $(U_s) = \exp(i\alpha_k\sigma^k)$

An element of the group is specified by the three parameters α_k , k=1,2,3 and the matrices σ^k are the corresponding generators of the group. Each has zero trace.

The algebra of the sigma matrices enables us to write these elements in closed form. Let us formally consider the α_k to make up a vector $\vec{\alpha} = \alpha\vec{e}$, and write the "scalar product" $\alpha_k\sigma^k$ as $\alpha\vec{e}_k \cdot \sigma$ We then find:

(2.9) $(\vec{e}_k \cdot \sigma)^2 = e_j\sigma^j e_k\sigma^k = e_j e_j = (1)$

since $\sigma^j\sigma^k + \sigma^k\sigma^j = 0$ and $(\sigma^i)^2 = 1$

The formal power series of $(U_s) = \exp(i\alpha_k\sigma^k)$ is

(2.10) $(U_s) = (1) + i\alpha(\vec{e} \cdot \sigma) + \dfrac{(i\alpha)^2}{2}(1) + ... =$

$\cos\alpha(1) + i\sin\alpha(\vec{e} \cdot \sigma)$

To establish the connection between the group $SU(2)$ and the rotation group $SO(3)$, we associate with each point $P = x = (x_1, x_2, x_3)$ in space the Hermitian matrix.

(2.11) $\underline{\underline{\mathbf{X}}}(\mathrm{t}) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$

This matrix has Tr $X = 0$ and det(X) = $-x_k x_k$.

Consider now an element U from $SU(2)$ and forming the matrix.

$$\underline{\underline{X}}' = \underline{UXU^{\#}}$$

$\underline{\underline{X}}'$ is also Hermitian, and Tr($UXU^{'}$) = Tr($U^{\#}UX$) = Tr(X) = 0, and $\underline{\underline{X}}'$ can be written in the general form.

(2.12) $$X'(x) = \begin{pmatrix} x'_3 & x'_1 - ix'_2 \\ x'_1 + ix'_2 & -x'_3 \end{pmatrix}$$

The x_k' is related to the x_k by a linear transformation. Also

$$\det(X') = \det(U)\det(X)\det(U^{\#}) = \det(X)\det(UU^{\#}) = \det(X)$$

Since det(X) = $-x_k x_k$. Then it follows that $x_k x_k$ = $.x_k' x_k'$.
Since the length of $x = (x_1, x_2, x_3)$ is preserved by the unitary transformation, the transformation must therefore belong to the rotation group SO(3), and we have then established a one to one correspondence between the groups SO(3) and SU(2).

As an example, we shall show that: $(U) = \exp(i\theta)\sigma^3) = (I)\cos(\theta) + i\sin(\theta)\sigma^3 = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$

(2.13) Corresponds to the rotation matrix: $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

This can be shown by matrix multiplication of $\underline{\underline{X}}' = \underline{UXU^{\#}}$, where we in the calculation for simplicity, replace $\frac{\theta}{2}\, by\, \theta$

$$X'(x) = \begin{pmatrix} x'_3 & x'_1 - ix'_2 \\ x'_1 + ix'_2 & -x'_3 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} =$$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} x_3 e^{-i\theta} & (x_1 - ix_2)e^{i\theta} \\ (x_1 + ix_2)e^{-i\theta} & -x_3 e^{i\theta} \end{pmatrix} = \begin{pmatrix} x_3 & (x_1 - ix_2)e^{i2\theta} \\ (x_1 + ix_2)e^{-i2\theta} & -x_3 \end{pmatrix} =$$

Replacing back $\theta\, by\, \frac{\theta}{2}$

$$\begin{pmatrix} x_3 & (x_1 - ix_2)(\cos\theta + i\sin\theta) \\ (x_1 + ix_2)(\cos\theta - i\sin\theta) & -x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_3 & x_1\cos\theta + x_2\sin\theta - i(-x_1\sin\theta + x_2\cos\theta) \\ (x_1\cos\theta + x_2\sin\theta) + i(-x_1\sin\theta + x_2\cos\theta) & -x_3 \end{pmatrix}$$

This is seen to comply with $x_1' = x_1\cos\theta + x_2\sin\theta$ and $x_2' = -x_1\sin\theta + x_2\cos\theta$

3. The group *SL*(2,C) and the proper Lorentz group

The set of all 2 x 2 matrices, with complex elements and with determinant equal to 1 forms a group under matrix multiplication. This group is denoted *SL*(2,C). It is related to the proper group of Lorentz transformation in much the same way as *SU*(2) is related to the group of proper rotations. We now relate each point $x = (x_0, x_1, x_2, x_3)$ in space-time to the general Hermitian

(3.1) $$X(t) = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

Having the determinant $\det(X(t)) = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2$, (since the mixed products cancel). Consider an element $\boldsymbol{M}$ of $SL(2,C)$ and the matrix $\boldsymbol{X}'$ given by:

(3.2) $$\boldsymbol{M}^{\#}\boldsymbol{X}'\boldsymbol{M} = \boldsymbol{X} \text{ or } \boldsymbol{X}' = (\boldsymbol{M}^{-1})^{\#}\,\boldsymbol{X}\,\boldsymbol{M}^{-1}$$

Then $\boldsymbol{X}'$ is also Hermitian and hence we may write

$$X(t) = \begin{pmatrix} x'_0 + x'_3 & x'_1 - ix'_2 \\ x'_1 + ix'_2 & x'_0 - x'_3 \end{pmatrix}$$

Where the marked coordinates are related to the unmarked by a linear transformation. Also

$$det(\boldsymbol{M}^{\#}\boldsymbol{X}'\boldsymbol{M}) = det(\boldsymbol{M}^{\#})\, det(\boldsymbol{X}')\det(\boldsymbol{M}) = \det(\boldsymbol{X}) \text{, so that}$$

$$\det(X') = x'_0{}^2 - x'_1{}^2 - x'_2{}^2 - x'_3{}^2 = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2$$

Hence the matrix $\boldsymbol{M}$, corresponds to a Lorentz transformation.

The Lorentz transformation connects an event seen from two inertial systems *K* and *K*', where *K'* moves with a sped *v*, relative to *K* along the *z* –axis. The choice of the *z* – axis (instead of usually the *x* axis), is caused by the formal similarity of the Lorentz with a ordinary rotation around the *z*-axis. The marked (') coordinates belong to *K*'.

(3.3) $$\begin{aligned} t' &= \frac{t - zv/c^2}{\sqrt{1 - v^2/c^2}} \\ x' &= x \\ y' &= y \\ z' &= \frac{z - vt}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

Introducing $x_0 = ct,\ \ x_1 = x,\ x_2 = y,\ x_3 = z,\ \beta = v/c,\ \ \gamma = \sqrt{1-\beta^2}$ and using $x_\mu,\ where\ \ \mu = (0,1,2,3)$, we may write the Lorentz transformation

$$(3.4)\qquad \begin{aligned} ct' &= \gamma(ct - \beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(z - \beta ct) \end{aligned} \qquad \text{Or written with the indexed coordinates} \qquad \begin{aligned} x_0' &= \gamma(ct - \beta z) \\ x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= \gamma(x_3 - \beta ct) \end{aligned}$$

To enhance the similarity with a proper rotation around the *z*-axis we put $\beta = v/c = \tanh\theta, \;\; and \;\; \gamma = \cosh\theta$. Such that $\beta\gamma = \sinh\theta$. Using this notation, we may write the Lorentz transformation:

$$(3.5)\qquad \begin{aligned} x_0' &= x_0 \cosh\theta - x_3 \sinh\theta \\ x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= -x_0 \sinh\theta + x_3 \cosh\theta \end{aligned}$$

Notice the formal resemblance with a proper rotation around the *z*-axis, except for the time coordinate.

$$(3.6)\qquad \begin{aligned} t' &= t \\ x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}$$

Transformation to an axis parallel system, but moving in an arbitrary direction is called a (Lorentz) *boost*. There is a linear relation between the coordinates x_μ and the moving coordinates x_μ'. This relation is specified by six parameters. There are three parameters to specify the orientation of the moving system, and three parameters to specify the velocity.
In the following, we shall often use Einstein's summation convention that whenever an index appears twice, summation over that index is implied.

The transformation *L* matrix between the two sets of coordinates can then be formally written:

$$(3.7)\qquad x_\mu' = L_{\mu\nu} x_\nu$$

In some books, special coordinates have an upper index. Although it is implied in Tensor Analysis, where upper and lower indices have difference significance, we prefer to keep the lower indices until now, since the upper indices may be confused with power index.

The matriices $L_{\mu\nu}$ form a group, called the proper Lorentz group.
The interval between two events: $x_\mu \;\; and \;\; x_\mu + \Delta x_\mu$ is given by

$$\Delta s^2 = \Delta x_0^{\;2} - \Delta x_1^{\;2} - \Delta x_2^{\;2} - \Delta x_3^{\;2}$$

It is a fundamental property of the Lorentz transformations, that the square of the interval length is an invariant. $\Delta s'^2 = \Delta s^2$.
The element of *SL*(2,*C*) that correspond to a Lorentz *boost* from above, can be written as

(3.8) $$P = \exp(\tfrac{\theta}{2}\sigma^3) = \cosh\tfrac{\theta}{2}(E) + \sinh\tfrac{\theta}{2}\sigma^3 = \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix}$$

This may be shown by matrix multiplication

(3.9)
$$X(t) = \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix}\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}\begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} =$$
$$\begin{pmatrix} (x_0 + x_3)e^{\theta/2} & (x_1 - ix_2)e^{\theta/2} \\ (x_1 + ix_2)e^{-\theta/2} & (x_0 - x_3)e^{-\theta/2} \end{pmatrix}\begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} =$$
$$\begin{pmatrix} (x_0 + x_3)e^{\theta} & (x_1 - ix_2) \\ (x_1 + ix_2) & (x_0 - x_3)e^{-\theta} \end{pmatrix} =$$
$$\begin{pmatrix} (x_0 + x_3)(\cosh\theta + \sinh\theta) & (x_1 - ix_2) \\ (x_1 + ix_2) & (x_0 - x_3)(\cosh\theta + \sinh\theta) \end{pmatrix} =$$
$$\begin{pmatrix} x_0 + x_3)(\cosh\theta + \sinh\theta) & (x_1 - ix_2) \\ (x_1 + ix_2) & (x_0 - x_3)(\cosh\theta + \sinh\theta) \end{pmatrix}$$

Making the substitution $\theta \to -\theta$, we find:

$$\begin{pmatrix} (x_0 + x_3)(\cosh\theta - \sinh\theta) & (x_1 - ix_2) \\ (x_1 + ix_2) & (x_0 - x_3)(\cosh\theta - \sinh\theta) \end{pmatrix} =$$

$$\begin{pmatrix} x_0\cosh\theta - x_3\sinh\theta - x_0\sinh\theta + x_3\cosh\theta & (x_1 - ix_2) \\ (x_1 + ix_2) & -x_0\sinh\theta + x_3\cosh\theta - x_0\sinh\theta + x_3\cosh\theta \end{pmatrix}$$

Which is seen to comply with:

$$\begin{aligned} x_0' &= x_0\cosh\theta - x_3\sinh\theta \\ x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= -x_0\sinh\theta + x_3\cosh\theta \end{aligned}$$

More generally a Lorentz boost from a frame K to a Frame K' moving with velocity $v = \tanh\theta$ moving in the direction $\boldsymbol{e}_v$ (a unit vector), is given by.

$$\boldsymbol{P} = \exp(\tfrac{\theta}{2}\vec{e}_v \cdot \sigma) = \cosh(\tfrac{\theta}{2})(E) + \sinh(\tfrac{\theta}{2})\vec{e}_v \cdot \sigma$$

Note that since the σ matrices are Hermitian then also the matrices $\boldsymbol{P}$ corresponding to any boost.

3.1 Transformation of the Pauli matrices

In discussing Lorentz transformations it is convenient to write σ^0 as the unit matrix., and we introduce the notation $\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ and $\tilde{\sigma}^\mu = (\tilde{\sigma}^0, -\tilde{\sigma}^1, -\tilde{\sigma}^2, -\tilde{\sigma}^3)$
We then have $X(x) = x_0\sigma^0 + x_k\sigma^k = x_\mu\sigma^\mu$ and $X'(x') = x_\mu'\tilde{\sigma}^\mu$
The relation

$$\boldsymbol{M}^{\#}\boldsymbol{X'M} = \boldsymbol{X}$$

Gives

$$x_\mu' M^{\#}\tilde{\sigma}^\mu M = x_\mu\tilde{\sigma}^\mu = L^\mu_\nu\tilde{\sigma}^\upsilon x_\mu'$$

since x_μ' is arbitrary we must have

$$M^{\#}\tilde{\sigma}^\mu M = L^\mu_\nu\tilde{\sigma}^\upsilon$$

And also

$$L^\mu_\nu = \tfrac{1}{2} tr(\tilde{\sigma}^\nu M^{\#}\tilde{\sigma}^\mu M)$$

We use the notation using an upper index: $x^\mu = (x^0, x^1.x^2, x^3) = (x_0, -x_1, -x_2, -x_3)$
Then

$$X_1 = x_0\sigma^0 - x_1\sigma^1 - x_2\sigma^2 - x_3\sigma^3 = x^\mu\sigma^\mu$$

Which also has the determinant $\det(X_1) = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2$, we can show that there exists a matrix $\boldsymbol{N}$, belonging to $SL(2,C)$, such that.

$$N^{\#}\sigma^\mu N = L^\mu_\nu\sigma^\upsilon$$

The matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ are evidently related. One can verify that when $\boldsymbol{M} = \boldsymbol{P}$, where $\boldsymbol{P}$ is given by (X,X), and corresponds to a Lorentz boost, we can take $\boldsymbol{N} = \boldsymbol{P}^{-1}$, and this will be true for a Lorentz boost in any direction. For a pure rotation of axes, we have $\boldsymbol{N} = \boldsymbol{M} = \boldsymbol{U}$, where U is a unitary matrix A general $\boldsymbol{M}$ can be found as the product of a rotation followed by a boost. $\boldsymbol{M} = \boldsymbol{PU}$
The corresponding $\boldsymbol{N}$, is then given by $\boldsymbol{N} = \boldsymbol{P}^{-1}\boldsymbol{U}$. Now $\boldsymbol{U}$ satisfy $\boldsymbol{U}^{\#}\boldsymbol{U} = \boldsymbol{E}$, and since $\boldsymbol{P}$ is Hermitian $\boldsymbol{P}^{\#} = \boldsymbol{P}$, hence: $\boldsymbol{NM}^{\#} = (\boldsymbol{P}^{-1}\boldsymbol{U})(\boldsymbol{U}^{\#}\boldsymbol{P}) = \boldsymbol{E}$, so $\boldsymbol{N}$ is the inverse of $\boldsymbol{M}^{\#}$.
The results (x,x) and (y,y) together with B19 are useful in constructing Lorentz scalars, vectors and higher order tensors.

4. The group *SU*(3)

The special unitary group $SU(3)$ is the group of Hermitian matrices with determinant equal to 1 . The discussion will lie on the treatment of $SU(2)$. An element in $SU(2)$ can formally be expressed as:

(4.1) $$\boldsymbol{U} = \exp(i\boldsymbol{H})$$

Where $\boldsymbol{H}$ is an Hermitian matrix. A general Hermitian matrix $\boldsymbol{U}$ i characterized by 3x3 = 9 real parameters.

The condition that det($\boldsymbol{U}$) = 1 reduces it to 8 parameters. Instead of the σ^k matrices in the previous section, we have 8 traceless matrices, which was introduced by Gell-Mann.

(4.2)
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

A general traceless Hermitian matrix is of the form

(4.3)
$$H = \alpha_1\lambda_1 + \alpha_2\lambda_2 + ... + \alpha_8\lambda_8$$
$$= \begin{pmatrix} \alpha_3 + \frac{1}{\sqrt{3}}\alpha_8 & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -\alpha_3 - \frac{1}{\sqrt{3}}\alpha_8 & \alpha_6 - i\alpha_7 \\ \alpha_1 + i\alpha_5 & \alpha_6 + i\alpha_7 & -\frac{2}{\sqrt{3}}\alpha_8 \end{pmatrix}$$

The matrices λ_α satisfy the commutation relations

(4.4)
$$[\lambda_a, \lambda_b] = 2i\sum_{c=1}^{8} f_{abc}\lambda_c$$

Where f_{abc} are the structure constants. The f_{abc} are odd in the interchange of any pair of indices, and the non vanishing f_{abc} are given for permutation of f_{abc} =1

$$f_{147} = f_{264} = f_{257} = f_{345} = f_{516} = f_{367} = \tfrac{1}{2}\ ,\ f_{458} = f_{678} = \sqrt{3}/2$$

The matrices also have the property: $Tr(\lambda_a, \lambda_b) = 2\delta_{ab}$
This can be verified by direct calculation.

Reference:W.N. Cottingham and D.A. Greenwod. The Standard Model of Particle Physics.